Q1/Sol'n:

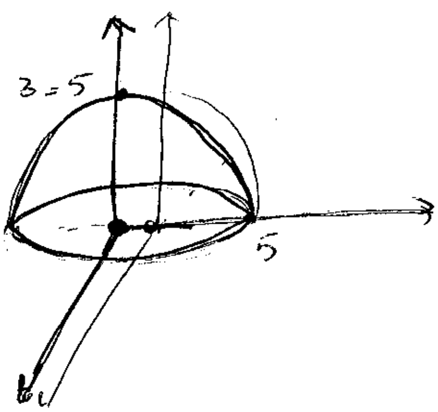
Before sketching the level curves, notice that

if $z = f(x,y) = \sqrt{25 - x^2 - y^2}$ then

$z^2 = 25 - x^2 - y^2$. So, $x^2 + y^2 + z^2 = 5^2$ which

is the equation for the sphere with radius 5,

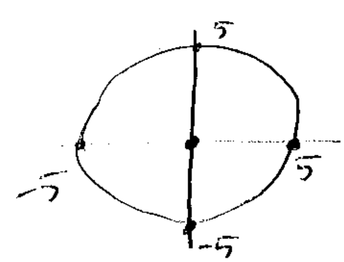
and centered at the origin i.e. $(0,0,0)$.



To depict the level sets, we need to write down corresponding algebraic equations:

For $z = 0$,

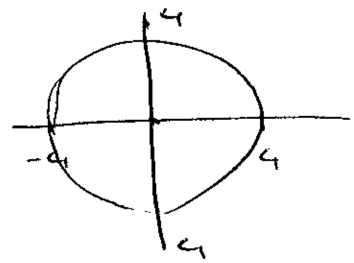
$0 = \sqrt{25 - x^2 - y^2} \Rightarrow x^2 + y^2 = 5^2$



Circle with radius 5
Centered at the origin.

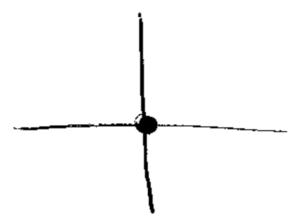
For $z=3$, we get following level set

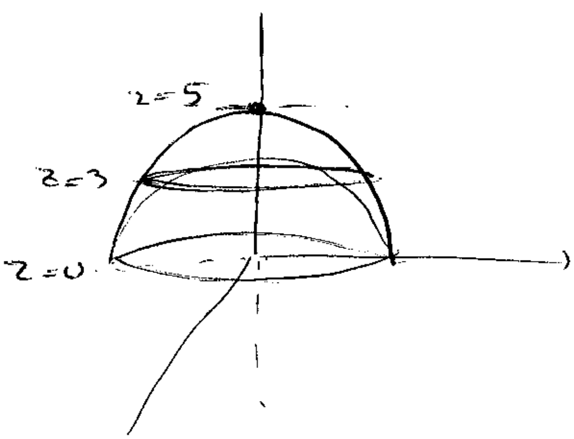
$$3=\sqrt{25-x^2-y^2} \Rightarrow x^2+y^2=4^2$$



For $z=5$, we get only a point for the level set.

$$5=\sqrt{25-x^2-y^2} \Rightarrow x^2+y^2=0 \Rightarrow x=y=0.$$





Q2 / sol'n :

Let $f_1(x,y) = \dfrac{x^2+e^{xy}}{x^2+y^3}$, $f_2(x,y) = \sin(xy^2)$ and

$f_3(x,y) = x\ln(xy)$

In order to compute $\dfrac{\partial f}{\partial x}$ we compute $\dfrac{\partial f_1}{\partial x}, \dfrac{\partial f_2}{\partial x}, \dfrac{\partial f_3}{\partial x}$

seperately and add these components.

(Same goes for parts b, c, and d.)

$$\frac{\partial f_1}{\partial x} = \frac{(x^2+y^3)(2x+\cancel{y}e^{xy}) - 2x\,(x^2+e^{xy})}{(x^2+y^3)^2} =$$

$$\frac{2x+\cancel{y}e^{xy}}{(x^2+y^3)^{\cancel{2}}} - 2x\,\frac{x^2+e^{xy}}{(x^2+y^3)^2}$$

$$\frac{\partial f_2}{\partial x} = y^2\cos(xy^2)$$

$$\frac{\partial f_3}{\partial x} = \ln(xy) + 1$$

So, $\dfrac{\partial f}{\partial x} = \dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_2}{\partial x} + \dfrac{\partial f_3}{\partial x}$.

