CALCULUS 1501B <u>HW2</u> SOLUTIONS

$\boxed{\#1}$ For positive integers m $\Gamma(m+1)=m!$

$24 = 2\cdot3\cdot4 = 4! = \Gamma(4+1) = \Gamma(5)$ $\quad n^2-4=5 \quad n^2=9 \quad n=\pm3 \quad \Rightarrow n=3 \ (n\in\mathbb{N})$

Answer: $\boxed{n=3}$

$\boxed{\#2}$ <u>Think</u>: as $n\to\infty$ $\quad a_n \to \frac{3}{1+\frac{1}{n^2}} = \frac{3}{1+0} = 3$

we need to show that $\forall\varepsilon>0\ \exists\, n_0\in\mathbb{N}$ s.t. $|a_n-3|<\varepsilon$ for all $n\geq n_0$

$$\left|\frac{3n^2}{n^2+1}-3\right|<\varepsilon$$

$$\Leftrightarrow |3n^2-3(n^2+1)| < \varepsilon(n^2+1)$$

$$\Leftrightarrow 3<\varepsilon(n^2+1)$$

$$\Leftrightarrow n^2 > \frac{3}{\varepsilon}-1$$

<u>Proof</u>: take any $\varepsilon>0$. Then $\exists\, n_0\in\mathbb{N}$ (specifically, any n_0 s.t. $n_0^2>\frac{3}{\varepsilon}-1$ will do)

s.t. $|a_n-3|<\varepsilon$ for $n\geq n_0$.

Indeed, if $n_0^2 > \frac{3}{\varepsilon}-1$ and $n\geq n_0$, then we have:

$$n^2\geq n_0^2 > \frac{3}{\varepsilon}-1 \Rightarrow n^2+1>\frac{3}{\varepsilon} \Rightarrow 3<\varepsilon(n^2+1) \Rightarrow \left|\frac{3n^2}{n^2+1}-3\right|<\varepsilon$$

∎

$\boxed{\#3}$ $\{a_n\}$ converges (product of two convergent sequences $4\cdot\left(\frac{2}{\pi}\right)^n$

Constant sequence

of the form r^n with $r=\frac{2}{\pi}$, $-1<r<1$)

$\{b_n\}$ converges (because $\lim\limits_{n\to\infty}|b_n| = \lim\limits_{n\to\infty}\frac{1}{n} = 0$)

$\{c_n\}$ diverges (because $\ln(n^2+3)-\ln n = \ln\frac{n^2+3}{n}$
$= \ln\left(n+\frac{3}{n}\right)\to\infty$ as $n\to\infty$)

$\{d_n\}$ converges (because $d_n=f(n)$ for $f(x)=\cos\left(e^{\frac{1}{x}}\right)$
and $\lim\limits_{x\to\infty} f(x) = \cos 1$)

$\boxed{\#4}$ Proof. The sequence of partial sums is $s_1 = -1$

$s_2 = -1+1 = 0$

$s_3 = -1+1+(-1) = -1$

$s_4 = -1+1+(-1)+1 = 0$

etc.

$$s_j = \begin{cases} -1 & \text{if } j \text{ is odd} \\ 0 & \text{if } j \text{ is even} \end{cases}$$

To show that the series diverges we need to show: $\forall L \in \mathbb{R} \quad L \neq \lim_{j\to\infty} s_j$

So, let L be a real number. Set $\varepsilon = \frac{1}{3}$.

Assume $\exists n_0 \in \mathbb{N}$ s.t. $|s_j - L| < \varepsilon = \frac{1}{3}$ for all $j \geq n_0$.

Choosing $j \geq n_0$ odd, we get: $|-1-L| < \frac{1}{3} \Rightarrow |L+1| < \frac{1}{3} \quad -1-\frac{1}{3} < L < -1+\frac{1}{3}$

choosing $j \geq n_0$ even, we get $|0-L| < \frac{1}{3} \Rightarrow |L| < \frac{1}{3} \quad -\frac{1}{3} < L < \frac{1}{3}$

We got: $-\frac{4}{3} < L < -\frac{2}{3} \quad -\frac{1}{3} < L < \frac{1}{3}$

Contradiction $\Rightarrow$ there is no such n_0.

$\Rightarrow L$ is not the limit of $\{s_j\}$.

$\Rightarrow \{s_j\}$ is divergent $\Rightarrow$ the series is divergent. ∎

$\boxed{\#5}$

$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ converges (geometric series with ratio $r = \frac{2}{3}$)

$\sum_{k=1}^{\infty} (1.2)^k$ diverges (geometric series with $r = 1.2 > 1$)

$\sum_{k=1}^{\infty} \frac{k(k^2+2)}{(k+1)(k+2)(k+3)}$ diverges (because $\lim_{k\to\infty} \frac{k(k^2+2)}{(k+1)(k+2)(k+3)} = 1 \neq 0$)

$\sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges (because $\sum_{n=1}^{\infty} \frac{1}{n+1} = \sum_{k=2}^{\infty} \frac{1}{k}$ p-series with $p=1$)