13 marks

1.

(4 marks) (a) State the Mean Value Theorem precisely.

If $a<b$, $f(x)$ is continuous on $[a,b]$,
differentiable on (a,b),
then $\exists c \in (a,b)$ s.t. $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

(4 marks) (b) Does there exist a function f which is differentiable everywhere and $f(-1) = 4$, $f(4) = 10$, $f'(x) \le 1$ for all x between -1 and 4? Explain why.

No.

Explanation: apply MVT on $[a,b] = [-1,4]$.
f is differentiable on $\mathbb{R} \Rightarrow f$ is cont. on $[-1,4]$ and differ. on $(-1,4)$.
By MVT $\exists c \in (-1,4)$ s.t. $f'(c) = \dfrac{10-4}{4-(-1)} = \dfrac{6}{5} > 1$.

(5 marks) (c) Show that any polynomial $f(x) = ax^3+bx^2+cx+d$ has at most 3 roots. In other words there are at most 3 real numbers c_1, c_2, c_3 such that $f(c_1) = f(c_2) = f(c_3) = 0$.

This is #21(a) from 4.2 p. 289.

Proof. By contradiction.

Assume such $f(x)$ has roots $a_1 < a_2 < a_3 < a_4$.

$f(a_1) = f(a_2) = f(a_3) = f(a_4)$

f is continuous and differentiable on $\mathbb{R}$

Apply Rolle Theorem on $[a_1, a_2], [a_2, a_3], [a_3, a_4]$

$\Rightarrow \exists\ b_1 \in (a_1, a_2), b_2 \in (a_2, a_3), b_3 \in (a_3, a_4)$
s.t. $f'(b_1) = 0$, $f'(b_2) = 0$, $f'(b_3) = 0$ $\quad b_1 < b_2 < b_3$

We got: $f'(x) = 3ax^2+2bx+c$ has at least 3 roots

- Contradiction

$\Rightarrow f$ has at most 3 zeros.

15 marks 2. Evaluate:

$f = x^2$
$g' = e^x$

(5 marks) (a) $\int x^2 e^x \, dx$ $= \underbrace{x^2}_{f}\underbrace{e^x}_{g} - \int \underbrace{(x^2)' e^x}_{f'g} dx = x^2 e^x - 2\int \underbrace{x}_{f}\underbrace{e^x}_{g'} dx$

integration by parts (twice)

$= x^2 e^x - 2\left(x e^x - \int x' e^x dx\right)$

$= \boxed{x^2 e^x - 2xe^x + 2e^x + C}$

(5 marks) (b) $\int \ln x \, dx$ $= \int \underbrace{\ln x}_{f} \cdot \underbrace{1}_{g'} dx = (\ln x)x - \int (\ln x)' x \, dx$

integration by parts

$g(x) = x$

$= x \ln x - \int \frac{1}{x} x \, dx$

$= \boxed{x \ln x - x + C}$

(5 marks) (c) $\int \frac{1}{(1+x^2)^2} dx$

$$\int \frac{1}{(1+x^2)^2} dx = \int \frac{1+x^2-x^2}{(1+x^2)^2} dx = \int \left(\frac{1}{1+x^2} - \frac{x^2}{(1+x^2)^2}\right) dx$$

$$= \arctan x - \int \frac{x^2}{(1+x^2)^2} dx$$

is found from

$$\arctan x + C = \int \underbrace{\frac{1}{1+x^2}}_{f} \cdot \underbrace{1}_{g'} dx = \frac{1}{1+x^2} x - \int \left(\frac{1}{1+x^2}\right)' x \, dx$$

$g(x) = x$

$$= \frac{x}{1+x^2} + 2\int \frac{x^2}{(1+x^2)^2} dx$$

$\Rightarrow \int \frac{x^2}{(1+x^2)^2} dx = \frac{1}{2}\arctan x - \frac{1}{2}\frac{x}{1+x^2} + C$

Answer: $\int \frac{1}{(1+x^2)^2} dx = \boxed{\frac{1}{2}\arctan x + \frac{1}{2}\frac{x}{1+x^2} + C}$

10 marks 3. Evaluate:

(5 marks) (a) $\int_1^3 \underbrace{x^3}_{g'} \underbrace{\ln x}_{f}\, dx = \left[(\ln x)\frac{x^4}{4}\right]_1^3 - \int_1^3 (\ln x)' \frac{x^4}{4}\, dx$

integration by parts

$g(x) = \frac{x^4}{4}$

$= \frac{1}{4}\left(3^4 \ln 3 - \ln 1\right) - \frac{1}{4}\int_1^3 \frac{1}{x} x^4\, dx$

$= \frac{81}{4}\ln 3 - \frac{1}{4}\int_1^3 x^3\, dx = \frac{81}{4}\ln 3 - \frac{1}{4}\left[\frac{x^4}{4}\right]_1^3$

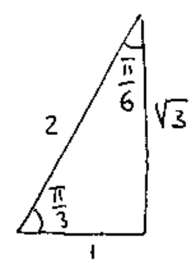
$= \frac{81}{4}\ln 3 - \frac{1}{16}(81-1) = \boxed{\frac{81}{4}\ln 3 - 5}$

(5 marks) (b) $\int_1^{\sqrt{3}} \arctan(1/x)\, dx$

integration by parts $= \int_1^{\sqrt{3}} \underbrace{\arctan\frac{1}{x}}_{f} \cdot \underbrace{1}_{g'}\, dx = \left[\left(\arctan\frac{1}{x}\right)x\right]_1^{\sqrt{3}} - \int_1^{\sqrt{3}} \left(\arctan\frac{1}{x}\right)' x\, dx$

