Symmetry, rearrangements and isoperimetry

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3. Continuous Steiner symmetrisation and symmetry of solutions to PDEs
4. Weighted rearrangements and isoperimetric inequalities

3. Continuous Steiner symmetrisation and symmetry of solutions to PDEs

We introduce a homotopy of rearrangements called continuous Steiner symmetrization. The tool is used to prove that non-negative solutions to some degenerate elliptic boundary value problems are symmetric. These symmetry results go beyond the classical ones obtained via the textbook Moving Plane Method.

Most of the material of this lecture can be found in the following articles. They also contain further references.

F.B., *Radial symmetry for nonnegative solutions of semilinear elliptic equations involving the p-Laplacian.* Progress in PDE, Vol. 1 (Pont-á-Mousson, 1997), Pitman Research Notes **383** (1998), 46–57.

F.B., *Continuous rearrangement and symmetry of solutions of elliptic problems.* Proc. Indian Acad. Sci. Math. Sci. **110** (2000), no.2, 157–204.

F.B., *Symmetry for a general class of overdetermined elliptic problems.* NoDEA **23** (2016), no.3, Art. 36, 16 pp.

3.1. Motivation

Consider the following variational problem:

$$(\mathbf{P}) \qquad J(v) := \int_{B_R} \left(\frac{1}{p} |\nabla v|^p - F(v) \right) dx \longrightarrow \text{Stat.!}, \\ v \in W_0^{1,p}(B_R),$$

where $p > 1$, $R > 0$ and $F \in C^1$ and satisfies some further conditions.

The critical points of **(P)** satisfy the Euler equation

$$(*) \qquad -\Delta_p u \equiv -\nabla \left(|\nabla u|^{p-2} \nabla u \right) = f(u) \quad \text{in } B_R, \\ u = 0 \quad \text{on } \partial B_R,$$

where $f = F'$. Using Schwarz symmetrization one can show that problem **(P)** has a non-negative radially symmetric (global) minimizer.

Questions :

Are all non-negative global minimizers radially symmetric?

What about local minimizers?

Are nonnegative solutions of (*) radially symmetric?

Here are some results.

1. If u is a non-negative global minimizer of **(P)**, then we must have

$$\|\nabla u\|_p = \|\nabla u^\star\|_p.$$

This implies that the super-level sets $\{u > t\}$ are balls for a.e. $t \in [0, \sup u)$ - although they are not necessarily concentric. See Brothers/Ziemer, 1989.

Next let u be a positive solution of (*).

2. Gidas, Ni, Nirenberg, 1979:
Let $p = 2$, $u \in C^2(\overline{B})$ and $f \in C^{0,1}$. Then u is radially symmetric and radially decreasing, that is, there is a function $U \in C^1[0, R]$, such that

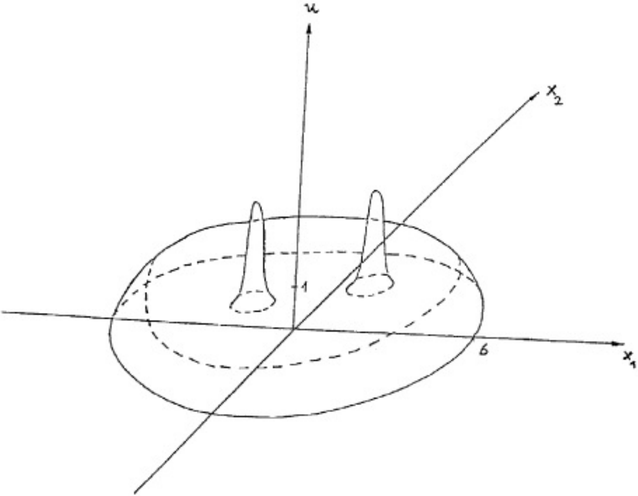
$$u(x) = U(|x|), \quad U'(r) < 0 \text{ for } 0 < r \le R. \qquad (**)$$

3. Damascelli, Pacella, 1998:
Let $1 < p < 2$, $u \in C^1(\overline{B_R})$ and $f \in C^{0,1}$. Then (**) holds.

There are further symmetry results, also for problems in $\mathbb{R}^N$, by Damascelli, Pacella, Sciunzi, Ramaswamy, Serrin, H. Zou and others. The tool in all these works is the **Moving Plane Method (MPM)**.

In the case $p > 2$ the (MPM) works only under additional assumptions on the solution.
Moreover, smoothness of f is sometimes insufficient to ensure radiality of solutions, as the next example shows.

