Pb 1 Since $3x^2+5x-2 = 3(x+2)(x-1/3)$, we can write

$$\frac{x-5}{3x^2+5x-2} = \frac{\frac{1}{3}(x-5)}{(x+2)(x-1/3)} = \frac{A}{x+2} + \frac{B}{x-1/3} \text{ for some } A, B.$$

In order to find A, B we write

$$\frac{\frac{1}{3}(x-5)}{(x+2)(x-1/3)} = \frac{A(x-1/3)+B(x+2)}{(x+2)(x-1/3)} = \frac{(A+B)x+(-\frac{1}{3}A+2B)}{(x+2)(x-1/3)}$$

which gives us the equations $\begin{cases} A+B = 1/3 \\ -\frac{1}{3}A + 2B = -5/3 \end{cases}$

by solving which we have $A = 1$, $B = -\frac{2}{3}$.

Therefore
$$\frac{x-5}{3x^2+5x-2} = \frac{1}{x+2} - \frac{\frac{2}{3}}{x-1/3} = \frac{1}{x+2} - \frac{2}{3x-1}.$$
$$= \frac{1}{2}\frac{1}{(1-(-\frac{x}{2}))} + \frac{2}{1-3x}.$$

Now we can use the power series representation

$$\frac{1}{1-x} = 1+x+x^2+\cdots = \sum_{i=0}^{\infty} x^i, \quad |x|<1$$ for both of the above terms:

$$\frac{1}{2}\,\frac{1}{1-\left(-\frac{x}{2}\right)} = \frac{1}{2}\sum_{i=0}^{\infty}\left(-\frac{x}{2}\right)^{i} = \frac{1}{2}\sum_{i=0}^{\infty}\frac{(-1)^{i}}{2^{i}}\,x^{i}, \quad \left|-\frac{x}{2}\right|<1,$$

and

$$\frac{2}{1-3x} = 2\sum_{i=0}^{\infty}(3x)^{i} = 2\sum_{i=0}^{\infty}3^{i}x^{i}, \quad |3x|<1.$$

since $\left|-\frac{x}{2}\right|<1$ is equivalent to $|x|<2$, and $|3x|<1$ is equivalent to $|x|<1/3$, the intersection where both of these conditions are satisfied is $|x|<1/3$.

Hence

$$\frac{1}{2}\,\frac{1}{1-\left(-\frac{x}{2}\right)} + \frac{2}{1-3x} = \frac{1}{2}\sum_{i=0}^{\infty}\frac{(-1)^{i}}{2^{i}}\,x^{i} + 2\sum_{i=0}^{\infty}3^{i}x^{i}$$

$$= \sum_{i=0}^{\infty}\left(\frac{(-1)^{i}}{2^{i+1}} + 2\,(3)^{i}\right)x^{i} \quad \text{for } |x|<1/3.$$

so the interval of convergence is $(-1/3, 1/3)$.

Pb 2 since $\frac{d}{dx}\left(\frac{1}{3}\,\frac{1}{1-3x}\right) = \frac{1}{(1-3x)^2}$ and

$$\frac{1}{3}\,\frac{1}{1-3x} = \frac{1}{3}\sum_{i=0}^{\infty}(3x)^i = \frac{1}{3}\sum_{i=0}^{\infty}3^i x^i, \quad |3x|<1,$$

we have:

$$\frac{1}{(1-3x)^2} = \frac{d}{dx}\left(\frac{1}{3}\sum_{i=0}^{\infty}3^i x^i\right) = \frac{1}{3}\sum_{i=1}^{\infty}3^i i\, x^{i-1}, \quad |3x|<1.$$

Therefore

$$\frac{x^5}{(1-3x)^2} = x^5\,\frac{1}{3}\sum_{i=1}^{\infty} i\,3^i x^{i-1} = \sum_{i=1}^{\infty} i\,3^{i-1} x^{i+4}, \quad |3x|<1.$$

Since $|3x|<1$ is equivalent to $|x|<1/3$, the radius of convergence (around 0) is $1/3$.

