1. $$\lim_{x \to 3} x^2 = 9$$

Given an arbitrary $\varepsilon > 0$, we wish to find $\delta > 0$ such that if $0 < |x-3| < \delta$ then $|x^2 - 9| < \varepsilon$.

Since $|x^2 - 9| = |x-3||x+3|$, by assuming that $\delta < 1$ we find a bound for $|x+3|$, namely that if $|x-3| < 1$ then $-1 < x-3 < 1$ and $2 < x < 4$, which implies that $5 < x+3 < 7$. Therefore if we pick $\delta < 1$ and $< \varepsilon/7$, $(0 < \delta <$ minimum of 1 and $\varepsilon/7)$, then $0 < |x-3| < \delta$ implies

$$|x^2 - 9| = |x-3||x+3| < \delta \cdot 7 < \frac{\varepsilon}{7} \cdot 7 = \varepsilon$$

which was desired.

2. Let $f(x) = 5x^2 + x$, by definition:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{5(x+h)^2 + (x+h) - (5x^2 + x)}{h}$$

$$= \lim_{h \to 0} \frac{5(x^2 + 2xh + h^2) + x + h - 5x^2 - x}{h}$$

$$= \lim_{h \to 0} \frac{5x^2 + 10xh + 5h^2 + x + h - 5x^2 - x}{h}$$

$$= \lim_{h \to 0} \frac{10xh + 5h^2 + h}{h}$$

$$= \lim_{h \to 0} (10x + 5h + 1) = 10x + 1.$$

3. $f(x) = 2^x (x^2+1)^3 + \arcsin x - \dfrac{x}{\sin x}$

$$f'(x) = 2^x \cdot \ln(2) \cdot (x^2+1)^3 + 2^x \cdot 3 \cdot 2x \cdot (x^2+1)^2$$

$$+ \frac{1}{\sqrt{1-x^2}} - \frac{\sin x - x \cos x}{(\sin x)^2}$$

$$= + \ln(2)\, 2^x (x^2+1)^3 + 6x\, 2^x (x^2+1)^2$$

$$- \csc x + x \cdot \cot x \cdot \csc x$$

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4. Using the Mean Value Theorem, since f is differentiable and therefore continuous also, we have

$$\frac{f(5) - f(1)}{5-1} = f'(c) \quad \text{for some } 1 < c < 5.$$

But $\dfrac{f(5)-f(1)}{5-1} = \dfrac{0-(-16)}{4} = \dfrac{16}{4} = 4$,

which contradicts the fact that $f'(x) < 3$ for all x. So such a function does not exist.

5.

$$\int x \cos(3x)\, dx$$

We use integration by parts

$$\int u\,dv = uv - \int v\,du.$$

setting $u = x$, $dv = \cos(3x)\,dx$ we have:

$$du = dx, \quad v = \frac{1}{3}\sin(3x), \text{ and}$$

$$\int x \cos(3x)\, dx = \frac{1}{3} x \sin(3x) - \int \frac{1}{3} \sin(3x)\, dx$$

$$= \frac{1}{3} x \sin(3x) - \frac{1}{3}\,\frac{-1}{3}\cos(3x) + C$$

$$= \frac{1}{3} x \sin(3x) + \frac{1}{9}\cos(3x) + C.$$

$$\int_1^8 x \ln x \, dx$$

We use integration by parts by setting

$$u = \ln x \qquad dv = x\,dx$$

$$\Rightarrow du = \frac{1}{x}\,dx \qquad v = \frac{x^2}{2}.$$

So $$\int_1^8 x \ln x \, dx = \frac{x^2}{2}.\ln x \Big|_{x=1}^{x=8} - \int_1^8 \frac{x^2}{2}\,\frac{1}{x}\,dx$$

$$= \frac{64}{2}.\ln 8 - \frac{1}{2}\int_1^8 x\,dx$$

$$= 32.\ln 2^3 - \frac{1}{2}\,\frac{1}{2}x^2 \Big|_{x=1}^{x=8}$$

$$= 3(32).\ln 2 - \frac{1}{4}(64-1)$$

$$= 96.\ln 2 - \frac{1}{4}(63)$$