Example 3.1.



Let $p > 1$ and $s > \frac{p}{p-1}$, and define

$$w(r) := \begin{cases} (1-r^2)^s & \text{if } r \le 1 \\ 0 & \text{if } r > 1 \end{cases},$$

$$v(r) := \begin{cases} 1 & \text{if } r < 5 \\ 1 - \left(\frac{r^2-25}{11}\right)^s & \text{if } 5 \le r \le 6 \end{cases}.$$

Choose $x^1, x^2 \in B_4$ with $|x^1 - x^2| \ge 2$, and set

$$u(x) := v(|x|) + w(|x - x^1|) + w(|x - x^2|) \quad \forall x \in B_6.$$

Then u is a solution of (*) with $R = 6$, where f is given by

$$f(u) := \begin{cases} \left(\frac{2s}{11}\right)^{p-1} \left[25 + 11\,(1-u)^{1/s}\right]^{(p/2)-1} \cdot (1-u)^{p-1-(p/s)} \cdot \\ \cdot \left[\frac{50}{11}(p-1)(s-1) + (2ps - 2s - p + N)(1-u)^{1/s}\right] \\ \qquad\qquad\qquad\qquad \text{if } \; 0 \le u \le 1 \\ (2s)^{p-1} \left[1 - (u-1)^{1/s}\right]^{(p/2)-1} \cdot (u-1)^{p-1-(p/s)} \cdot \\ \cdot \left[-2(s-1)(p-1) + (2ps - 2s - p + N)(u-1)^{1/s}\right] \\ \qquad\qquad\qquad\qquad \text{if } \; 1 \le u \le 2 \end{cases} .$$

Note that $f \in C^{\infty}((0,1) \cap (1,2)) \cap C[0,2)$.
Breaking of symmetry takes place at the level $u = 1$ where $f(1) = 0$.

Let us consider 3 cases:

(i) Let $p = 2$ (Laplacian case) and $s > 2$.
Then $f \in C^{1-(2/s)}[0,2]$, but $f \notin C^{0,1}[0,2]$.

(ii) If $p \in (1,2)$ and $s > \frac{p}{p-1}$, or if $p > 2$ and $s \in (\frac{p}{p-1}, \frac{p}{p-2})$, then $f \in C^{p-1-(p/s)}[0,2)$, but $f \notin C^{0,1}[0,2)$.

(iii) If $p > 2$ and $s \geq \frac{p}{p-2}$, then $f \in C^{0,1}[0,2]$.

3.2. Continuous Steiner symmetrization

In this section we describe a continuous homotopy that connects given sets and functions with their Steiner symmetrizations.

History: Polya/Szegö 1951, McNabb 1967, Brascamp/Lieb/Luttinger 1974, Kawohl 1989, Solynin 1990, F.B. 1995-.

3.2.1. Continuous rearrangement on $\mathbb{R}$:

First we rearrange an interval continuously. Let $I = [x-r, x+r]$, $x \in \mathbb{R}$, $r > 0$. We set

$$S_t(I) := [xe^{-t} - r, xe^{-t} + r], \quad t \in [0, +\infty]. \qquad (*)$$

Note that this implies $S_0(I) = I$, $S_\infty(I) =: I^* = [-r, r]$, $|S_t(I)|_1 = |I|_1$ and the following **semigroup property**,

$$S_{t+s}(I) = S_t(S_s(I)), \quad (s, t \in [0, +\infty].)$$

(We use the convention that $t + \infty = +\infty + t = +\infty$ for every $t \in [0, +\infty]$.)

Let $\mathcal{M}_1$ be the set of all measurable subsets of $\mathbb{R}$. We have the following

Theorem 3.2. *There exists a unique family of rearrangements*

$$S_t : \mathcal{M}_1 \to \mathcal{M}_1, \quad (t \in [0, +\infty]),$$

such that the following holds:

(i) *If* $I = [x - r, x + r]$ *is an interval, (*$x \in \mathbb{R}$*,* $r > 0$*), then* $S_t(I) = [xe^{-t} - r, xe^{-t} + r]$*;*

(ii) *If* $M \in \mathcal{M}_1$ *and* $s, t \in [0, +\infty]$*, then* $S_{t+s}(M) = S_t(S_s(M))$ *, (semigroup property).*

