1

(a) The function $f(x) = \dfrac{1}{x^2+4x+9} = \dfrac{1}{(x+2)^2+5}$ is

non-negative, continuous and decreasing on $[1,\infty)$, and

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{(x+2)^2+5}\,dx = \frac{1}{\sqrt{5}} \tan^{-1}\frac{x+2}{\sqrt{5}} \Big|_{x=1}^{\infty}$$

$$= \frac{1}{\sqrt{5}}\frac{\pi}{2} - \frac{1}{\sqrt{5}}\tan^{-1}\frac{3}{\sqrt{5}} \quad \text{is convergent.}$$

Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+5n+9}$ is convergent by the

integral test.

One could also argue that

$$0 \le \frac{1}{x^2+4x+9} \le \frac{1}{x^2} \quad \text{for} \quad 1 \le x < \infty$$

and use the comparison test to show that

$$\int_1^\infty \frac{1}{x^2+4x+9}\,dx \quad \text{is convergent.}$$

(b) The function $f(x) = \dfrac{1}{2\sqrt{x}-1}$ is non-negative, continuous and decreasing on $[1,\infty)$. Also we have

$$0 \le \frac{1}{2\sqrt{x}} \le \frac{1}{2\sqrt{x}-1}$$

. Therefore by the comparison test $\displaystyle\int_1^\infty \frac{dx}{2\sqrt{x}-1}$ is divergent since

$$\int_1^\infty \frac{dx}{2\sqrt{x}} = \frac{1}{2}\,\frac{x^{1/2}}{1/2} = x^{1/2}\Big|_{x=1}^{x=\infty}$$

is divergent.

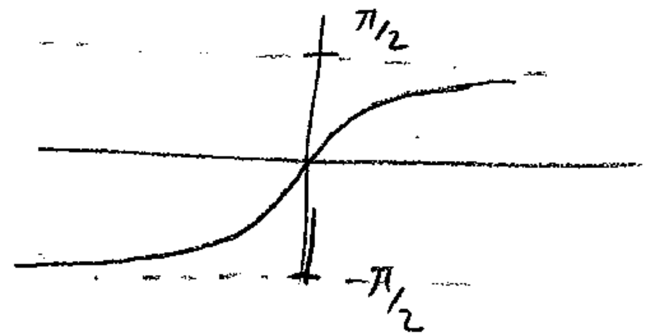
Hence, by the integral test $\displaystyle\sum_{n=1}^\infty \frac{1}{2\sqrt{n}-1}$ is divergent.

Note that one could also use the fact that

$$\int \frac{1}{2\sqrt{x}-1}\,dx = \frac{1}{2}\left(-1+2\sqrt{x}+\ln\left|-1+2\sqrt{x}\right|\right)+C.$$

2

(a) Since the graph of $f(x) = \tan^{-1} x$ looks like



,

for $n = 1, 2, 3, 4, \ldots$ we have

$$0 \leq \tan^{-1}(n) \leq \pi/2 .$$

So

$$0 \leq \frac{\tan^{-1}(n)}{n^3} \leq \frac{\pi/2}{n^3}, \quad n = 1, 2, 3, \ldots .$$

Since $\sum \frac{\pi/2}{n^3} = \frac{\pi}{2} \sum \frac{1}{n^3}$ is convergent by the p-series test, it follows from the comparison test that

$$\sum_{n=1}^{\infty} \frac{\tan^{-1}(n)}{n^3}$$ is convergent.

(b) Since $0 \leq \frac{1}{n} \leq \frac{1}{n-0.3}$

and $\sum \frac{1}{n}$ is divergent by the p-series test, it follows from the comparison test that $\sum \frac{1}{n-0.3}$ is divergent.

3/

(a) We compare the sequence $a_n = \frac{e^n+n}{5^n+3} > 0$ with $b_n = \frac{e^n}{5^n} > 0$ in the limit:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim \frac{\frac{e^n+n}{5^n+3}}{\frac{e^n}{5^n}} = \lim \frac{5^n e^n + n5^n}{5^n e^n + 3e^n}$$

$$= \lim_{n\to\infty} \frac{1+\frac{n}{e^n}}{1+\frac{3}{5^n}} = \frac{1+0}{1+0} = 1 > 0$$ because using

L'hospital: $\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$, $\lim_{x\to\infty} \frac{3}{5^x} = \lim_{x\to\infty} \frac{3}{5^x \ln 5} = 0$.

