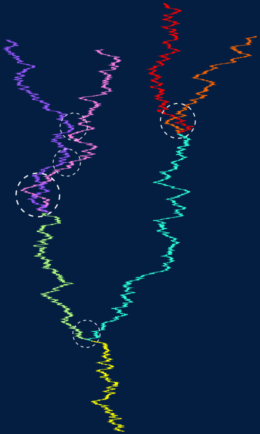


Branching Brownian motion with decay of mass and the non-local Fisher-KPP equation

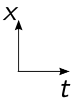
Sarah Penington

Joint work with Louigi Addario-Berry and Julien Berestycki

University of Bath

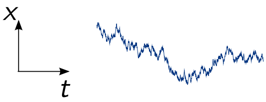
Branching Brownian motion

- Start with a single individual with an Exp(1) lifetime
- The individual moves according to (one-dimensional) Brownian motion
- When the individual dies, it produces two offspring individuals
- Each new individual has an independent Exp(1) lifetime and moves independently according to a Brownian motion until it dies and has offspring and so on.



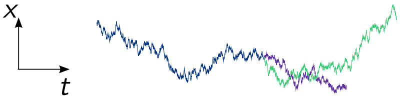
Branching Brownian motion

- Start with a single individual with an Exp(1) lifetime
- The individual moves according to (one-dimensional) Brownian motion
- When the individual dies, it produces two offspring individuals
- Each new individual has an independent Exp(1) lifetime and moves independently according to a Brownian motion until it dies and has offspring and so on.



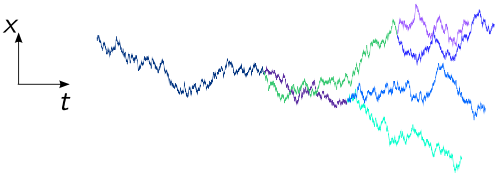
Branching Brownian motion

- Start with a single individual with an Exp(1) lifetime
- The individual moves according to (one-dimensional) Brownian motion
- When the individual dies, it produces two offspring individuals
- Each new individual has an independent Exp(1) lifetime and moves independently according to a Brownian motion until it dies and has offspring and so on.



Branching Brownian motion

- Start with a single individual with an Exp(1) lifetime
- The individual moves according to (one-dimensional) Brownian motion
- When the individual dies, it produces two offspring individuals
- Each new individual has an independent Exp(1) lifetime and moves independently according to a Brownian motion until it dies and has offspring and so on.



BBM with decay of mass

Individuals within distance μ of each other have to share resources so their masses decay.

$N(t)$ is number of particles at time t.
Locations of particles given by $\mathrm{X}(t) = (X_i(t), 1 \leq i \leq N(t))$.
Masses of particles given by $\mathrm{M}(t) = (M_i(t), 1 \leq i \leq N(t))$.

Let $\zeta_\mu(t,x) = \frac{1}{2\mu} \sum_{\{i : |X_i(t) - x| \in (0,\mu)\}} M_i(t)$.

$M_i(t)$ decays at rate $M_i(t)\zeta_\mu(t, X_i(t))$ so

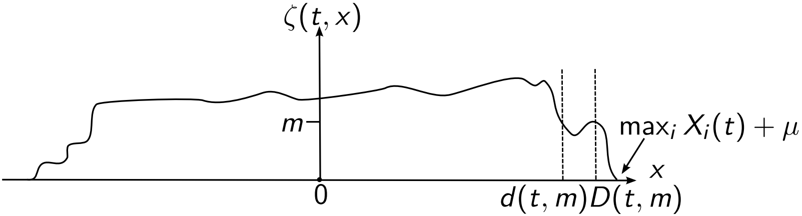
$$M_i(t) = \exp\left(-\int_0^t \zeta_\mu(s, X_{i,t}(s))\, ds\right)$$

where $X_{i,t}(s)$ is the location of the ancestor of $X_i(t)$ at time s.

Total mass increases through branching.

Front location

Let $d(t,m) = \min\{x > 0 : \zeta_\mu(t,x) < m\}$ and
$D(t,m) = \max\{x : \zeta_\mu(t,x) > m\}$.



If $x \geq \max_i X_i(t) + \mu$ then $\zeta_\mu(t,x) = 0$.

