Measure Theory 4122B/9022B, 2nd Assignment
Due April 10th, 2013

1. Exercise 12, page 92: Show that there are $f \in L^1(\mathbb{R}^d, m)$ and a sequence $\{f_n\}$ with $f_n \in L^1(\mathbb{R}^d, m)$ such that

$$||f_n - f||_{L^1} \to 0,$$

but $f_n(x) \to f(x)$ for no x.

2. Exercise 22 (and a part of 21), page 94: Prove that if $f \in L^1(\mathbb{R}^d, m)$ and

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi}\, dx,$$

then $\hat{f}$ is a continuous function and $\hat{f}(\xi) \to 0$ as $\xi \to \infty$.

3. Exercise 23, page 94: As an application of the Fourier transform, show that there does not exist a function $I \in L^1(\mathbb{R}^d, m)$ such that

$$f * I = f \qquad \text{for all } f \in L^1(\mathbb{R}^d, m).$$

4. Exercise 25, page 95: Show that for any $\epsilon > 0$, the function $F(\xi) = \frac{1}{(1+|\xi|^2)^\epsilon}$ is the Fourier transform of a function in $L^1(\mathbb{R}^d, m)$. (See the hint in the textbook.)

5. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Show that for any $1 \leq p < q$, we have $L^q(X, \mu) \subset L^p(X, \mu)$. Let $\ell^p(\mathbb{Z})$ denote the L^p space of the integers equipped with the counting measure. Show that $\ell^p(\mathbb{Z}) \subset \ell^q(\mathbb{Z})$ for any $1 \leq p < q$.

6. Exercise 18, page 42: Prove that every measurable function is the limit a.e. of a sequence of continuous functions.