Spectral stability of solitary waves in the nonlinear Dirac equation in the nonrelativistic limit

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We consider the nonlinear Dirac equation in $\mathbb{R}^n$, $n \geqslant 1$, with the scalar self-interaction, known as the Soler model [Iva38, Sol70]:

$$\mathrm{i}\partial_t \psi = D_m \psi - f(\psi^* \beta \psi)\beta\psi, \qquad \psi(x,t) \in \mathbb{C}^N, \quad x \in \mathbb{R}^n,$$

where the Dirac operator is $D_m = -\mathrm{i}\alpha \cdot \nabla + \beta m$, $m > 0$, with the self-adjoint $N \times N$ Dirac matrices $\alpha = (\alpha^\jmath)_{1 \leqslant \jmath \leqslant n}$ and β chosen so that $D_m^2 = -\Delta + m^2$. The nonlinearity is represented by a real-valued function $f \in C^1(\mathbb{R}\backslash\{0\})$ such that $f(\tau) = |\tau|^k + O(|\tau|^K)$ for $\tau \to 0$, with $0 < k < K$.

We study the point spectrum of the linearization at a solitary wave solution $\phi_\omega(x)e^{-\mathrm{i}\omega t}$, focusing on the spectral stability, that is, the absence of eigenvalues with nonzero real part. For $n = 1$ and $n \geqslant 3$, we prove the spectral stability of solitary waves in the nonrelativistic limit $\omega \lesssim m$ for the charge-subcritical cases $k \lesssim 2/n$ and for the "charge-critical case" $k = 2/n$ (if $K > 4/n$). For technical reasons, we can not consider the values $k \gtrsim 0$, and we only have partial results in the dimension $n = 2$.

An important part of the stability analysis is the proof of the absence of bifurcations of nonzero-real-part eigenvalues from the embedded threshold points at $\pm 2m\mathrm{i}$. Our approach is based on constructing a new family of exact bi-frequency solitary wave solutions $\phi_\omega(x)e^{-\mathrm{i}\omega t} + \chi_\omega(x)e^{\mathrm{i}\omega t}$ in the Soler model, on using this family to determine the multiplicity of $\pm 2\omega\mathrm{i}$ eigenvalues of the linearized operator, and on the analysis of the behaviour of "nonlinear eigenvalues": characteristic roots of holomorphic operator-valued functions [Kel51].

The talk is based on the articles [BC16, BC17a] and the preprint [BC17b].

References

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