Maximum particle in BBM

Theorem (Bramson)

The rightmost particle location $\max_{i\geq 1} X_i(t)$ *has median* $med(t)$ *which satisfies*

$$med(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + O(1).$$

Theorem (Hu and Shi)

Almost surely

$$\limsup_{t\to\infty} \frac{|\max_{i\geq 1} X_i(t) - med(t)|}{\log t} < \infty.$$

Front location for BBM with decay of mass

Theorem (Addario-Berry, P.)

Let $c^* = 3^{4/3}\pi^{2/3}/2^{7/6}$. *Then for* $m \in (0,1)$, *almost surely,*

$$\limsup_{t\to\infty} \frac{\sqrt{2}t - d(t,m)}{t^{1/3}} \geq c^* \quad \textit{and} \quad \liminf_{t\to\infty} \frac{\sqrt{2}t - D(t,m)}{t^{1/3}} \leq c^*.$$

There are

- large times t at which the first low-density region lags at least distance $c^* t^{1/3} + o(t^{1/3})$ behind the rightmost particle
- large times t at which there is some high-density region within distance $c^* t^{1/3} + o(t^{1/3})$ of the rightmost particle.

Front location for BBM with decay of mass

Theorem (Addario-Berry, Berestycki, P.)

Let $c^* = 3^{4/3}\pi^{2/3}/2^{7/6}$. *Then for* $m \in (0, m^*]$, *almost surely,*

$$\limsup_{t\to\infty} \frac{\sqrt{2}t - D(t,m)}{t^{1/3}} \geq c^* \quad \textit{and} \quad \liminf_{t\to\infty} \frac{\sqrt{2}t - d(t,m)}{t^{1/3}} \leq c^*.$$

Question: Do we have that for $m \in (0,1)$, almost surely

$$\lim_{t\to\infty} \frac{\sqrt{2}t - D(t,m)}{t^{1/3}} = c^* \quad \text{and} \quad \lim_{t\to\infty} \frac{\sqrt{2}t - d(t,m)}{t^{1/3}} = c^* \ ?$$

Density self-correction

Recall that $\zeta_\mu(t,x) = \frac{1}{2\mu}\sum_{\{i:|X_i(t)-x|\in(0,\mu)\}} M_i(t)$ and $\frac{d}{dt}M_i(t) = -M_i(t)\zeta_\mu(t,X_i(t))$.

Heuristically,

$$\frac{\mathrm{d}}{\mathrm{d}t}\zeta_\mu(t,x) \approx \zeta_\mu(t,x) - \frac{1}{2\mu}\sum_{\{i:|X_i(t)-x|\in(0,\mu)\}} M_i(t)\cdot\zeta_\mu(t,X_i(t))\,.$$

- If $\zeta_\mu(t,y) \ll 1$ for all y s.t. $|x-y|<\mu$, get exponential growth.
- If $\zeta_\mu(t,y) \gg 1$ for all y s.t. $|x-y|<\mu$, get exponential decay.

Upper bound

Fix $c \in (0, c^*)$ and let $g(s) = \sqrt{2}s - cs^{1/3}$ for $s \geq 0$.

Proposition (Jaffuel)

There exists $\delta = \delta(c) > 0$ such that for t sufficiently large

$$\mathbb{P}\left(\exists i \leq N(t) \text{ s.t. } X_{i,t}(s) \geq g(s) \forall s \leq t\right) \leq e^{-\delta t^{1/3}}.$$

Upper bound

Fix $c \in (0, c^*)$ and let $g(s) = \sqrt{2}s - cs^{1/3}$ for $s \geq 0$.

Proposition (Jaffuel)

There exists $\delta = \delta(c) > 0$ such that for t sufficiently large

$$\mathbb{P}\left(\exists i \leq N(t) \text{ s.t. } X_{i,t}(s) \geq g(s) \forall s \leq t\right) \leq e^{-\delta t^{1/3}}.$$

Proposition (using Jaffuel and Roberts)

For any $C > 0$, there exists $\delta = \delta(c, C) > 0$ such that for t sufficiently large

$$\mathbb{P}\left(\exists i \leq N(t) \text{ s.t. } \mathrm{Leb}(\{s \leq t : X_{i,t}(s) \leq g(s)\}) \leq Ct^{1/3}\right) \leq e^{-\delta t^{1/3}}.$$

Upper bound

Fix $c \in (0, c^*)$ and let $g(s) = \sqrt{2}s - cs^{1/3}$ for $s \geq 0$.