$$\frac{\partial^2 f_1}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f_1}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{2x + ye^{xy}}{x^2+y^3} - 2x\,\frac{x^2+e^{xy}}{(x^2+y^3)^2}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{2x+ye^{xy}}{x^2+y^3}\right) - \frac{\partial}{\partial y}\left(2x\,\frac{x^2+e^{xy}}{(x^2+y^3)^2}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{2x+ye^{xy}}{(x^2+y^3)}\right) - 2x\,\frac{\partial}{\partial y}\left(\frac{x^2+e^{xy}}{(x^2+y^3)^2}\right)$$

$$= \frac{(x^2+y^3)((1+xy)e^{xy}) - 3y^2(2x+ye^{xy})}{(x^2+y^3)^2} - 2x\,\frac{(x^2+y^3)^2(xe^{xy}) - 3y^2(x^2+y^3)(x^2+e^{xy})}{(x^2+y^3)^4}$$

$$= \frac{e^{xy}(x^5y - x^4 + 2x^3y^4 + 12xy^2 - 3x^2y^3 - 2y^6 + xy^7) + 6xy^2(x^2-y^3)}{(x^2+y^3)^3}$$

$$\frac{\partial^2 f_2}{\partial x \partial y} = 2y\cos(xy^2) - 2xy^3\sin(xy^2)$$

$$\frac{\partial^2 f_3}{\partial x \partial y} =$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f_1}{\partial x \partial y} + \frac{\partial^2 f_2}{\partial x \partial y} + \frac{\partial^2 f_3}{\partial x \partial y}$$

Q3 / sol'n :

If (a,b) is a critical point for $f(x,y)$, then we have to have

$$\nabla f \Big|_{(a,b)} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)\Big|_{(a,b)} = (0,0)$$

Hence
$$\frac{\partial f}{\partial x}\Big|_{\substack{x=a\\y=b}} = 2x - 2y - 8\Big|_{(a,b)} = 2a - 2b - 8 = 0 \quad (\mathrm{I})$$

and
$$\frac{\partial f}{\partial y}\Big|_{\substack{x=a\\y=b}} = 3y^2 - 2x + 7 = 3b^2 - 2a + 7 = 0 \quad (\mathrm{II})$$

From equation (I), we get $2a = 2b + 8$

or equivalently $a = b + 4$. Plugging this into (II),

it yields $3b^2 - 2b - 8 + 7 = 0$ or $3b^2 - 2b - 1 = 0$.

Thus $b = +1$ and $b = \frac{-1}{3}$. Also, $a = 5$, and $a = \frac{11}{3}$

Therefore $(5,1)$ and $\left(\frac{11}{3}, \frac{-1}{3}\right)$

are critical points of $f(x,y)$.

Note that $f_{xx} = 2$ and $f_{yy} = 6y$ and

$f_{xy} = f_{yx} = -2$.

$$Df(a,b) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy}^2 \Big|_{(a,b)} =$$

$$2(6y) - (-2)^2 = 12y - 4 \Big|_{(a,b)} = 12b - 4$$

Now, at the point $(5,1)$;

$Df(5,1) = 12(1) - 4 = 8 > 0$ and $f_{xx}(5,1) = 2 > 0$

and at the point $\left(\frac{11}{3}, \frac{-1}{3}\right)$;

$Df\left(\frac{11}{3}, \frac{-1}{3}\right) = 12\left(\frac{-1}{3}\right) - 4 = -8$

So, $(5,1)$ is a relative minimum and $\left(\frac{11}{3}, \frac{-1}{3}\right)$ is a saddle point for the function $f(x,y)$.

Q4 / sol'n:

At minimum point of $f(x,y) = x^2 + y^2 - xy$

subject to the constraint ~~[illegible]~~

$$g(x,y) = 2x + y - 14 = 0$$

we must have $\nabla f \parallel \nabla g$.

But $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2x-y, 2y-x) \parallel \nabla g = (2,1)$

This yields

$$2x - y = 2\lambda$$
$$2y - x = \lambda$$

and

$$2x + y - 14 = 0 .$$

Solving these equations gives us $\lambda = 3$

and $x = 5\frac{\lambda}{3}$, $y = 4\frac{\lambda}{3}$.

Thus $x = 5$, $y = 4$, and

$f(5,4) = ~~[illegible]~~\ 5^2 + 4^2 - 5 \cdot 4 = 21$

This is minimum value for f.