10 marks 1. If $a_n = (-1)^n \dfrac{n}{n+1}$, is $\{a_n\}_{n=1}^{\infty}$ monotonic? Is it bounded? Explain.

$\{a_n\}_{n=1}^{\infty}$ is not monotonic because if n is odd $a_n = -\dfrac{n}{n+1} < 0$ and if n is even $a_n = \dfrac{n}{n+1} > 0$ which means that

$a_1 \le a_2$, $a_3 \le a_2$, $a_3 \le a_4$, $a_5 \le a_4$, ... So $\{a_n\}$ is neither increasing nor decreasing.

Since $|a_n| = \left|(-1)^n \dfrac{n}{n+1}\right| = \left|\dfrac{n}{n+1}\right| = \dfrac{n}{n+1} \le 1$, for all $n = 1, 2, 3, \dots$, $\{a_n\}$ is a bounded seque-nce.

5 marks 2. Let $a_n = \frac{1}{2^n}$. Use the formal ϵ, N definition to show that $\lim_{n\to\infty} a_n = 0$.

For a given $\varepsilon > 0$, we wish to find an integer $N > 0$ s.t. for any $n > N$ we have $\left|\frac{1}{2^n} - 0\right| < \varepsilon$. In order to have $\left|\frac{1}{2^n}\right| < \varepsilon$, we study this inequality:

$$\left|\frac{1}{2^n}\right| < \varepsilon \Rightarrow \frac{1}{2^n} < \varepsilon \Rightarrow 2^n > \frac{1}{\varepsilon} \Rightarrow n > \log_2 1/\varepsilon$$

$$\Rightarrow n > \log_2 \varepsilon^{-1} = -\log_2 \varepsilon = -\frac{\ln(\varepsilon)}{\ln(2)}.$$

This is suggesting that we should pick

$N = \left\lceil -\frac{\ln(\varepsilon)}{\ln(2)} \right\rceil$ = the smallest integer greater than or equal to $-\frac{\ln(\varepsilon)}{\ln(2)}$.

If $n > N \Rightarrow n > -\frac{\ln(\varepsilon)}{\ln(2)} \Rightarrow n > \log_2 \varepsilon^{-1} = \log_2 1/\varepsilon$

$$\Rightarrow 2^n > 1/\varepsilon \Rightarrow \frac{1}{2^n} < \varepsilon \Rightarrow \left|\frac{1}{2^n} - 0\right| < \varepsilon.$$

Note that if $-\frac{\ln(\varepsilon)}{\ln(2)}$ is negative $(\varepsilon > 1)$, we can choose $N = 1$.

5 marks 3. Use the Squeeze Theorem to determine $\lim_{n\to\infty} \frac{n+\ln n}{n^2+(-1)^n}$.

First we show that $\ln x \leq x$ if $x \geq 1$: let

$f(x) = x - \ln x$, $f(1) = 1 - 0 = 1 > 0$

$f'(x) = 1 - \frac{1}{x} \geq 0$ if $x \geq 1$.

So $f(x) > 0$ if $x \geq 1$. Now using the above inequality we have

$$\frac{n}{n^2+1} \leq \frac{n+\ln n}{n^2+(-1)^n} \leq \frac{2n}{n^2-1} \quad \text{for } n=1,2,3,\ldots$$

Since $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$ and $\lim_{n\to\infty} \frac{2n}{n^2-1} = 0$,

it follows from the Squeeze Theorem that

$$\lim_{n\to\infty} \frac{n+\ln n}{n^2+(-1)^n} = 0.$$

4.

5 marks (a) Give example of a series $\sum_{n=1}^{\infty} a_n$ such that $\lim_{n\to\infty} a_n = 0$, but the series diverges.

$\sum_{n=1}^{\infty} \frac{1}{n}$: $\lim \frac{1}{n} = 0$ but $\sum_{n=1}^{\infty} \frac{1}{n}$

is divergent by the p-series test $(p=1)$.

5 marks (b) Let $\sum_{n=1}^{\infty} a_n$ be a series with only positive terms, and let $S_N = \sum_{n=1}^{N} a_n$ be the partial sum of the first N terms of the series (i.e., the partial sum of order N). Prove that if $S_N < 5 - \sin(N^2)$, then the series $\sum a_n$ converges.

Since $a_n > 0$ for all n, the sequence of partial sums $S_N = a_1 + a_2 + \dots + a_N$ is increasing; also $S_N < 5 - \sin(N^2) \leq 5 + 1 = 6$ for all N, which shows that $\{S_N\}$ is bounded above. Therefore by the Monotonic Sequence Theorem $\{S_N\}$ is convergent, which means that the series $\sum a_n$ converges to a point on the real line.

5. Determine if each of the following series is convergent or divergent. Be clear about any test for convergence/divergence you apply.

5 marks (a) $\sum_{n=1}^{\infty} \frac{3^{n^2}}{n!}$

Let $a_n = \frac{3^{n^2}}{n!}$. We apply the Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{3^{(n+1)^2}}{(n+1)!}}{\frac{3^{n^2}}{n!}} \right| = \lim_{n\to\infty} \frac{3^{(n+1)^2}\; n!}{3^{n^2}\;(n+1)!}$$

$$= \lim_{n\to\infty} \frac{3^{n^2+2n+1}}{3^{n^2}\;(n+1)} = \lim_{n\to\infty} \frac{3^{2n+1}}{n+1} = \lim_{x\to\infty} \frac{3^{2x+1}}{x+1} \left(= \frac{\infty}{\infty}\right)$$

$$\overset{\text{L'Hospital}}{=} \lim_{x\to\infty} \frac{2\;3^{2x+1}\ln(3)}{1} = \infty.$$ So the series is divergent.