Note that from the above requirements it follows that $S_0(M) = M$ and

$$S_\infty(M) = M^* = \left[-\frac{1}{2}|M|_1, +\frac{1}{2}|M|_1\right] \ (= \text{ symmetrization of } M.)$$

Let us construct the family $\{S_t(M)\}_{t\geq 0}$ when M is a simple set, $M = \cup_{k=1}^{m} I_k$, with mutually disjoint intervals I_k. Then rule (*) gives

$$S_t(M) := \cup_{k=1}^{m} S_t(I_k),$$

as long as the intervals $S_t(I_k)$ do not yet overlap.
One (or more) of these intervals will collide at some value $t = t_0 > 0$. That is, $S_{t_0}(M)$ is a simple set $M' = \cup_{k=1}^{m'} I'_k$, with mutually disjoint intervals I'_k and $m' < m$.
Then, for $t > t_0$, we have $S_t(M) := S_{t-t_0}(M')$, by the semigroup property.
Continuing in this manner we arrive after a finite number of steps at some value $t' > 0$ at which $S_{t'}(M)$ is a single interval I''. Then, for $t \in (t', +\infty]$, the semigroup property and rule (*) give $S_t(M) := S_{t-t'}(I'')$.

If $M \in \mathcal{M}_1$, then we choose any sequence $\{M_k\}_{k=1}^{\infty}$ of simple sets that converges in measure to M. Then $S_t(M)$ is the limit of the sequence $\{S_t(M_k)\}_{k=1}^{\infty}$, for every $t \in [0, +\infty]$.

3.2.2. Continuous Steiner symmetrization of sets

For $M \in \mathcal{M}$ we write

$$M(x') := \{x_1 : (x_1, x') \in M\}, \quad (x' \in \mathbb{R}^{N-1}).$$

We define set transformations $S_t : \mathcal{M} \to \mathcal{M}$, ($t \in [0, +\infty]$), by

$$S_t(M) := \left\{(x_1, x') : x_1 \in S_t\left(M(x')\right), x' \in \mathbb{R}^{N-1}\right\}, \quad (M \in \mathcal{M}).$$

Then the mappings S_t are rearrangements, and the semigroup property and

$$S_\infty(M) = M^* (= \text{ Steiner symmetrization of } M)$$

hold for every $M \in \mathscr{M}$. The family $\{S_t\}_{t \geq 0}$ is called a **continuous Steiner symmetrization (CStS)**.

3.2.3. Continuous Steiner symmetrization of functions

We define mappings $S_t : \mathscr{S}_+ \to \mathscr{S}_+$ by

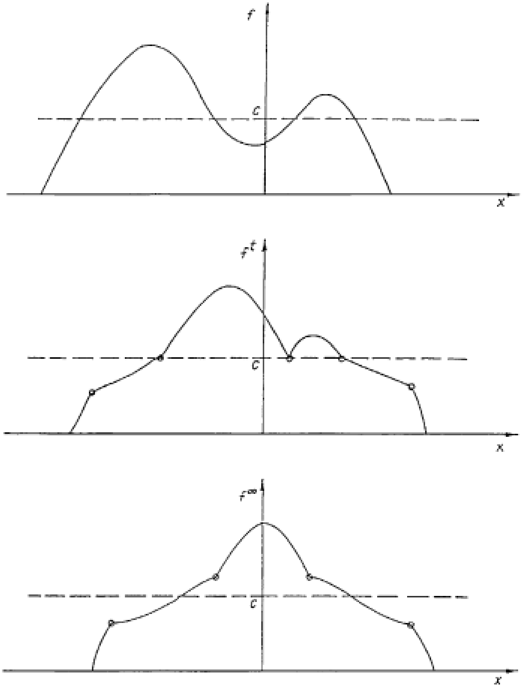
$$\begin{aligned} S_t(u)(x) \quad &:= \quad \int_0^\infty \chi\left(S_t\{u > s\}\right)(x)\, ds, \\ &\qquad (x \in \mathbb{R}^N,\ t \in [0, +\infty],\ u \in \mathscr{S}_+). \end{aligned}$$

Then the S_t's are rearrangements, $S_0(u) = u$, and the semigroup property and

$$S_\infty(u) = u^*(= \text{ Steiner symmetrization of } u),$$

hold for every $u \in \mathscr{S}_+$.

From now on we will use the shorthands $S_t(u) =: u^t$ and $S_t(M) =: M^t$ for the CStS of sets and functions.



f
c
x
f^t
c
x
f^∞
c
x

3.3. Properties of CStS

1. Despite of the last picture, CStS 'improves' the regularity of functions:

If u is Lipschitz continuous with Lipschitz constant L, then so is u^t, with Lipschitz constant less than or equal to L, for every $t \in [0, +\infty]$.