Proposition (Jaffuel)

There exists $\delta = \delta(c) > 0$ such that for t sufficiently large

$$\mathbb{P}\left(\exists i \leq N(t) \text{ s.t. } X_{i,t}(s) \geq g(s) \forall s \leq t\right) \leq e^{-\delta t^{1/3}}.$$

Proposition (using Jaffuel and Roberts)

For any $C > 0$, there exists $\delta = \delta(c, C) > 0$ such that for t sufficiently large

$$\mathbb{P}\left(\exists i \leq N(t) \text{ s.t. } \mathrm{Leb}(\{s \leq t : X_{i,t}(s) \leq g(s)\}) \leq Ct^{1/3}\right) \leq e^{-\delta t^{1/3}}.$$

Proposition

For any $m > 0$, almost surely

$$\limsup_{t \to \infty} \frac{\sqrt{2}t - d(t, m)}{t^{1/3}} \geq c^*.$$

Lower bound

Proposition (Roberts)

There exists $C^* < \infty$ a.s. such that for all t,

$$\#\{i : \forall s \in [0,t], X_{i,t}(s) > \sqrt{2}s - c^* s^{1/3} + \frac{c^* s^{1/3}}{\log^2(s+e)} - C^*\} \geq 1$$

Lower bound

Proposition (Roberts)

There exists $C^* < \infty$ a.s. such that for all t,

$$\#\{i : \forall s \in [0,t], X_{i,t}(s) > \sqrt{2}s - c^* s^{1/3} + \frac{c^* s^{1/3}}{\log^2(s+e)} - C^*\} \geq 1$$

Proposition

There exists Z such that for all t sufficiently large,

$$\mathbb{P}\left(\sup\{\zeta_\mu(s,x) : 0 \leq s \leq t, x \in \mathbb{R}\} > Z \log t\right) \leq t^{-4}.$$

Lower bound

Proposition (Roberts)

There exists $C^* < \infty$ a.s. such that for all t,

$$\#\{i : \forall s \in [0,t], X_{i,t}(s) > \sqrt{2}s - c^* s^{1/3} + \frac{c^* s^{1/3}}{\log^2(s+e)} - C^*\} \geq 1$$

Proposition

There exists Z such that for all t sufficiently large,

$$\mathbb{P}\left(\sup\{\zeta_\mu(s,x) : 0 \leq s \leq t, x \in \mathbb{R}\} > Z \log t\right) \leq t^{-4}.$$

Proposition

For any $m \in (0,1)$, almost surely

$$\liminf_{t\to\infty} \frac{\sqrt{2}t - D(t,m)}{t^{1/3}} \leq c^*.$$

PDE approximation - non-local Fisher-KPP equation

For $\delta > 0$, let $z_\delta(t,x) = \frac{1}{2\delta} \sum_{\{i : |X_i(t) - x| < \delta\}} M_i(t)$.

Theorem (Addario-Berry, Berestycki, P.)

For $t \geq 1$, let $\delta(t) = t^{-1/5}$. Let u denote the solution to

$$\begin{cases} \frac{\partial u}{\partial s} = \frac{1}{2}\Delta u + u(1 - \phi_\mu * u), & s > 0, \quad x \in \mathbb{R}, \\ u(0,x) = z_{\delta(t)}(t,x), & x \in \mathbb{R}, \end{cases}$$

where $\phi_\mu(y) = \frac{1}{2\mu} \mathbb{1}_{\{|y| \leq \mu\}}$ and $\phi_\mu * u(s,x) = \frac{1}{2\mu} \int_{x-\mu}^{x+\mu} u(s,y) dy$.