5 marks (b) $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

Let $a_n = \frac{1}{2^n-1} > 0$. By the limit comparison test, using $b_n = \frac{1}{2^n} > 0$, we have:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = 1 > 0;$$

since $\sum \frac{1}{2^n}$ is convergent, $\sum \frac{1}{2^n-1}$ is also convergent.

6. Determine if each of the following series converges absolutely, converges conditionally, or diverges. Be clear about any test for convergence/divergence you apply.

5 marks (a) $\sum_{n=1}^{\infty}(-1)^n \frac{n}{1+\ln n}$

Since $\lim_{n\to\infty} \frac{n}{1+\ln(n)} = \lim_{x\to\infty} \frac{x}{1+\ln(x)}$

$\overset{\text{L'Hospital}}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{x}} = \lim_{x\to\infty} x = \infty,$

$\lim (-1)^n \frac{n}{1+\ln}$ is not 0, and the series is divergent.

5 marks (b) $\sum_{n=1}^{\infty}(-1)^{n-1} \frac{(\ln n)^2}{n}$

Let $b_n = \frac{(\ln n)^2}{n}$. This sequence is decreasing since:

$f(x) = \frac{(\ln x)^2}{x} \Rightarrow f'(x) = \frac{2\frac{1}{x}(\ln x)x - (\ln x)^2}{x^2} = \frac{2\ln x\,(1-\ln x)}{x^2}$

$f'(x) \leq 0$ if $x > e$.

Also $\lim_{n\to\infty} b_n = 0$ since $\lim_{x\to\infty} \frac{(\ln x)^2}{x} = \lim_{x\to\infty} \frac{2\frac{1}{x}\ln x}{1}$

$= \lim_{x\to\infty} \frac{2\ln x}{x} = \lim_{x\to\infty} \frac{\frac{2}{x}}{1} = 0.$

Therefore $\sum (-1)^{n-1} \frac{(\ln n)^2}{n}$ is convergent by the alternating series test.

10 marks 7. Use the power series representation of $\frac{1}{1-x}$ to evaluate $\sum_{n=1}^{\infty} n\left(\frac{1}{3}\right)^{n-1}$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{if} \quad |x| < 1.$$

so $$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}.$$

Therefore by setting $x = 1/3$ we have:

$$\frac{1}{\left(1-\frac{1}{3}\right)^2} = \sum_{n=1}^{\infty} n\left(\frac{1}{3}\right)^{n-1}$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} n\left(\frac{1}{3}\right)^{n-1} = \frac{1}{\left(\frac{2}{3}\right)^2} = \frac{9}{4}.$$

10 marks 8. Given $\sum_{n=1}^{\infty} \frac{1}{2^n\sqrt{n}}(x-1)^n$, what is the interval of convergence of the series?

We apply the Ratio Test: let $a_n = \frac{1}{2^n\sqrt{n}}(x-1)^n$,

then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{2^{n+1}\sqrt{n+1}}(x-1)^{n+1}}{\frac{1}{2^n\sqrt{n}}(x-1)^n}\right|$

$= \lim_{n\to\infty}\left|\frac{\sqrt{n}}{2\sqrt{n+1}}\right||x-1| = \frac{1}{2}|x-1|$

Therefore if $\frac{1}{2}|x-1|<1 \Leftrightarrow |x-1|<2$ the series $\sum_{n=1}^{\infty}\frac{1}{2^n\sqrt{n}}(x-1)^n$ is convergent and if $\frac{1}{2}|x-1|>1 \Leftrightarrow |x-1|>2$, it is divergent.

Now we consider the endpoints $|x-1|=2 \Leftrightarrow x=3$ or $x=-1$:

$x=3 \Rightarrow \sum \frac{1}{2^n\sqrt{n}}(3-1)^n = \sum\frac{1}{\sqrt{n}}$ is divergent by the p-series test $(p=\frac{1}{2})$;

$x=-1 \Rightarrow \sum\frac{1}{2^n\sqrt{n}}(-1-1)^n = \sum\frac{(-1)^n}{\sqrt{n}}$ is convergent by the alternating series test. So the interval of convergence of the series is $[-1,3)$.

9.

5 marks (a) Evaluate $\int e^{-x^2}\,dx$ as a power series centred at the origin. Write the first three nonzero terms of the series.

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{6!} + \cdots$$

$$\Rightarrow \int e^{-x^2}\,dx = \int \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}\,dx = \sum_{n=0}^{\infty} \int \frac{(-x^2)^n}{n!}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \int \frac{x^{2n}}{n!}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + C$$

$$= x - \frac{x^3}{3} + \frac{x^5}{10} - \cdots + C.$$

5 marks (b) Determine the interval of convergence of the power series found in Part (a) above.

The series is $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)\,n!}$.

Let $a_n = (-1)^n \dfrac{x^{2n+1}}{(2n+1)n!}$. By the Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1} x^{2n+3}}{(2n+3)\,(n+1)!}}{\dfrac{(-1)^n x^{2n+1}}{(2n+1)\,n!}} \right| =$$

$$\lim_{n\to\infty} \frac{(2n+1)\;n!\;|x^{2n+3}|}{(2n+3)\,(n+1)!\;|x^{2n+1}|} = \lim_{n\to\infty} \frac{(2n+1)}{(2n+3)(n+1)}\,|x^2| = 0 < 1$$

for all x; so the interval of convergence is $(-\infty,\infty)$.