$g(x) = x$

$= \sqrt{3}\arctan\frac{1}{\sqrt{3}} - \arctan 1 - \int_1^{\sqrt{3}} \frac{1}{1+\left(\frac{1}{x}\right)^2}\left(-x^{-2}\right)x\, dx$



$= \sqrt{3}\frac{\pi}{6} - \frac{\pi}{4} + \int_1^{\sqrt{3}} \frac{x}{x^2+1}\, dx$

substitution $u = x^2+1$

$du = 2x\,dx$

$= \sqrt{3}\frac{\pi}{6} - \frac{\pi}{4} + \frac{1}{2}\int_2^4 \frac{1}{u}\, du = \boxed{\sqrt{3}\frac{\pi}{6} - \frac{\pi}{4} + \frac{1}{2}(\ln 4 - \ln 2)}$

8 marks 4.

Write out the form of the partial fraction decomposition for the following rational functions without computing the coefficients:

(4 marks) (a) $\dfrac{1+6x}{(x-1)^3(2x+5)}$

$$\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}+\frac{D}{2x+5}$$

(4 marks) (b) $\dfrac{3x^2+5x+2}{x(x-1)(x^2+2x+10)^2}$

$x^2+2x+10$ is irreducible (because $2^2-40<0$)

Answer: $\dfrac{A}{x}+\dfrac{B}{x-1}+\dfrac{Cx+D}{x^2+2x+10}+\dfrac{Ex+F}{(x^2+2x+10)^2}$

12 marks 5.

Evaluate:

(6 marks) (a) $\displaystyle\int \frac{5x+1}{x^2-3x+2}\,dx$

$$x^2-3x+2=(x-1)(x-2)$$

$$\frac{5x+1}{(x-1)(x-2)}=\frac{A}{x-1}+\frac{B}{x-2}$$

$$5x+1=A(x-2)+B(x-1)$$

$$\begin{cases} A+B=5 \\ -2A-B=1 \end{cases} \Rightarrow A=-6,\ B=11$$

$$\int \frac{5x+1}{x^2-3x+2}\,dx=\int\left(-\frac{6}{x-1}+\frac{11}{x-2}\right)dx=\boxed{-6\ln|x-1|+11\ln|x-2|+C}$$

(6 marks) (b) $\displaystyle\int \frac{x^3+4}{x^2+4}\,dx$

Divide x^3+4 by x^2+4 $\Rightarrow$ $x^3+4=x(x^2+4)-4x+4$

$$\int \frac{x^3+4}{x^2+4}\,dx=\int \frac{x(x^2+4)-4x+4}{x^2+4}\,dx=\int x\,dx-4\int\frac{x}{x^2+4}\,dx+4\int\frac{1}{x^2+4}\,dx$$

$$=\boxed{\frac{x^2}{2}-2\ln(x^2+4)+2\tan^{-1}\frac{x}{2}+C}$$

12 marks 6.

Determine whether the following improper integrals are convergent and compute their values for the ones that are convergent:

(6 marks) (a) $\int_4^\infty \frac{1}{(x-2)^{3/2}}\,dx$

$$= \lim_{b\to\infty} \int_4^b (x-2)^{-3/2}\,dx$$

$$= \lim_{b\to\infty} \left[\frac{(x-2)^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}\right]_4^b = \lim_{b\to\infty}\left[\frac{-2}{\sqrt{x-2}}\right]_4^b$$

$$= -2\lim_{b\to\infty}\left(\frac{1}{\sqrt{b-2}} - \frac{1}{\sqrt{4-2}}\right) = \boxed{\sqrt{2} \text{ Convergent}}$$

(6 marks) (b) $\int_0^1 \frac{\ln x}{x}\,dx$

$$= \lim_{a\to 0+} \int_a^1 \frac{\ln x}{x}\,dx$$

substitution $u = \ln x$

$du = \frac{1}{x}dx$

$dx = x\,du$

$$= \lim_{a\to 0+} \int_{\ln a}^{\ln 1} u\,du$$

$$= \lim_{a\to 0+} \int_{\ln a}^{0} u\,du = \lim_{a\to 0+}\left[\frac{u^2}{2}\right]_{\ln a}^{0}$$

$$= \lim_{a\to 0+} \frac{1}{2}\left(0 - (\ln a)^2\right) = -\infty \quad \boxed{\text{Divergent}}$$

10 marks 7. Use the Comparison Theorem to determine whether the following integrals are convergent or divergent:

(5 marks) (a) $\displaystyle\int_1^\infty \frac{3+e^{-x^2}}{x}\,dx$

$$0 \le \frac{3}{x} \le \frac{3+e^{-x^2}}{x} \quad \text{for } x \ge 1$$

Continuous

$$\int_1^X \frac{3}{x}\,dx \text{ is divergent}$$

$$\underset{CT}{\Longrightarrow} \int_1^\infty \frac{3+e^{-x^2}}{x}\,dx \text{ is divergent}$$

(5 marks) (b) $\displaystyle\int_1^\infty \frac{x}{x^3+1}\,dx$

$$0 \le \frac{x}{x^3+1} \le \frac{x}{x^3} = \frac{1}{x^2} \quad \text{for } x \ge 1$$

Continuous

$$\int_1^\infty \frac{1}{x^2}\,dx \text{ is convergent}$$

$$\underset{CT}{\Longrightarrow} \int_1^\infty \frac{x}{x^3+1}\,dx \text{ is convergent}$$