For $T < \infty$, there exists $C < \infty$ such that for t sufficiently large,

$$\mathbb{P}\left(\sup_{s \in [0,T], x \in \mathbb{R}} \left| z_{\delta(t)}(t+s,x) - u(s,x) \right| \geq C\delta(t) \right) \leq t^{-n}.$$

PDE approximation - non-local Fisher-KPP equation

Proposition

For any $\alpha < 1$ and $n \in \mathbb{N}$, for t sufficiently large,

$$\mathbb{P}\left(\max_{i \leq N(t)} M_i(t) \geq t^{-\alpha}\right) \leq t^{-n}.$$

Proposition (Feynman-Kac formula)

Let u denote the solution to

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u(1 - \phi_\mu * u), & t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

Then for $t \geq 0$, $x \in \mathbb{R}$,

$$u(t,x) = e^t \mathbb{E}_x\left[e^{-\int_0^t \phi_\mu * u(t-s, B(s)) ds} u_0(B(t))\right].$$

Front location for the non-local Fisher-KPP equation

Suppose $u_0 \in L^\infty(\mathbb{R})$, $u_0 \geq 0$, $u_0 \not\equiv 0$ and u_0 is compactly supported, and take $\phi \in L^1(\mathbb{R})$, $\phi \geq 0$, $\phi(0) > 0$, $\int_{-\infty}^{\infty} \phi(x)dx = 1$. Let u denote the solution to

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u(1 - \phi * u), \quad t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}, \end{cases}$$

where $\phi * u(t,x) = \int_{-\infty}^{\infty} \phi(y)u(t,x-y)dy$.

Theorem (P.)

If there exists $\alpha > 2$ such that for r sufficiently large, $\int_r^\infty \phi(x)dx \leq r^{-\alpha}$, then there exists $A < \infty$ such that

$$\liminf_{t\to\infty} \inf_{x\in[0,\sqrt{2}t - \frac{3}{2\sqrt{2}}\log t - A(\log\log t)^3]} u(t,x) > 0$$

$$\textit{and} \quad \lim_{t\to\infty} \sup_{x \geq \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t + 10\log\log t} u(t,x) = 0.$$

Front location for the non-local Fisher-KPP equation

Suppose $u_0 \in L^\infty(\mathbb{R})$, $u_0 \geq 0$, $u_0 \not\equiv 0$ and u_0 is compactly supported, and take $\phi \in L^1(\mathbb{R})$, $\phi \geq 0$, $\phi(0) > 0$, $\int_{-\infty}^{\infty} \phi(x)dx = 1$. Let u denote the solution to

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u(1 - \phi * u), \quad t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}, \end{cases}$$

where $\phi * u(t,x) = \int_{-\infty}^{\infty} \phi(y)u(t,x-y)dy$.

Theorem (Bouin, Henderson, Ryzhik)

If there exists $\alpha > 2$ such that for r sufficiently large, $\int_r^\infty \phi(x)dx \leq r^{-\alpha}$, then

$$\liminf_{t\to\infty} \inf_{x \in [0, \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t]} u(t,x) > 0$$

$$\textit{and} \quad \lim_{A\to\infty} \limsup_{t\to\infty} \sup_{x \geq \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + A} u(t,x) = 0.$$

Behaviour behind the front

Suppose $u_0 \in L^\infty(\mathbb{R})$, $u_0 \geq 0$ and $u_0 \not\equiv 0$. Let u denote the solution to

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u(1 - \phi_\mu * u), \quad t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}, \end{cases}$$

where $\phi_\mu(y) = \frac{1}{2\mu}\mathbb{1}_{\{|y|\leq\mu\}}$ and $\phi_\mu * u(t,x) = \frac{1}{2\mu}\int_{x-\mu}^{x+\mu} u(t,y)dy$.

Theorem (Addario-Berry, Berestycki, P.)

There exists $\mu^* > 0$ such that for any $\mu \in (0, \mu^*]$, $c \in (0, \sqrt{2})$,

$$\lim_{t\to\infty} \sup_{x\in[-ct,ct]} |u(t,x) - 1| = 0.$$

Behaviour behind the front

Let $\zeta_\mu(t,x) = \frac{1}{2\mu}\sum_{\{i:|X_i(t)-x|\in(0,\mu)\}} M_i(t)$ and

$$M_i(t) = \exp\left(-\int_0^t \zeta_\mu(s, X_{i,t}(s))\,ds\right).$$

Theorem (Addario-Berry, Berestycki, P.)

There exists $\mu^* > 0$ such that for $\mu \in (0,\mu^*]$, $c \in (0,\sqrt{2})$, $\epsilon > 0$, for t sufficiently large,

$$\mathbb{P}\left(\sup_{s\geq t}\sup_{|x|\leq cs} |\zeta_\mu(s,x) - 1| \geq \epsilon\right) \leq t^{-n}.$$

Question: What happens for large μ?