Symmetry, rearrangements and isoperimetry

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Plan of the course:

Thursday 7 March, 3-5pm

1. An introduction to rearrangements

2. Polarization and symmetry of solutions to variational problems

Thursday 14 March, 3-5pm

3. Continuous Steiner symmetrization and symmetry of solutions to PDEs

4. Weighted rearrangements and isoperimetric inequalities

1. An introduction to rearrangements

We set up the general framework of rearrangements and their role in the calculus of variations. In particular, we give a list of symmetrizations (Schwarz-, Steiner- and cap symmetrisation) and some integral inequalities a la Hardy-Littlewood and Polya-Szegö.

Most of the material of this lecture can be found in the following articles. They also contain many further references on rearrangements.

F. BROCK, *Rearrangements and applications to symmetry problems in PDE.* Handbook of differential equations: stationary partial differential equations. Vol. IV, 1–60, Elsevier/North-Holland, Amsterdam, 2007.

F. BROCK, A. YU. SOLYNIN, *An approach to symmetrization via polarization.* Trans. Amer. Math. Soc. **352** (2000), no. 4, 1759–1796.

1.1. Motivation

Consider a variational problem of the form

$$(\mathbf{P}) \qquad J(v) \equiv \int\limits_{\Omega} \Big(\frac{1}{p}|\nabla v|^p - F(x,v)\Big)\, dx \longrightarrow \text{ Stat. ! }, \quad v \in K,$$

where K is a closed subset of $W^{1,p}(\Omega),\ p > 1$, and Ω is a domain in $\mathbb{R}^N$. The non-negative minimizers of **(P)** describe stable - so-called **ground** - states of equilibria, as they appear in many physical applications.

We ask for **symmetries** of the solutions of **(P)**, if F and Ω have certain "symmetries".

Often the set of admissible functions K involves **constraints** which have one of the following forms:

$$\begin{aligned}
& v \geq 0, \\
& \int_\Omega G(v)\,dx = 0, \quad (G \text{ continuous}), \\
& |\{v > 0\}| = C, \quad (C > 0).
\end{aligned}$$

As we shall see, **rearrangements** are those transformations

$$v \longmapsto Tv$$

that preserve these constraints. Moreover, some of these rearrangements turn v into a function Tv which has symmetry properties, and

$$J(Tv) \leq J(v) \quad \forall v \in K.$$

This allows to show that problem **(P)** has indeed symmetric solutions.

1.2. Notation

$$\mathbb{R}_0^+ = [0, +\infty)$$

$x = (x_1, \ldots, x_N)$ points in $\mathbb{R}^N$

$$x \cdot y = \sum_{i=1}^{N} x_i y_i$$

$$|x| = \sqrt{x \cdot x}$$

$\mathscr{M}$ set of all Lebesgue-measurable - measurable in short - sets of $\mathbb{R}^N$

$$M_1 \Delta M_2 = (M_1 \setminus M_2) \cup (M_2 \setminus M_1), \quad (M_1, M_2 \in \mathscr{M})$$

$|M|$ measure of a set $M \in \mathscr{M}$

$\| \cdot \|_{p,M}$ norm in the space $L^p(M)$, ($M \in \mathcal{M}$)

$$\| \cdot \|_p = \| \cdot \|_{p,\mathbb{R}^N}, \ (1 \leq p \leq +\infty)$$

If Ω is an open set in $\mathbb{R}^N$, and if $u \in L^\infty(\Omega)$, we define the **modulus of continuity**, $\omega_{u,\Omega}$, by

$$\omega_{u,\Omega}(t) := \sup\{|u(x)-u(y)| : \ x, y \in \Omega, \ |x-y| < t\}, \quad (t > 0).$$

(Here and in the following $\sup$ $(\inf)$ means ess sup (ess inf).)
We also write $\omega_{u,\mathbb{R}^N} = \omega_u$.
Note that if $u \in C(\Omega)$ then u is equicontinuous on Ω iff $\lim_{t \searrow 0} \omega_{u,\Omega}(t) = 0$.

$W^{1,p}(\Omega)$ is the usual Sobolev space, and $W_0^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the space $W^{1,p}(\Omega)$.

Usually we extend measurable functions $u : \Omega \to \mathbb{R}$ by zero outside Ω, so that $W_0^{1,p}(\Omega) \subset W^{1,p}(\mathbb{R}^N)$ in that sense.

$C_0^{0,1}(\Omega)$ is the set of Lipschitz functions with compact support in Ω.

The lower index "$+$" indicates the corresponding subset of non-negative functions, e.g. $L_+^p(\mathbb{R}^N)$, $W_{0+}^{1,p}(\Omega)$, $C_{0+}^{0,1}(\Omega), \ldots$

A function $G : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is called a **Young function** if G is continuous and convex with $G(0) = 0$.

1.3. Rearrangements, general properties

We often treat measurable sets only in a.e. sense, that is we identify a set M with its equivalence class given by all measurable sets $\widetilde{M}$ with $|M\Delta\widetilde{M}| = 0$. If $M_1, M_2 \in \mathscr{M}$, we write

$$\begin{aligned} M_1 = M_2 \quad &\Longleftrightarrow \quad |M_1 \Delta M_2| = 0, \quad \text{and} \\ M_1 \subset M_2 \quad &\Longleftrightarrow \quad |M_1 \setminus M_2| = 0. \end{aligned}$$

A set transformation $T : \mathscr{M} \longrightarrow \mathscr{M}$ is called a **rearrangement** if, ($M, M_1, M_2 \in \mathscr{M}$),

$$\begin{aligned} &M_1 \subset M_2 \Longrightarrow TM_1 \subset TM_2, \quad \text{(monotonicity)}, \\ &|M| = |TM|, \quad \text{(equimeasurability)}. \end{aligned}$$

Also, for any $M \in \mathscr{M}$, the set TM is called a **rearrangement of** M.

The notion of rearrangement is reserved for *measurable* sets. If a rearrangement has certain regularizing properties, as for instance the symmetrizations and the polarization (see the next section), then one can introduce pointwise representatives for the set TM when M is open or compact.