Pb 3

Using the power series representations

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} \quad \text{for all } x, \text{ and}$$

$$\ln(1+x) = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{x^i}{i} \quad \text{for } |x|<1, \text{ we have:}$$

$$e^{2x} = \sum_{i=0}^{\infty} \frac{(2x)^i}{i!} = \sum_{i=0}^{\infty} \frac{2^i x^i}{i!} \quad \text{for all } x, \text{ and}$$

$$\ln(1-x) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}(-x)^i}{i} = \sum_{i=1}^{\infty} \frac{(-1)^{2i-1} x^i}{i} = -\sum_{i=1}^{\infty} \frac{x^i}{i}$$

for $|-x|<1$.

Hence

$$e^{2x} + \ln(1-x) = \sum_{i=0}^{\infty} \frac{2^i x^i}{i!} - \sum_{i=1}^{\infty} \frac{x^i}{i}$$

$$= 1 + \sum_{i=1}^{\infty} \left(\frac{2^i}{i!} - \frac{1}{i} \right) x^i \quad \text{for } |x|<1,$$

and the latter is the Maclaurin series of

$$e^{2x} + \ln(1-x).$$

Pb 4 The Taylor series of $f(x)$ at $a=5$ is of the form:

$$\sum_{i=0}^{\infty} \frac{f^{(i)}(5)}{i!}(x-5)^i.$$ So we need to find higher derivatives of $f(x)=\ln(x)$ at $a=5$:

$$f(x)=\ln(x) \Rightarrow f'(x)=\frac{1}{x}$$

$$f''(x) = -\frac{1}{x^2}$$

$$f'''(x)= \frac{2}{x^3}$$

$$f^{(4)}(x) = \frac{-6}{x^4}$$

$$f^{(5)}(x) = \frac{24}{x^5}$$

$$\vdots$$

$$f^{(i)}(x) = (-1)^{i+1}\frac{(i-1)!}{x^i} \quad \text{for } i \geq 1.$$

Hence:

$$\sum_{i=0}^{\infty} \frac{f^{(i)}(5)(x-5)^i}{i!} = \ln(5) + \sum_{i=1}^{\infty} \frac{(-1)^{i+1}(i-1)!}{i!\,5^i}(x-5)^i$$

$$= \ln(5) + \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i\,5^i}(x-5)^i$$

Pb 5

Using $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, for all x,

we have:

$$\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{(2n+1)!\, 4^{2n+1}} = \sum_{n=0}^{\infty} (-1)^n \frac{(\pi/4)^{2n+1}}{(2n+1)!} = \sin(\pi/4) = \frac{1}{\sqrt{2}}.$$

Similarly, using

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \dots, \quad |x| < 1,$$

we have:

$$-\frac{(\ln 2)^3}{3} + \frac{(\ln 2)^5}{5} - \frac{(\ln 2)^7}{7} + \cdots$$

$$= \arctan(\ln 2) - \ln 2.$$

Note that $-1 < \ln 2 < 1$ since $\frac{1}{e} < 2 < e$.

Pb 6 Note that we can use

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \text{ for all } x, \text{ to find the}$$

Maclaurin series

$$x \cos x = x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!} \quad (\text{for all } x)$$

So the Taylor polynomial of degree 5 at $a=0$ for $f(x) = x \cos x$ should be equal to

$$T_5(x) = x - \frac{x^3}{2} + \frac{x^5}{4!} = x - \frac{x^3}{2} + \frac{x^5}{24}.$$

However, in order to find the accuracy of the estimation, we need to have a bound on $|f^{(6)}(x)|$ for $-\frac{1}{2} \le x \le \frac{1}{2}$ so that we can have a bound on the error:

$$R_5(x) = f(x) - T_5(x) = x \cos x - \left(x - \frac{x^3}{2} + \frac{x^5}{24}\right).$$

We know from Taylor's inequality that

$$|R_5(x)| \le \frac{M}{6!} \left(\frac{1}{2}\right)^6, \text{ where } M \text{ satisfies } |f^{6}(x)| \le M$$

for $-\frac{1}{2} \le x \le \frac{1}{2}$.

