1. Consider the sequence defined by the recursion $a_{n+1} = \sqrt{a_n}$.

   (a) Show that if $0 < a_0 < 1$, then a_n is convergent.

   Solution.
   Observe that if $x \in (0,1)$, then (i) $\sqrt{x} \in (0,1)$ and (ii) $\sqrt{x} > x$. Thus if $a_n \in (0,1)$ then $0 < a_n < a_{n+1} < 1$. By induction (formal proof not necessary), if our sequence begins with $a_0 \in (0,1)$ it will be bounded and increasing. By the monotone sequence theorem it will be convergent.

   (b) Given that a_n is convergent and $0 < a_0 < 1$, evaluate $L = \lim_{n\to\infty} a_n$.

   Solution.
   Since $f(x) = \sqrt{x}$ is continuous on its domain we have

$$\begin{aligned} L &= \lim_{n\to\infty} a_{n+1} \\ &= \lim_{n\to\infty} \sqrt{a_n} \\ &= \sqrt{\lim_{n\to\infty} a_n} \\ &= \sqrt{L} \end{aligned}$$

   L therefore solves the equation $L = \sqrt{L}$, which is equivalent to $L^2 = L$ or $L(L-1) = 0$. There are two possibilities, namely $L = 0$ and $L = 1$. Since a_n is increasing we must have $L \geq a_0 > 0$, and therefore $L = 1$.

2. Evaluate the sum of the (convergent) series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n(n+1)}$.

   Solution
   Noting that $\frac{2}{n(n+1)} = \frac{2}{n} - \frac{2}{n+1}$ we find that the partial sums of this series

are

$$\begin{aligned}
s_N &= \sum_{n=1}^{N} \frac{2}{n(n+1)} \\
&= \sum_{n=1}^{N} \left[\frac{2}{n} - \frac{2}{n+1}\right] \\
&= \sum_{n=1}^{N} \frac{2}{n} - \sum_{n=1}^{N} \frac{2}{n+1} \\
&= \left[\frac{2}{1} + \frac{2}{2} + \ldots + \frac{2}{N-1} + \frac{2}{N}\right] - \left[\frac{2}{2} + \frac{2}{3} + \ldots + \frac{2}{N} + \frac{2}{N+1}\right] \\
&= 2 - \frac{2}{N+1}
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{2}{n(n+1)} &= \lim_{N\to\infty} s_N \\
&= \lim_{N\to\infty} \left(2 - \frac{2}{N+1}\right) \\
&= 2
\end{aligned}$$

3. Determine whether the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n+1}$ is absolutely convergent, conditionally convergent or divergent. Justify your answer.

Solution

The series is divergent since $\lim_{n\to\infty} (-1)^{n+1} \frac{n}{n+1}$ does not exist (even terms converge to -1, odd terms to 1).

4. Let s denote the sum of the (convergent) infinite series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

(a) It can be shown that $1+\frac{1}{2^4}+\frac{1}{3^4}+\ldots+\frac{1}{10^4}=1.0820\ldots$. Use this fact to derive an upper and lower bound for s.

Solution
If $R_n=\frac{1}{(n+1)^4}+\frac{1}{(n+2)^4}+\ldots$ we know that

$$\int_{n+1}^{\infty}\frac{1}{x^4}\,dx\le R_n\le\int_{n}^{\infty}\frac{1}{x^4}\,dx\ .$$

And since $\int_n^\infty \frac{1}{x^4}\,dx=\frac{1}{3n^3}$ this yields the following

$$\frac{1}{3\cdot 11^3}\le R_{10}\le\frac{1}{3\cdot 10^3}\ .$$

Adding $s_{10}=1.0820\ldots$ to each side gives

$$1.0820\ldots+\frac{1}{3\cdot 11^3}\le s\le 1.0820\ldots\frac{1}{3\cdot 10^3}\ .$$

(b) In (a) we used 10 terms to estimate s. How many terms would we need to use in order to ensure that the resulting error was no larger than $\frac{10^{-6}}{3}$?