The definition implies, ($M_1, M_2 \in \mathscr{M}$),

$$
\begin{aligned}
&T(M_1 \cap M_2) \subset TM_1 \cap TM_2,\\
&T(M_1 \cup M_2) \supset TM_1 \cup TM_2,\\
&|M_1 \setminus M_2| \geq |TM_1 \setminus TM_2|,\\
&|M_1 \Delta M_2| \geq |TM_1 \Delta TM_2|.
\end{aligned}
$$

Next we introduce rearrangements of measurable functions $u : \mathbb{R}^N \to \mathbb{R}$. We will usually not distinguish between u and its equivalence class given by all measurable functions which differ from u on a nullset. We define

$$\mathscr{S}_+ := \{u : \mathbb{R}^N \to \mathbb{R}_0^+, \text{ measurable}, |\{u > t\}| < +\infty \ \ \forall t > 0\}.$$

Note that $L_+^p(\mathbb{R}^N)$, $W_+^{1,p}(\mathbb{R}^N)$, $(1 \le p < +\infty)$, and $C_{0+}^{0,1}(\mathbb{R}^N)$ are subsets of $\mathscr{S}_+$.
If $u \in \mathscr{S}_+$, its **distribution function** μ_u is given by

$$\mu_u(t) := |\{u > t\}|, \quad (t \in \mathbb{R}_0^+).$$

Note, μ_u is non-increasing and right-continuous with $\mu_u(t) = 0 \ \forall > \sup u$, and $\mu_u(t) < \infty \ \ \forall t \in (0, +\infty)$.

We will say that two functions $u, v \in \mathcal{S}_+$ are **equidistributed**, $u \sim v$, if $\mu_u(t) = \mu_v(t) \ \ \forall t > 0$. %

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The distribution function of μ_u - that is, the right-continuous inverse of μ_u - is called the **symmmetric decreasing rearrangement** of u and is denoted by $u^\sharp$.

Note, $u^\sharp$ is a non-increasing, right-continuous function on $\mathbb{R}_0^+$ with $u^\sharp(0) = \sup u$, $\lim_{s \to +\infty} u^\sharp(s) = 0$, and

$$u^\sharp(s) = \inf\{t \geq 0 : \; \mu_u(t) \leq s\}, \qquad (s \geq 0).$$

Let T be a rearrangement and $u \in \mathscr{S}_+$. We define a function $Tu \in \mathscr{S}_+$ by

$$Tu(x) := \sup \left\{ t \in \mathbb{R}_0^+ : \ x \in T\{u > t\} \right\}, \quad (x \in \mathbb{R}^N).$$

From this one obtains

$$\begin{aligned} \{Tu > t\} &= T\{u > t\} \text{ and} \\ \{Tu \geq t\} &= T\{u \geq t\}, \quad (t > 0). \end{aligned}$$

The function Tu is also called a **rearrangement of u**.

Note that

$$u \le v \implies Tu \le Tv, \quad \text{(monotonicity)},$$
$$|\{u > t\}| = |\{Tu > t\}| \;\; \forall t > 0, \quad \text{(equimeasurability)},$$

and in the special case that u is a characteristic function of a set $M \in \mathscr{M}$, i.e. $u = \chi(M)$, we have

$$T\chi(M) = \chi(TM).$$

Moreover, if $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is non-decreasing with $\varphi(0) = 0$, then

$$T\left(\varphi(u)\right) = \varphi\left(Tu\right).$$

The definition of Tu can be written in a more compact form using the so-called layer-cake formula,

$$u(x) = \int_0^{+\infty} \chi(\{u > t\})(x)\, dt, \quad (x \in \mathbb{R}^N),$$

i.e. u is the superposition of the characteristic functions of its superlevel sets. Note that the integral is à la Bochner, i.e., if $0 = t_0^k < t_1^k < \ldots < t_k^k$,
where $k \in \mathbb{N}$, $\max\{t_i^k - t_{i-1}^k\} : 1 \le i \le k\} \to 0$, $t_k^k \to +\infty$ as $k \to +\infty$, then

$$\sum_{i=1}^{k} \chi(\{u > t_i^k\})(t_i^k - t_{i-1}^k) \longrightarrow u \qquad \text{in measure.}$$

Then

$$Tu(x) = \int_0^{+\infty} \chi(T\{u > t\})(x)\, dt, \quad (x \in \mathbb{R}^N).$$

Note, if $u \in L^1_+(\mathbb{R}^N)$ then we have that

$$\int_{\mathbb{R}^N} u\,dx = \int_0^\infty |\{u > t\}|\,dt,$$

which is a variant of Fubini's Theorem.
Then the above formulas yield

$$\int_{\mathbb{R}^N} u\,dx = \int_{\mathbb{R}^N} Tu\,dx.$$

A more general property is the following one. It is often called **Cavalieri's principle**.

Theorem 1.1. *Let T be a rearrangement, $f : \mathbb{R}_0^+ \to \mathbb{R}$ continuous or non-decreasing with $f(0) = 0$, $u \in \mathscr{S}_+$ and $f(u) \in L^1(\mathbb{R}^N)$. Then*

$$\int\limits_{\mathbb{R}^N} f(u)\,dx = \int\limits_{\mathbb{R}^N} f(Tu)\,dx.$$

Note that the Theorem follows for non-decreasing f from the previous remarks. We drop the proof in the case of continuous functions f.

The next Theorem has many applications, too. It has been shown by Crowe, Rosenbloom and Zweibel in 1986 for the Schwarz symmetrization. However, their proof carries over to arbitrary rearrangements without difficulties.

Theorem 1.2. *Let* $F \in C((\mathbb{R}_0^+)^2)$*,* $F(0,0) = 0$*, and*

$$F(A,B) - F(a,B) - F(A,b) + F(a,b) \geq 0$$
$$\forall a,b,A,B \ \text{ with } \ 0 \leq a \leq A,\ 0 \leq b \leq B, \qquad (*)$$

and let T *be a rearrangement. Furthermore, let* $u, v \in \mathscr{S}_+$ *and* $F(u,0),\ F(0,v),\ F(u,v) \in L^1(\mathbb{R}^N)$*. Then*

$$\int_{\mathbb{R}^N} F(u,v)\,dx \leq \int_{\mathbb{R}^N} F(Tu,Tv)\,dx.$$

Note, if $F \in C^2$, then property (*) is equivalent to

$$\frac{\partial^2 F(s,t)}{\partial s \partial t} \geq 0 \quad \forall s,t \geq 0.$$

Important special cases of Theorem 1.2. are given for $F(u,v) = uv$ and $F(u,v) = -G(|u-v|)$, where G is any Young function.