Therefore we need to compute $f^{(6)}(x)$:

$$f(x) = x\cos x,$$

$$f'(x) = \cos x - x\sin x,$$

$$f''(x) = -\sin x - \sin x - x\cos x = -2\sin x - x\cos x,$$

$$f^{(3)}(x) = -2\cos x - \cos x + x\sin x = -3\cos x + x\sin x,$$

$$f^{(4)}(x) = 3\sin x + \sin x + x\cos x = 4\sin x + x\cos x,$$

$$f^{(5)}(x) = 4\cos x - x\sin x + \cos x = 5\cos x - x\sin x,$$

$$f^{(6)}(x) = -5\sin x - \sin x - x\cos x = -6\sin x - x\cos x.$$

Since $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f^{(3)}(0) = -3$, $f^{(4)}(0) = 0$, $f^{(5)}(0) = 5$, we have:

$$T_5(x) = \sum_{i=0}^{5} \frac{f^{(i)}(0)}{i!} x^i = x - \frac{3}{3!}x^3 + \frac{5x^5}{5!}$$

$$= x - \frac{x^3}{2} + \frac{x^5}{24}$$, as computed above.

For the bound on $|f^{(6)}(x)|$, we show that

$$|f^{(6)}(x)| = |-6\sin x - x\cos x| = |6\sin x + x\cos x| \le$$

$6\sin(0.5) + (0.5)\cos(0.5) \approx 3.31534 < 3.5$ for all x in $[-0.5, 0.5]$.

In order to justify the latter, note that the function $g(x) = 6\sin x + x\cos x$ is an odd function, in other words

$$g(-x) = 6\sin(-x) + (-x)\cos(-x)$$
$$= -6\sin x - x\cos x = -g(x).$$

Moreover we show that $g(x)$ is increasing on $[0, 0.5]$ (which implies that g is increasing on $[-0.5, 0.5]$ since it is odd) by proving that $g'(x) \geq 0$ for any x in $[0, 0.5]$.

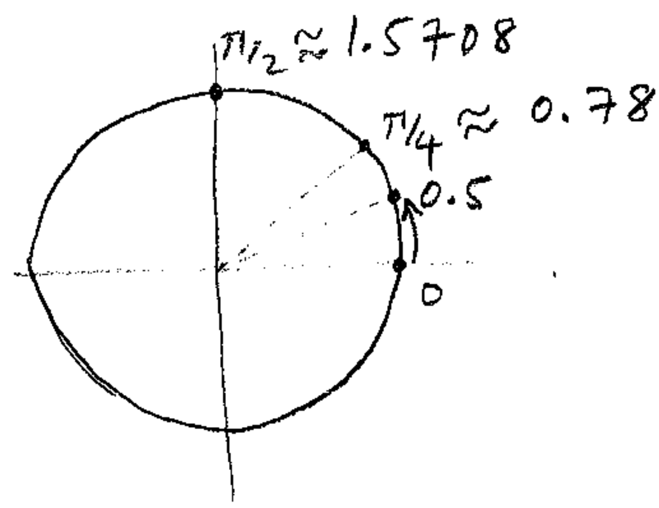
The reason is that

$$g'(x) = 6\cos x + \cos x - x\sin x = 7\cos x - x\sin x$$

and if $0 \leq x \leq 0.5$ then

$$7\cos x - x\sin x \geq 7\cos(\pi/4) - \frac{\pi}{4}\sin\left(\frac{\pi}{4}\right) = \left(7 - \frac{\pi}{4}\right)\frac{1}{\sqrt{2}} > 0$$

considering



(Note that $\sin x$ is increasing on $[0, \pi/2]$.)

Therefore the maximum of $|f^{(6)}(x)| = |6\sin x + x\cos x|$ in $[-0.5, 0.5]$ is attained at $x = 0.5$ (and at -0.5) which means that for any x in $[-0.5, 0.5]$

$$|f^{(6)}(x)| \le |6\sin(0.5) + 0.5\cos(0.5)| \approx 3.31534 < 3.5,$$

as it was claimed above. Hence it follows from Taylor's inequality that

$$|R_6(x)| \le \frac{3.5}{6!}\left(\frac{1}{2}\right)^6 = 0.0000759549.$$