Solution Since $R_n\le\int_n^\infty\frac{1}{x^4}\,dx=\frac{1}{3n^3}$, it suffices to ensure that $\frac{1}{3n^3}\le\frac{10^{-6}}{3}$. This will occur if and only if $n\ge 100$, thus we need at least 100 terms.

5. Determine whether the series $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}+7}{\sqrt{n^3+3n-1}}$ converges or diverges.

Solution
When n is large, we would expect that

$$\frac{\sqrt{n}+7}{\sqrt{n^3+3n-1}}\approx\frac{\sqrt{n}}{\sqrt{n^3}}=\frac{n^{1/2}}{n^{3/2}}=\frac{1}{n}\ .$$

To verify this let $b_n = \frac{1}{n}$ and observe that

$$\begin{aligned}
\frac{a_n}{b_n} &= na_n \\
&= \frac{n\sqrt{n}+7n}{\sqrt{n^3+3n-1}} \\
&= \frac{n^{3/2}+7n}{\sqrt{n^3+3n-1}} \\
&= \frac{n^{3/2}\left(1+7n^{-1/2}\right)}{n^{3/2}\sqrt{1+3n^{-2}-n^{-3}}} \\
&= \frac{1+7n^{-1/2}}{\sqrt{1+3n^{-2}-n^{-3}}}
\end{aligned}$$

It is clear now that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, and since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} a_n$ also diverges by limit comparison.

6. Determine whether the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n\, n!}{(2n)!}$ is absolutely convergent, conditionally convergent or divergent.

Solution
Let $a_n = \frac{(-2)^n n!}{(2n)!}$ so that $|a_n| = \frac{2^n n!}{(2n)!}$ and

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \frac{2^{n+1}}{2^n} \cdot \frac{(n+1)!}{n!} \cdot \frac{(2n)!}{(2(n+1))!} \\
&= 2 \cdot (n+1) \cdot \frac{1}{(2n+2)(2n+1)} \\
&= \frac{1}{2n+1}
\end{aligned}$$

Thus $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 0 < 1$, and the series is absolutely convergent by the Ratio Test.

7. Determine whether the series $\displaystyle\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{n\ln(n)}$ is absolutely convergent, conditionally convergent or divergent.

Solution

To check absolute convergence let $f(x)=\frac{1}{x\ln(x)}$ and observe that f is clearly positive and decreasing on $[2,\infty)$. Moreover the substitution $u=\ln(x)$ yields

$$\int_2^\infty \frac{1}{x\ln(x)}\,dx=\int_{\ln(2)}^\infty \frac{1}{u}\,du=\infty\ .$$

Therefore the series $\sum_{n=2}^\infty \frac{1}{n\ln(n)}$ diverges by the Integral Test, and our series is not absolutely convergent.

To check convergence let $b_n=\frac{1}{n\ln(n)}$ and observe that (i) b_n is clearly decreasing and (ii) $\lim_{n\to\infty}b_n=0$. Therefore the series $\sum_{n=2}^\infty(-1)^{n+1}\frac{1}{n\ln(n)}$ converges by the Alternating Series Test.

Therefore the series $\sum_{n=2}^\infty(-1)^{n+1}\frac{1}{n\ln(n)}$ is conditionally convergent.

8. (a) Is the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n+1}}$ conditionally or absolutely convergent? Justify your answer.

Solution

The series is convergent by the Alternating Series Test. But $\sum_{n=1}^\infty \frac{1}{\sqrt{n+1}}$ clearly diverges, so this convergence is not absolute. Therefore the series is conditionally convergent.

(b) How many terms would be required in order to estimate the sum of the series in part (a) with an error that does not exceed 10^{-4}?

Solution

Since this series satisfies the conditions of the Alternating Series Test we know that

$$|R_n| \leq b_{n+1} = \frac{1}{\sqrt{n+2}} \ .$$

Thus in order to ensure that $|R_n| \leq 10^{-4}$ it suffices to ensure that $\frac{1}{\sqrt{n+2}} \leq 10^{-4}$, which requires $n \geq 10^8 - 2$. Thus if we use at least $10^8 - 2$ terms we can be sure the resulting error does not exceed 10^{-4}.