Corollary 1.3. **(i)** *If* $u, v \in L^2_+(\mathbb{R}^N)$, *then*

$$\int_{\mathbb{R}^N} uv\,dx \le \int_{\mathbb{R}^N} TuTv\,dx, \quad \textit{(Hardy-Littlewood inequality).}$$

(ii) *If* $u, v \in \mathscr{S}_+$ *and* $G(|u-v|) \in L^1(\mathbb{R}^N)$ *for some Young function* G, *then*

$$\int_{\mathbb{R}^N} G(|u-v|)\,dx \ge \int_{\mathbb{R}^N} G(|Tu - Tv|)\,dx.$$

In particular, if $u, v \in L^p_+(\mathbb{R}^N)$, *(*$1 \le p < +\infty$*), then*

$$\|u-v\|_p \ge \|Tu - Tv\|_p, \quad \textit{(Nonexpansivity in } L^p\textit{).}$$

Let us show only the Hardy-Littlewood inequality:
If $u, v \in L^2_+(\mathbb{R}^N)$, then

$$\begin{aligned}
\int_{\mathbb{R}^N} uv\,dx &= \int_{\mathbb{R}^N}\int_0^\infty \chi(\{u>s\})(x)\,ds \int_0^\infty \chi(\{v>t\})(x)\,dt\,dx \\
&= \int_0^\infty\int_0^\infty\int_{\mathbb{R}^N} \chi(\{u>s\})(x)\chi(\{v>t\})(x)\,dx\,ds\,dt \\
&= \int_0^\infty\int_0^\infty |\{u>s\}\cap\{v>t\}|\,ds\,dt \\
&\le \int_0^\infty\int_0^\infty |\{Tu>s\}\cap\{Tv>t\}|\,ds\,dt \\
&= \int_{\mathbb{R}^N} TuTv\,dx.
\end{aligned}$$

Rearrangements are nonexpansive in L^∞, too.

Corollary 1.4. *Let $u, v \in \mathscr{S}_+ \cap L^\infty(\mathbb{R}^N)$, and let T be a rearrangement. Then*

$$\|Tu - Tv\|_\infty \le \|u - v\|_\infty.$$

Proof : Let $C := \|u - v\|_\infty$. Then we have $-C \le u - v \le C$ a.e. on $\mathbb{R}^N$. By the monotonicity this means that

$$Tv - C = T(v - C) \le Tu \le T(v + C) = Tv + C, \qquad \text{a.e. on } \mathbb{R}^N,$$

and the assertion follows. $\square$

1.4. A list of rearrangements

1.4.1. Schwarz symmetrization

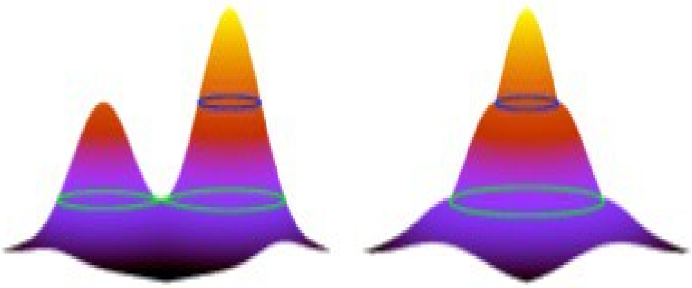
Let $M \in \mathscr{M}$. If $|M| < \infty$, then let $M^\star$ be the ball B_R with $|B_R| = |M|$, and if $|M| = \infty$, then let $M^\star = \mathbb{R}^N$.
Correspondingly, for any function $u \in \mathscr{S}_+$ we introduce $u^\star$ by the layer-cake formula with $Tu = u^\star$ and $T\{u > \lambda\} = \{u > \lambda\}^\star$. An equivalent definition is

$$u^\star(x) := u^\sharp(\kappa_N |x|^N), \quad (x \in \mathbb{R}^N).$$

(Here $\kappa_N = |B_1|$.)
The objects $M^\star$ and $u^\star$ are called the **Schwarz symmetrizations** of M and u, respectively.

Note that $u^\star$ is **radially symmetric and radially non-increasing**, that is, $u^\star$ depends on the radial distance $|x|$ only, and is non-increasing in $|x|$. The superlevel sets $\{u^\star > t\}$ are balls centered at zero, and they have the same measure as $\{u > t\}$, ($t \in \mathbb{R}$).

Note also that two functions $u, v \in \mathscr{S}_+$ are equidistributed, $u \sim v$, iff $u^\star = v^\star$.



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1.4.2. Steiner symmetrization

We write $x = (x_1, x')$ for points in $\mathbb{R}^N$, $|N|_1$ for the one-dimensional measure of a set $N \subset \mathbb{R}$ and $l(x')$ for the line $\{x = (x_1, x') : \; x_1 \in \mathbb{R}\}$, ($x' \in \mathbb{R}^{N-1}$).
If $M \in \mathscr{M}$, its **Steiner symmetrization**, M^*, is given by

$$M^* := \{x = (x_1, x') : \; 2|x_1| < |M \cap l(x')|_1, \; x' \in \mathbb{R}^{N-1}\}.$$

If $u \in \mathscr{S}_+$, its **Steiner symmetrization**, u^*, is given by the layer-cake formula with $Tu = u^*$ and $T\{u > t\} = \{u > t\}^*$.
An equivalent definition is

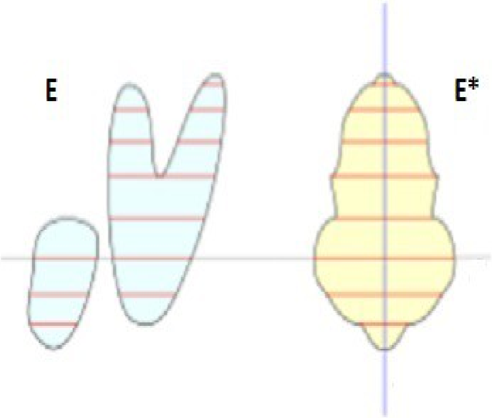
$$\begin{aligned} u^*(x_1, x') \quad &:= \quad \sup \left\{ t \in \mathbb{R} : \; 2|x_1| < |\{u(\cdot, x') > t\}|_1 \right\}, \\ & \qquad (x = (x_1, x') \in \mathbb{R}^N). \end{aligned}$$

(Here $\{u(\cdot, x') > t\}$ is a short-hand for $\{x_1 : \; u(x_1, x') > t\}$.)