There using the limit comparison test,
either both Σa_n and Σb_n are convergent
or both divergent. Since

$$\sum_{n=0}^{\infty} \frac{e^n}{5^n} = \sum_{n=0}^{\infty} \left(\frac{e}{5}\right)^n = \frac{1}{1-\frac{e}{5}} \quad \left(\text{because } \left|\frac{e}{5}\right| < 1\right),$$

$$\Sigma a_n = \Sigma \frac{e^n + n}{5^n + 3}$$ is also convergent.

(b) We compare the sequence $a_n = \dfrac{n\sqrt{n} + \cos n}{n^2\sqrt{n} + n + 1} > 0$
in the limit with $b_n = \frac{1}{n} > 0$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{n\sqrt{n} + \cos n}{n^2\sqrt{n} + n + 1}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{n^2\sqrt{n} + n\cos n}{n^2\sqrt{n} + n + 1}$$

$$= \lim_{n\to\infty} \frac{1 + \dfrac{\cos n}{n\sqrt{n}}}{1 + \dfrac{1}{n\sqrt{n}} + \dfrac{1}{n^2\sqrt{n}}} = 1 > 0.$$ So by the limit
comparison test, since $\Sigma b_n = \Sigma \frac{1}{n}$ is divergent,

$$\Sigma a_n = \Sigma \frac{n\sqrt{n} + \cos n}{n^2\sqrt{n} + n + 1}$$ is also divergent.

4/

Since $b_n = \frac{1}{e^{2n}} = e^{-2n}$ is decreasing

and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{e^{2n}} = 0$, by the

alternating series test, $\sum (-1)^{n-1} b_n = \sum (-1)^{n-1} e^{-2n}$

is convergent. As for the absolute convergence,

this series is absolutely convergent using

geometric series:

$$\sum_{n=1}^{\infty} \left| (-1)^{n-1} e^{-2n} \right| = \sum_{n=1}^{\infty} \frac{1}{e^{2n}} = \sum_{n=1}^{\infty} \left(\frac{1}{e^2}\right)^n$$

$$= \frac{1}{e^2} \sum_{n=1}^{\infty} \left(\frac{1}{e^2}\right)^{n-1} = \frac{1}{e^2} \sum_{n=0}^{\infty} \left(\frac{1}{e^2}\right)^n = \frac{1}{e^2} \frac{1}{1-\frac{1}{e^2}}$$

Note that we are using the fact that

$\left|\frac{1}{e^2}\right| < 1.$

5/ $b_n = \frac{1}{n^{0.2}}$ is decreasing and

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^{0.2}} = 0$. Therefore by the alternating series test $\sum_{n=1}^{\infty} (-1)^n b_n = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^{0.2}}$ is convergent. But this series is not absolutely convergent since

$\sum \left| (-1)^n \frac{1}{n^{0.2}} \right| = \sum \frac{1}{n^{0.2}}$ which is divergent by the p-series test $(p = 0.2 < 1)$.

Hence the series $\sum (-1)^n \frac{1}{n^{0.2}}$ is conditionally convergent.

6/ (a) $a_n = \dfrac{n^3}{(-2)^n}$

In order to use the Ratio Test, we compute:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)^3}{(-2)^{n+1}}}{\frac{n^3}{(-2)^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^3}{n^3} \frac{1}{(-2)} \right| = \frac{1}{2} < 1.$$

Since $\lim \left| \frac{a_{n+1}}{a_n} \right| = 1/2 < 1$, it follows from the Ratio Test that $\sum a_n = \sum \frac{n^3}{(-2)^n}$ is absolutely convergent.

(b) $$a_n = \left(\frac{2 \ln n + e^{-n}}{\ln n + |\sin(n/3)|} \right)^{en}$$

In order to apply the root test, we compute:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left|\left(\frac{2\ln n + e^{-n}}{\ln n + |\sin(n/3)|}\right)^{en}\right|^{1/n}$$

$$= \lim_{n\to\infty} \left(\frac{2\ln n + e^{-n}}{\ln n + |\sin(n/3)|}\right)^{e}$$

$$= \lim_{n\to\infty} \left(\frac{2 + \frac{1}{e^n \ln n}}{1 + \frac{|\sin(n/3)|}{\ln n}}\right)^{e}$$

$$= 2^e > 1.$$

Since $\lim \sqrt[n]{|a_n|} = 2^e > 1$, it follows that the series is divergent.