Measure Theory 4122B/9022B, 1st Assignment
Due March 25th, 2013

1. Exercise 14, page 41: The purpose of this exercise is to show that covering by a finite number of intervals will not suffice in the definition of the outer measure m_*. The outer Jordan content $J_*(E)$ of a set E in $\mathbb{R}$ is defined by

$$J_*(E) = \inf \sum_{j=1}^{N} |I_j|,$$

where the inf is taken over every finite covering $E \subset \cup_{j=1}^{N} I_j$ by intervals I_j.
(a) Prove that $J_*(E) = J_*(\bar{E})$ for every set $E \subset \mathbb{R}$.
(b) Exhibit a countable set $E \subset [0, 1]$ such that $J_*(E) = 1$ while $m_*(E) = 0$.

2. Exercise 23, page 43: Suppose $f(x, y)$ is a function on $\mathbb{R}^2$ that is separately continuous: for each fixed variable, f is continuous in the other variable. Prove that f is measurable on $\mathbb{R}^2$.

3. Exercise 25, page 43: An alternative definition of measurability is as follows: E is measurable if for every $\epsilon > 0$ there is a closed set F contained in E with $m_*(E \setminus F) < \epsilon$. Show that this definition is equivalent with the one given in the text.

4. Suppose E_1 and E_2 are a pair of compact sets in $\mathbb{R}^d$ with $E_1 \subset E_2$, and let $a = m(E_1)$ and $b = m(E_2)$. Prove that for any c with $a < c < b$, there is a compact set E with $E_1 \subset E \subset E_2$ and $m(E) = c$.

5. Exercise 37, page 45: Suppose Γ is a curve $y = f(x)$ in $\mathbb{R}^2$, where f is continuous. Show that $m(\Gamma) = 0$.

6. Let X be a set and $\mathcal{M}$ a non-empty collection of subsets of X. Prove that if $\mathcal{M}$ is closed under complements and countable unions of disjoint sets, then M is a σ-algebra.

7. Let X be an uncountable set, and let $\mathcal{M}$ be the collection of all subsets of X such that either E or E^c is at most countable, and define $\mu(E) = 0$ in the first case, and $\mu(E) = 1$ in the second case. Prove that $\mathcal{M}$ is a σ-algebra and μ is a measure on it. Describe the corresponding measurable functions and their integrals.

Let $(X, \mathcal{M}, \mu)$ be a complete measure space.

8. Let $f_n : X \to [0, \infty], n = 1, 2, 3, \ldots,$ be a sequence of measurable functions such that $f_1(x) \geq f_2(x) \geq f_3(x) \geq \cdots$ and $\lim f_n(x) = f(x)$ for all $x \in X$, and $f_1 \in L^1(X, \mu)$. Prove that

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Also, show that the condition $f_1 \in L^1(X, \mu)$ cannot be omitted.

9. Suppose $f \in L^1(X, \mu)$. Prove that for each $\epsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f| \, d\mu < \epsilon$ for any $E \in \mathcal{M}$ with $\mu(E) < \delta$.