The function u^* is even in the variable x_1 and nonincreasing in x_1 for $x_1 \geq 0$, and

$$|\{u(\cdot, x') > t\}|_1 = |\{u^*(\cdot, x') > t\}|_1 \qquad (*)$$
$$\forall t \in \mathbb{R} \text{ and for a.e. } x' \in \mathbb{R}^{N-1},$$

the set on the right-hand side of (*) being an interval centered at zero.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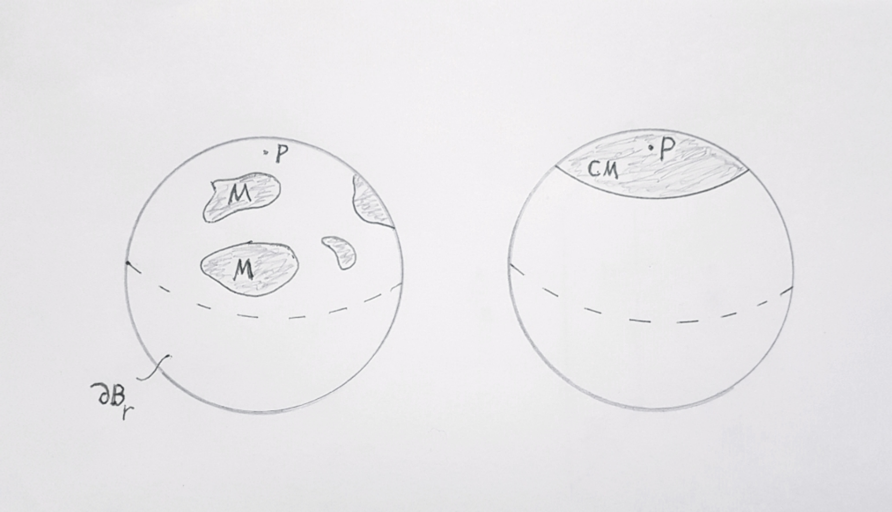
E*

1.4.3. Cap symmetrization

Let $P := (1, 0, \dots, 0)$ - the 'north pole'. If $M \in \mathscr{M}$, then there is for a.e. $r > 0$ a unique value $\rho \geq 0$ such that the spherical cap $B_\rho(P) \cap \partial B_r$ has the same $(N-1)$-Lebesgue measure as $M \cap \partial B_r$. We denote this spherical cap by $CM(r)$. The set

$$CM := \{x \in \mathbb{R}^N : \, x \in CM(r), \, r > 0\}$$

is called the cap symmetrization of M.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M
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Furthermore, if $u : \mathbb{R}^N \to \mathbb{R}$ is measurable, we define its **cap symmetrization**, Cu, by

$$Cu(x) := \sup \Big\{ t \in \mathbb{R} : \, x \in C\{u > t\} \Big\}.$$

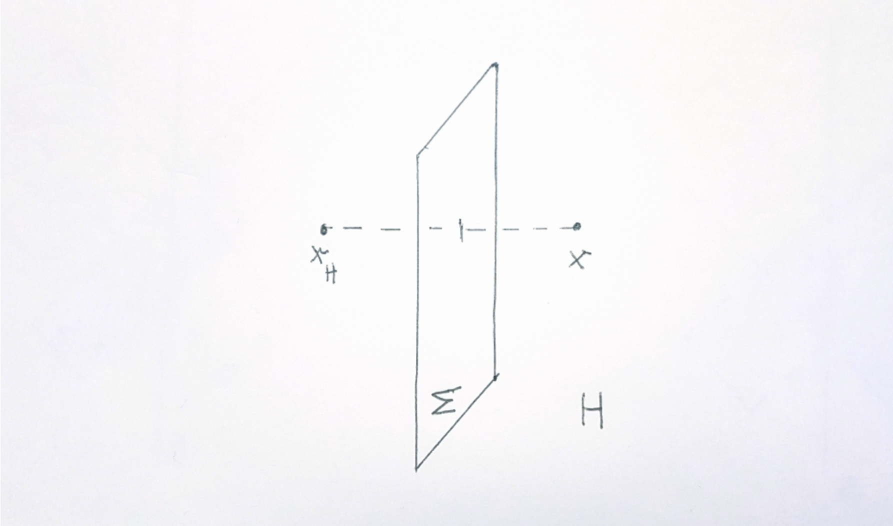
Note, the superlevel sets $\{Cu > t\} \cap \partial B_r$ are spherical caps centered at P and have the same $(N-1)$-measure as $\{u > t\} \cap \partial B_r$, ($r > 0$, $t \in \mathbb{R}$). Hence Cu depends only on the radial distance $r = |x|$ and on the geographical lattitude $\theta_1 := \arccos(x_1/|x|)$ only, and is nonincreasing in $\theta_1 \in [0, \pi]$.
Note also, the cap symmetrization is frequently referred to as **foliated Schwarz symmetrization**.

1.4.4. Polarization

Now we study a simple rearrangement that can be used to prove many functional inequalities for symmetrizations.

Let Σ be some $(N-1)-$ dimensional affine hyperplane in $\mathbb{R}^N$ and assume that H is one of the two open half-spaces into which $\mathbb{R}^N$ is subdivided by Σ. For any point $x \in \mathbb{R}^N$ let σx denote its reflection in $\Sigma = \partial H$. Furthermore, if $u : \mathbb{R}^N \to \mathbb{R}$ is measurable, then we define its **polarization (with respect to H)**, u_H, by

$$u_H(x) := \begin{cases} \max\{u(x);\, u(\sigma x)\} & \text{if } \ x \in H \\ \min\{u(x);\, u(\sigma x)\} & \text{if } \ x \in \mathbb{R}^N \setminus H. \end{cases}$$


Σ
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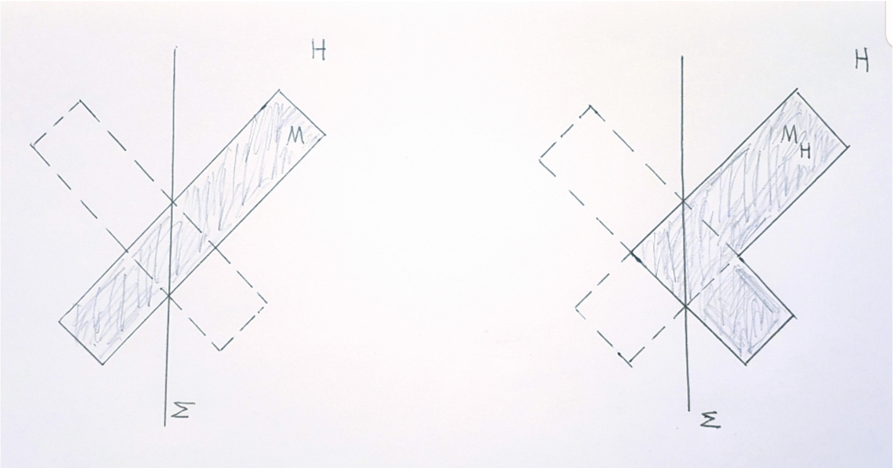
If $M \in \mathscr{M}$ we define its **polarization (w.r.t. H)**, M_H, via its characteristic function,

$$\chi(M_H) := \Big(\chi(M)\Big)_H.$$

Note that polarization is also referred to as **two-point rearrangement**.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The following properties are easy to prove:

Theorem 1.5. *Let H be a half-space, Ω a domain in $\mathbb{R}^N$ and $1 \leq p < +\infty$.*
(i) *Assume $\Omega = \sigma\Omega$. If $u \in C(\Omega) \cap L^\infty(\Omega)$, or $u \in C(\overline{\Omega}) \cap L^\infty(\Omega)$, then so does u_H and*

$$\omega_{u_H,\Omega} \leq \omega_{u,\Omega}.$$

In particular, if u is Lipschitz continuous with Lipschitz constant L, then so is u_H, with Lipschitz constant $\leq L$.
(ii) *Assume again $\Omega = \sigma\Omega$. If $u \in W_0^{1,p}(\Omega)$, then so is u_H and*

$$\|\nabla u\|_p = \|\nabla u_H\|_p.$$

(iii) *If $u \in W_{0+}^{1,p}(\Omega)$, respectively $u \in C_{0+}^{0,1}(\Omega)$, then $u_H \in W_{0+}^{1,p}(\Omega_H)$, respectively $u \in C_{0+}^{0,1}(\Omega_H)$.*

Now we state a convolution-type inequality.

Theorem 1.6. *Let H be a half-space, $w : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ measurable and non-increasing, and let $u, v \in \mathscr{S}_+$. Then*

$$\iint_{\mathbb{R}^{2N}} u(x)v(y)w(|x-y|)\,dxdy \le \iint_{\mathbb{R}^{2N}} u_H(x)v_H(y)w(|x-y|)\,dxdy,$$

provided that one of the integrals converges.

Proof: Let $x, y \in H$. Then

$$|x-y| = |\sigma x - \sigma y| \leq |\sigma x - y| = |x - \sigma y|.$$

Hence, using the monotonicity of w, an elementary calculation yields

$$\begin{aligned}
& u(x)v(y)w(|x-y|) + u(\sigma x)v(y)w(|\sigma x - y|) + \\
& u(x)v(\sigma y)|)w(|x-\sigma y|) + u(\sigma x)v(\sigma y)|)w(|\sigma x - \sigma y|) \\
\leq\ & u_H(x)v_H(y)w(|x-y|) + u_H(\sigma x)v_H(y)w(|\sigma x - y|) + \\
& u_H(x)v_H(\sigma y)|)w(|x-\sigma y|) + u_H(\sigma x)v_H(\sigma y)|)w(|\sigma x - \sigma y|).
\end{aligned}$$

Then an integration over $H \times H$ proves the claim. $\square$

1.5. Inequalities for Schwarz symmetrization

Many integral inequalities for symmetrizations can be proved exploiting related properties for polarizations and using an approximation procedure. Our aim is to present some ideas of such a program. For convenience we will restrict ourselves to the case of Schwarz symmetrization.

In the following, let $\mathcal{H}_0$ denote the set of half-spaces H in $\mathbb{R}^N$ such that $0 \in H$.

The following separation property is crucial.

Lemma 1.6. *Let $u \in L^p_+(\mathbb{R}^N)$ for some $p \in [1, \infty)$ and suppose that $u \neq u^\star$. Then there exists $H \in \mathcal{H}_0$ such that*

$$\|u_H - u^\star\|_p < \|u - u^\star\|_p.$$

Proof : If $H \in \mathcal{H}_0$ then we have that $u = (u^\star)_H$, and an elementary analysis shows that

$$\begin{aligned} & |u_H(x) - u^\star(x)|^p + |u_H(\sigma x) - u^\star(\sigma x)|^p \\ \leq \; & |u(x) - u^\star(x)|^p + |u(\sigma x) - u^\star(\sigma x)|^p \quad \forall x \in H. \end{aligned} \tag{$*$}$$

An integration of this over H then leads to $\|u_H - u^\star\|_p \leq \|u - u^\star\|_p$. Therefore to prove the claim it suffices to show that, for a suitable choice of H, inequality (*) becomes strict on a subset of H of positive measure. Since $u \neq u^\star$, we find some number $c > 0$ such that $|\{u > c\} \Delta \{u^\star > c\}| > 0$.
Let x^1 and x^2 density points of the sets $\{u > c\} \setminus \{u^\star > c\}$ and $\{u^\star > c\} \setminus \{u > c\}$, respectively.

We choose a halfspace H such that $x^1 = \sigma x^2$ and $x^2 \in H$. (Note that from $u^\star(x^1) \leq c < u^\star(x^2)$ it follows that $0 \in H$.) Hence there is a subset K of H of positive measure containing x^2 such that

$$u^\star(x) > c \geq u(x), \quad u^\star(\sigma x) \leq c < u(\sigma x) \qquad \forall x \in K.$$

But this means that the inequality (*) becomes strict on the set K. $\square$

Next we show that the modulus of continuity does not increase under Schwarz symmetrization.

Theorem 1.7. *Let* $u \in \mathscr{S}_+ \cap L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$*. Then* $u^\star \in C(\mathbb{R}^N)$ *and*

$$\omega_{u^\star} \leq \omega_u \ . \qquad\qquad (*)$$

Proof: Assume first that $u \in C^{0,1}_{0+}(B_R)$ for some $R > 0$. Let

$$A^\star(u) := \{v \in C^{0,1}_{0+}(B_R) : \, v \sim u, \, \omega_{v,B_R} \leq \omega_{u,B_R}\},$$

and $\delta := \inf\{\|v - u^\star\|_2 : \, v \in A^\star(u)\}$.
By Arzelá's Theorem and by the nonexpansivity of the Schwarz symmetrization in L^2, δ is attained for some $U \in A^\star(u)$. Assume that $\delta > 0$.