9. Determine the radius and interval of convergence for the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(2x-7)^n}{3n+1}$.

Solution

To begin note that $(2x-7)^n = 2^n \left(x - \frac{7}{2}\right)^n$, and we see that the series is centered at $a = \frac{7}{2}$. Now fix $x \neq \frac{7}{2}$ and let $a_n = \frac{(2x-7)^n}{3n+1}$, so that

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \frac{|2x-7|^{n+1}}{|2x-7|^n} \cdot \frac{3n+1}{3(n+1)+1} \\ &= |2x-7| \cdot \frac{3n+1}{3n+4} \end{aligned}$$

Thus

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |2x-7| \cdot \lim_{n\to\infty} \frac{3n+1}{3n+4} = |2x-7| \ ,$$

and the Ratio Test ensures that our series converges whenever $|2x-7| < 1$, or $\left|x - \frac{7}{2}\right| < \frac{1}{2}$, and diverges whenever $\left|x - \frac{7}{2}\right| > \frac{1}{2}$. Therefore the radius of convergence is $R = \frac{1}{2}$.

In order to determine the interval of convergence we must check the endpoints $\frac{7}{2} \pm \frac{1}{2}$, which are simply 3 and 4. When $x = 3$ the series becomes $\sum_{n=1}^{\infty} (-1)^n \frac{1}{3n+1}$, which converges by the Alternating Series Test. When $x = 4$ the series becomes $\sum_{n=1}^{\infty} \frac{1}{3n+1}$, which diverges by comparison (limit or otherwise) with the harmonic series. Thus the interval of convergence is $[3, 4)$.

10. (a) Express $\frac{x}{1+x^4}$ as a power series. Be sure to indicate the radius and interval of convergence.

Solution

Using the geometric series (and assuming $|x| < 1$, so that $|-x^4| < 1$) we find that

$$\begin{aligned} \frac{1}{1+x^4} &= \frac{1}{1-(-x^4)} \\ &= \sum_{n=0}^{\infty} \left(-x^4\right)^n \\ &= \sum_{n=0}^{\infty} (-1)^n x^{4n} \\ &= 1 - x^4 + x^8 - x^{12} + x^{16} - \dots \end{aligned}$$

Thus

$$\begin{aligned} \frac{x}{1+x^4} &= x \sum_{n=0}^{\infty} (-1)^n x^{4n} \\ &= \sum_{n=0}^{\infty} (-1)^n x^{4n+1} \\ &= x - x^5 + x^9 - x^{13} + x^{17} - \dots \end{aligned}$$

The radius and interval of convergence are 1 and $(-1, 1)$.

(b) Estimate $\int_0^1 \frac{x}{1+x^4}\, dx$ using the first three (non-zero) terms of an appropriate series.

Solution

Integrating term-by-term we obtain

$$\begin{aligned}
\int_0^1 \frac{x}{1+x^4}\,dx &= \int_0^1 \left[x - x^5 + x^9 - x^{13} + x^{17} - \ldots\right]\,dx \\
&= \int_0^1 x\,dx - \int_0^1 x^5\,dx + \int_0^1 x^9\,dx - \ldots \\
&= \frac{1}{2} - \frac{1}{6} + \frac{1}{10} - \ldots
\end{aligned}$$

Thus an estimate based on the first three terms is simply

$$\int_0^1 \frac{x}{1+x^4}\,dx \approx \frac{1}{2} - \frac{1}{6} + \frac{1}{10} = \frac{13}{30}\,.$$

Note also that

$$\int \frac{x}{1+x^4}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{4n+2}\,,$$

whose interval of convergence is $(-1, 1]$, so that this term-by-term integration is in fact permitted.

11. Suppose that $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \dfrac{2}{3}$. Evaluate $\lim\limits_{n\to\infty} a_n$.

Solution

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{2}{3}$, it follows that $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \frac{2}{3}$ as well (the function $f(x) = |x|$ is continuous on all of $\mathbb{R}$). Since $\frac{2}{3} < 1$ the Ratio Test ensures that the series $\sum_{n=1}^{\infty} a_n$ converges, and therefore $\lim_{n\to\infty} a_n = 0$.