There also holds **Polya-Szegö's principle**:

If $G : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is convex, with $G(0) = 0$, and if u is Lipschitz continuous with compact support, then

$$\int_{\mathbb{R}^N} G(|\nabla u|)\, dx \geq \int_{\mathbb{R}^N} G(|\nabla u^t|)\, dx, \qquad (t \in [0, +\infty]).$$

Moreover, if $u \in W_+^{1,p}(\mathbb{R}^N)$, ($1 \leq p < +\infty$), then so is u^t and

$$\|\nabla u^t\|_p \leq \|\nabla u\|_p, \quad (t \in [0, +\infty)).$$

2. The mapping $t \longmapsto u^t$ is continuous in $L^p_+(\mathbb{R}^N)$ and continuous from the right in $W^{1,p}_+(\mathbb{R}^N)$, $(1 \leq p < +\infty)$. Moreover, if $u \in C^{0,1}_{0+}(B_R)$, with Lipschitz constant L, then

$$|u^t(x) - u(x)| \leq RLt, \qquad (x \in B_R,\ t \in (0, +\infty]).$$

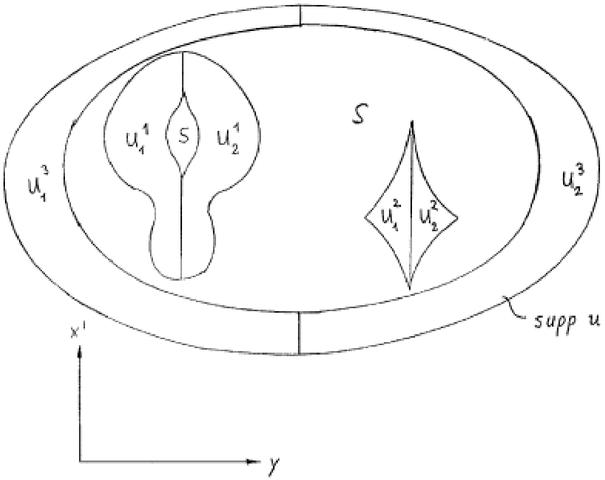
3.4. Local symmetry

In this section we discuss certain partial symmetries of smooth functions.

Henceforth we will write $\nabla' = (\partial/\partial x_2, \ldots, \partial/\partial x_N)$.

Let Ω be a domain in $\mathbb{R}^N$ with $\Omega = \Omega^*$ and $u \in C^1_+(\overline{\Omega})$ with $u = 0$ on $\partial\Omega$. We will say that u is **locally symmetric in x_1-direction**, if it has the following property:

$(\mathbf{S})_\mathbf{1}$ If (x^-, x^+) is a pair of points in Ω with
$x^- = (x_1^-, x'), x^+ = (x_1^+, x')$, $x_1^- < x_1^+$ and
$u(x^-) = u(x^+) < u(y, x')$ for all $y \in (x_1^-, x_1^+)$,
then:
$u_{x_1}(x^+) = -u_{x_1}(x^-)$ and
$\nabla' u(x^+) = \nabla' u(x^-)$.



S
U_1^1
S
U_2^1
U_1^3
U_2^3
U_1^2
U_2^2
supp u
x^1
y

We will say that u is **locally symmetric in every direction**, if it has the symmetry property $(\mathbf{S})_1$ w.r.t. every rotated coordinate system.

One can show the following

Theorem 3.3. *Let u be locally symmetric in every direction. Then u has the following symmetry property:*

$(\mathbf{S})$ $\quad \Omega = \cup_{k=1}^{m} A_k \cup S$,

where the A_k's and S are mutually disjoint,

$A_k = B_{R_k}(z_k) \setminus \overline{B_{r_k}(z_k)},\ z_k \in \Omega,\ 0 \le r_k < R_k,$