Then we find by Lemma 1.6. a halfspace $H \in \mathcal{H}_0$ such that $\|U_H - u^\star\|_2 < \delta$. Since $(B_R)_H = B_R$, $U_H \sim u$ and $\omega_{U_H,B_R} \leq \omega_{U,B_R}$, we have that $U_H \in C^{0,1}_{0+}(B_R)$ and $U_H \in A^\star(u)$. But this contradicts to the minimality of U. Hence $\delta = 0$ and thus $U = u^\star$, and inequality (*) follows in this case.
In the general case we choose a sequence $\{u_n\} \subset C^{0,1}_{0+}(\mathbb{R}^N)$ converging to u in $C(\mathbb{R}^N)$. Then we have that $(u_n)^\star \to u^\star$ in $C(\mathbb{R}^N)$, and (*) follows. $\square$

Next we deal with norm inequalities in $W^{1,p}$ for rearrangements. Such inequalities are often referred to as **Polya-Szegö's Principle** in the literature.
A decisive role in the proofs plays the weak lower semi-continuity of the norms.

Theorem 1.8. *Let* $u \in W^{1,p}(\mathbb{R}^N) \cap \mathscr{S}_+$ *for some* $p \in [1, +\infty]$. *Then* $u^\star \in W^{1,p}(\mathbb{R}^N) \cap \mathscr{S}_+$ *and*

$$\|\nabla u\|_p \geq \|\nabla u^\star\|_p. \qquad (*)$$

Note that inequality (*) for $p = 1$ is equivalent to the classical isoperimetric inequality.

Proof of Theorem 1.8.: For convenience we restrict ourselves to the case $p \in (1, +\infty)$.
First assume $u \in C^{0,1}_{0+}(B_R)$ for some $R > 0$, and

$$B^\star(u) := \{v \in C^{0,1}_{0+}(B_R) : \; v \sim u, \, \omega_v \leq \omega_u, \, \|\nabla v\|_p \leq \|\nabla u\|_p\},$$

Furthermore, let $\delta = \inf\{\|v - u^\star\|_2 : \; v \in B^\star(u)\}$.

In view of the weak lower semicontinuity of the norm and the nonexpansivity of the Schwarz symmetrization, $B^\star(u)$ is weakly closed in $C^{0,1}_{0+}(B_R)$.
Hence there exists some $U \in B^\star(u)$ with $\delta = \|U - u^\star\|_2$.
Since $\|\nabla U_H\|_p = \|\nabla U\|_p \ \forall H \in \mathcal{H}_0$, we may then argue as in the proof of Theorem 1.7. to obtain that $\delta = 0$, and thus $U = u^\star$. From this (*) follows in this case.
In the general case we choose a sequence $\{u_n\} \subset C^{0,1}_{0+}(\mathbb{R}^N)$ which converges to u in $W^{1,p}(\mathbb{R}^N)$. Then we have that $\|\nabla u_n\|_p \geq \|\nabla (u_n)^\star\|_p$, $n = 1, 2, \ldots$. Hence we find a subsequence $\{(u_n')^\star\}$ and a function $v \in W^{1,p}(\mathbb{R}^N)$ such that $(u_n')^\star \rightharpoonup v$ weakly in $W^{1,p}(\mathbb{R}^N)$. By the nonexpansivity of the Schwarz symmetrization in L^p it follows that $(u_n)^\star \to u^\star$ in $L^p(\mathbb{R}^N)$, so that $v = u^\star$. Finally, the weak lower semicontinuity of the norm gives
$\|\nabla u^\star\|_p \leq \liminf \|\nabla (u_n')^\star\|_p \leq \lim \|\nabla u_n\|_p = \|\nabla u\|_p$.
The Theorem is proved. $\square$

Using similar arguments one can show many further integral inequalities involving functions, their gradients and symmetrizations. We give just one more example for the Schwarz symmetrization without proof.

Theorem 1.9. *Let H be a half-space, $w : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ measurable and non-increasing, and let $u, v \in \mathscr{S}_+$. Then*

$$\iint\limits_{\mathbb{R}^{2N}} u(x)v(y)w(|x-y|)\, dxdy \;\leq\; \iint\limits_{\mathbb{R}^{2N}} u^\star(x)v^\star(y)w(|x-y|)\, dxdy,$$

provided that one of the integrals converges.

2. Polarization and symmetry of solutions to variational problems

In this chapter we use polarization in order to prove symmetry results for minimizers of some elliptic variational problems. We emphasize that this part does not require any knowledge about the symmetrization inequalities of the previous chapter.

F. BROCK, *Rearrangements and applications to symmetry problems in PDE.* Handbook of differential equations: stationary partial differential equations. Vol. IV, 1–60, Elsevier/North-Holland, Amsterdam, 2007.

T. WETH, *Symmetry of solutions to variational problems for nonlinear elliptic equations via reflection methods.* Jahresber. Dtsch. Math.-Ver. **112** (2010), no. 3, 119–158.

2.1. Identification of symmetry

Polarizations can be used to identify symmetry.
We will use the following notation.
Let $\mathcal{H}$ the set of all affine half-spaces of $\mathbb{R}^N$, and

$$\begin{aligned}
\mathcal{H}_0 &:= \{H \in \mathcal{H} :\ 0 \in H\}, \\
\mathcal{H}^* &:= \{H \in \mathcal{H} :\ H = \{x :\ x_1 > \lambda\} \text{ or } \ H = \{x :\ x_1 < \lambda\}, \\
&\qquad\qquad \text{for some } \lambda \in \mathbb{R}\}, \\
\mathcal{H}_0^* &:= \{H \in \mathcal{H}^* :\ 0 \in H\}, \\
C\mathcal{H} &:= \{H \in \mathcal{H} :\ 0 \in \partial H\}, \ \text{ and} \\
C\mathcal{H}_P &:= \{H \in C\mathcal{H} :\ P \in H\}.
\end{aligned}$$

The following properties are easy to check:

1. Let $u \in \mathscr{S}_+$. Then we have $u = u^\star$ iff $u = u_H$ for all $H \in \mathcal{H}_0$. Furthermore, we have $u = u^*$ iff $u = u_H$ for all $H \in \mathcal{H}_0^*$.

2. Let $u : \mathbb{R}^N \to \mathbb{R}$ be measurable. Then we have $u = Cu$ iff $u = u_H$ for all $H \in C\mathcal{H}_P$.

More complicated are the following criteria. We omit their proof.

Theorem 2.1. *Let $u \in L^p(\mathbb{R}^N)$ for some $p \in [1, \infty)$.*

(i) *Let $|\{u > 0\}| > 0$. If, for every $H \in \mathcal{H}^*$, there holds either $u = u_H$ or $\sigma u = u_H$, then there is a point $\xi = (\lambda_0, 0, \dots, 0) \in \mathbb{R}^N$, ($\lambda_0 \in \mathbb{R}$), such that $u \in \mathcal{S}_+$ and $u(\cdot - \xi) = u^*(\cdot)$. In particular, if $u = u_H \ \forall H \in \mathcal{H}_0^*$ then $u \in \mathcal{S}_+$ and $u = u^*$.*

(ii) *Let $|\{u > 0\}| > 0$. If, for every $H \in \mathcal{H}$, there holds either $u = u_H$ or $\sigma u = u_H$, then there is a point $\xi \in \mathbb{R}^N$ such that $u \in \mathcal{S}_+$ and $u(\cdot - \xi) = u^\star(\cdot)$. In particular, if $u = u_H \ \forall H \in \mathcal{H}_0$ then $u \in \mathcal{S}_+$ and $u = u^\star$.*

(iii) *If, for every $H \in C\mathcal{H}$, there holds either $u = u_H$ or $\sigma_H u = u_H$, then there is a rotation ρ about zero such that $u(\rho \, \cdot) = Cu(\cdot)$. In particular, if $u = u_H \ \forall H \in C\mathcal{H}_P$ then $u = Cu$.*

2.2. Symmetry of solutions to variational problems

Our proofs rely on polarization and the following **Principle of Unique Continuation (PUC)**:

Theorem 2.2. *Let Ω be a domain in $\mathbb{R}^N$, and let $u \in W^{2,2}(\Omega)$ satisfy*

$$-\Delta u + cu = 0 \quad \text{in } \Omega,$$

where $c \in L^\infty(\Omega)$. Furthermore, suppose that there is a nonempty open subset U of Ω such that $u \equiv 0$ in U. Then $u \equiv 0$ in Ω.

First we consider a problem in a domain with rotational symmetry.

Let $\Omega_1 = B_{R_2} \setminus \overline{B_{R_1}}$, or $\Omega_1 = B_{R_2}$, for some $R_2 > R_1 \geq 0$, and let F, G, H functions satisfying

$$F, G, H \in C^2(\mathbb{R}),\ F' =: f,\ G' =: g,\ H' =: h,$$
$$|f(t)|, |g(t)| \leq c(1 + |t|^p),\ |h(t)| \leq c(1 + |t|^q),$$
with $c > 0$, $1 \leq p < (N+2)/(N-2)$,
$1 \leq q < N/(N-2)$, if $N \geq 3$, and p, q finite if $N = 2$,
g does not vanish on intervals of $\mathbb{R}$,

and let

$$K_1 = \{v \in W^{1,2}(\Omega_1) : \int_{\Omega_1} G(v)\, dx = 1\}.$$

We consider the following variational problem

$$(\mathbf{P_1}) \qquad J_1(v) \equiv \int\limits_{\Omega_1} \Big(\frac{1}{2}|\nabla v|^2 - F(v) \Big)\, dx + \int\limits_{\partial\Omega_1} H(v)\, dS \longrightarrow \text{Inf !},$$
$$v \in K_1.$$

If u is a minimizer of $(\mathbf{P_1})$ then standard variational calculus shows that

$$-\Delta u = f(u) + \alpha g(u) \text{ in } \Omega_1,$$

$$\partial u/\partial \nu + h(u) = 0 \text{ on } \partial\Omega_1, \ (\nu : \text{ exterior normal}),$$

$$\int_{\Omega_1} \Big(\nabla u \nabla \varphi - f(u)\varphi\Big)\, dx + \int_{\partial\Omega_1} h(u)\varphi \, dS = 0, \quad \text{and}$$

$$\int_{\Omega_1} \Big(|\nabla \varphi|^2 - (f'(u) + \alpha g'(u))\varphi^2\Big)\, dx + \int_{\partial\Omega_1} h'(u)\varphi^2 \, dS \geq 0$$

$$\text{for every } \varphi \in W^{1,2}(\Omega_1) \text{ satisfying } \int_{\Omega_1} g(u)\varphi \, dx = 0, \text{ and}$$

$$u \in W^{2,2}(\Omega_1) \cap C^1(\overline{\Omega_1}),$$

with some $\alpha \in \mathbb{R}$.
Note that problem $(\mathbf{P_1})$ with $F(v) = -(1/2)v^2$, $G(v) = |v|^{p+1}$ and $H \equiv 0$, has been extensively studied.

We first recall the well-known result that in case of the unconstrained problem with Dirichlet boundary (or Neumann) boundary conditions, that is, $H \equiv 0$ and $K_1 = W_0^{1,2}(\Omega_1)$ (respectively $K_1 = W^{1,2}(\Omega)$), minimizers u of $(\mathbf{P_1})$ are radial. The idea is to show that the functions $v_{ij} := x_i u_{x_j} - x_j u_{x_i}$, $(i \neq j)$, vanish. Note that that proof carries over to our more general boundary condition, as well.

Theorem 2.3. *Let u a be a minimizer of $(\mathbf{P_1})$. Then there is a rotation about zero ρ such that $u(\rho\,\cdot) = Cu(\cdot)$.*

Proof : We will assume that u is not constant in Ω_1 - since otherwise the assertion is trivially true.

First we claim that u cannot be constant on open subsets of Ω_1. Indeed, if $u \equiv c$ on an open set $U \subset \Omega_1$, then $f(c) + \alpha g(c) = 0$. Setting $v := u - c$ we have that $-\Delta v = c(x)v$ in Ω_1, with some bounded function c, while $v \equiv 0$ in U. The (PUC) then tells us that $v \equiv 0$ throughout Ω_1, a contradiction.
Let $H \in C\mathcal{H}$. Then the results of Chapter 1 show that $J(u_H) = J(u)$ and $u_H \in K_1$, so that u_H is a minimizer, too. This means that u_H satisfies the same Euler equations, with the Lagrangian multiplier α possibly replaced by some other number α'.
We claim that actually $\alpha = \alpha'$. Indeed, there is an open set $V \subset \Omega_1$ with $u = u_H$ or $\sigma u = u_H$ on V. Then $\alpha g(u) = \alpha' g(u_H)$, respectively $\alpha g(\sigma u) = \alpha' g(u)$, in V. Since u, respectively σu, is not constant on V, this means that $\alpha = \alpha'$, by the non-degeneracy condition for g.

Finally, setting $v := u - u_H$, respectively $v := \sigma u - u_H$, we find that v satisfies $-\Delta v = c(x)v$ in Ω_1, with some bounded function c, while $v \equiv 0$ in V. By the (PUC) this means that $v \equiv 0$, or equivalently $u \equiv u_H$, respectively $\sigma u \equiv u_H$ throughout Ω_1. Since H was arbitrary, the assertion follows from Theorem 2.1. $\square$

Note, Theorem 2.3. remains valid in an unbounded domain $\Omega_1 = \mathbb{R}^N \setminus \overline{B_{R_1}}$ and/or if the functions F, G and H depend additionally on $|x|$ and satisfy appropriate growth conditions.

One can also obtain symmetry results for global and local minimizers, and also for stationary solutions of $(\mathbf{P_1})$, including situations with more than one integral constraint and/or with Dirichlet boundary conditions.

Next we study a problem in the entire space. Let F, G as before, except that the growth conditions are replaced by

$$|F(t)|, |G(t)| \leq c(|t|^p + |t|^q), \quad \text{with } c > 0 \text{ and } p, q \geq 1,$$

and let

$$K_2 := \{v \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N) : \nabla v \in L^2(\mathbb{R}^N), \int_{\mathbb{R}^N} G(v)\, dx = d\},$$

where $d \in \mathbb{R}$.
We consider the following variational problem:

$$(\mathbf{P_2}) \qquad J(v) \equiv \int_{\mathbb{R}^N} \Big(\frac{1}{2}|\nabla v|^2 - F(v) \Big)\, dx \longrightarrow \text{Inf !}, \quad v \in K_2.$$

Assume that u is a solution of $(\mathbf{P_2})$. Then u satisfies

$$-\Delta u = f(u) + \alpha g(u) \quad \text{ on } \mathbb{R}^N,$$

for some $\alpha \in \mathbb{R}$.

If in addition, u is bounded, then standard elliptic estimates show that

$$u \in W^{2,2}_{loc}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N), \quad \text{and}$$
$$\lim_{|x|\to\infty} u(x) = 0.$$

Proceeding as in the previous proof and working with halfspaces $H \in \mathcal{H}$, we then obtain the following

Theorem 2.4. *Let u a be bounded minimizer of $(\mathbf{P_2})$ and $|\{u > 0\}| > 0$. Then there is a point $\xi \in \mathbb{R}^N$ such that $u(\cdot - \xi) = u^\star(\cdot)$. In particular, u does not change sign.*

Our method is also applicable to local minimizers of some variational problems.

We will say that u is a *local* minimizer of $(\mathbf{P_2})$ if there exists a number $\varepsilon > 0$ such that

$$\|u-v\|_p + \|u-v\|_q + \|\nabla(u-v)\|_2 < \varepsilon \quad \forall v \in K_2.$$

The next result resembles those that have been obtained for stationary solutions of $(\mathbf{P_2})$ using the well-known Moving Plane Method (MPM). However, the (MPM) requires either an asymptotic estimate of the solution u at infinity, or some 'nice' additional condition on the nonlinearity $f(t) + \alpha g(t)$ near 0. Such information is not needed here.

Theorem 2.5. *Let u be a non-negative and bounded local minimizer of $(\mathbf{P_2})$. Then there is a point $\xi \in \mathbb{R}^N$ such that $u(\cdot - \xi) = u^\star(\cdot)$.*

Sketch of the proof : We write $H_\lambda = \{x : x_1 > \lambda\}$ and $u_{H_\lambda} =: u_\lambda$. Further, let σ_λ denote reflexion in the hyperplane $\{x_1 = \lambda\}$.
First observe that $u_\lambda \in K_2$ and $J_2(u_\lambda) = J_2(u) \ \forall \lambda \in \mathbb{R}$.
Since u is is nonnegative, one can show that $u_\lambda \to u$ in $L^p(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$ as $\lambda \to -\infty$. Hence we find a number $\lambda_1 \in \mathbb{R}$ such that u_λ is a local minimizer, too, $\forall \lambda \leq \lambda_1$.
Clearly we may assume that $u_\lambda \not\equiv \sigma_\lambda u$ for these λ.
Then, using the (PUC) as in the proof of Theorem 2.3. we show that $u = u_\lambda$ for $\lambda \leq \lambda_1$. Similarly one proves that there exists a number λ_2, ($\lambda_2 > \lambda_1$), such that $\sigma_\lambda u = u_\lambda$ $\forall \lambda \geq \lambda_2$.

Now let

$$\lambda^* := \max\{\lambda \in \mathbb{R} : \ u = u_\mu \ \forall \mu \in (-\infty, \lambda)\}.$$

By continuity, we have that $\lambda^* \in [\lambda_1, \lambda_2]$ and $u = u_\lambda$ $\forall \lambda \in (-\infty, \lambda^*]$. From this it follows that $u_\lambda \to u$ in $L^p(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$ and $\nabla u_\lambda \to \nabla u$ in $L^2(\mathbb{R}^N)$ as $\lambda \to \lambda^*$. Hence there is a number $\delta > 0$ such that u_λ is a local minimizer, too, $\forall \lambda \in [\lambda^*, \lambda^* + \delta]$. Using the (PUC) once again, it follows that either $u = u_\lambda$ or $\sigma_\lambda u = u_\lambda$ for each of these λ.
On the other hand, we cannot have $u = u_\lambda$ on an interval $[\lambda^*, \lambda^* + \delta']$, ($0 < \delta' \leq \delta$), in view of the maximality of λ^*. By continuity, it then follows that $u = u_{\lambda^*} = \sigma_{\lambda^*} u$.

Hence $u = u_\lambda$ for $\lambda \leq \lambda^*$ and $\sigma_\lambda u = u_\lambda$ for $\lambda \geq \lambda^*$, which means that $u(x_1 - \lambda^*, x') = u^*(x_1, x')$ on $\mathbb{R}^N$.
We may repeat these considerations in any rotated coordinate system, thus obtaining that if $H \in \mathcal{H}$ then either $u = u_H$ or $\sigma u = u_H$. The assertion then follows from Theorem 2.1. $\square$

Remark 2.6. Polarizations can also be used in proving symmetry properties of sign-changing solutions to variational problems in $\mathbb{R}^N$ which are associated to **degenerate elliptic operators**, such as the p-Laplacian. In such a situation the (PUC) is not applicable, but instead one relies on the regularity of the solution. This allows to show that the solutions are **locally symmetric** in the sense that the superlevel sets $\{u > t\}$ and the sublevel sets $\{u < -t\}$, $(t > 0)$, are balls. Using this and the Strong Maximum Principle for the degenerate elliptic operators, one can prove in some cases that u is actually non-negative and radially symmetric about some point.

More on local symmetry for solutions of elliptic equations will follow in my next lecture, in the context of **Continuous Steiner Symmetrization**.