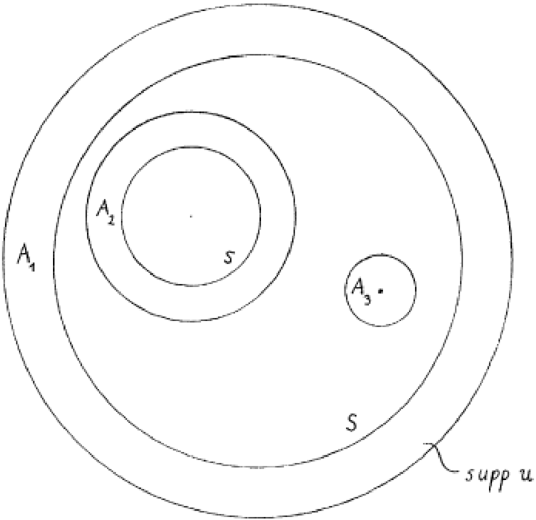
$u(x) = U_k(|x - z_k|)\ \forall x \in A_k,$

for some function $U_k \in C^1[r_k, R_k]$,

$U_k'(r) < 0\ \forall r \in (r_k, R_k),$ *and*

$\nabla u = 0$ *on S.*

Note that the function u from our example 3.1. is locally symmetric in every direction.



A1
A2
S
A3
S
supp u

3.5. Identification of symmetry

An important feature of the CStS is that a quantitative version of Polya-Szegö's principle can help to identify local symmetry of smooth functions.

Theorem 3.4. *Let $G \in C^1[0, +\infty)$ be strictly convex with $G(0) = 0 \leq G'(0)$, and let $u \in C^1(\overline{B_R})$, $u \geq 0$ in B_R and $u = 0$ on ∂B_R. Assume that*

$$\lim_{t \to 0} \frac{1}{t} \int_{B_R} \big(G(|\nabla u^t|) - G(|\nabla u|) \big)\, dx = 0. \qquad (*)$$

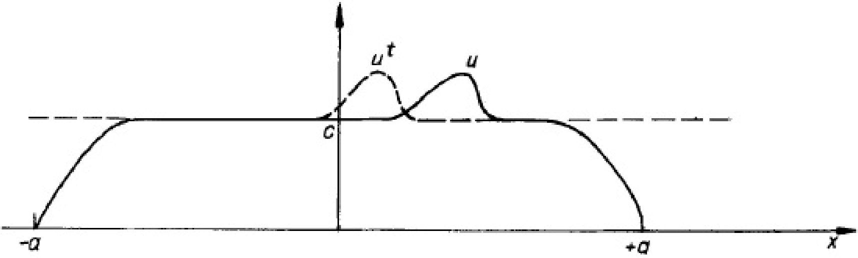
Then u is locally symmetric in every direction.

The idea of the proof of Theorem 3.4. is that the CStS 'improves locally' the symmetry of smooth functions.

More specifically, let (x^-, x^+) be a pair of points as in property $(\mathbf{S})_\mathbf{1}$ and such that $u_{x_1}(x^-) \neq 0$ and $u_{x_1}(x^+) \neq 0$. Defining a number $D(x^-, x^+)$ as

$$D(x^-, x^+) := \frac{d}{dt} \left\{ \frac{G(|\nabla u^t(x^-)|)}{|u^t_{x_1}(x^-)|} + \frac{G(|\nabla u^t(x^+)|)}{|u^t_{x_1}(x^+)|} \right\} \Bigg|_{t=0},$$

a calculation shows that $D(x^-, x^+) \leq 0$, with equality only if $u_{x_1}(x^-) = -u_{x_1}(x^+)$ and $\nabla' u(x^-) = \nabla' u(x^+)$.
On the other hand, if condition (*) is satisfied, then one can show that $D(x^-, x^+) = 0$ at every such pair of points (x^-, x^+).



u^t
u
c
-a
+a
x

3.6. Symmetry results

In the following, let $g \in C[0,+\infty) \cap C^1(0,+\infty)$ with $g(0) = 0$ and $g'(z) > 0$ for $z > 0$.
We will use the convention that $g(|y|)\frac{y}{|y|} = 0$ if $y = 0 \in \mathbb{R}^N$.
Our first symmetry result is

Theorem 3.5. *Let $f \in C[0,+\infty)$, and let $u \in C^1(\overline{B_R})$ be a non-negative distributional solution of*

$$(\mathbf{Q}) \begin{cases} -\nabla\left(g(|\nabla u|)|\nabla u|^{-1}\nabla u\right) = f(u) & \textit{in } B_R, \\ u = 0 & \textit{on } \partial B_R. \end{cases}$$

Then u is locally symmetric in every direction.

Sketch of the proof:
Define

$$\begin{aligned} F(v) &:= \int_0^v f(s)\,ds, \quad (v \geq 0), \text{ and} \\ G(z) &:= \int_0^z g(\zeta)\,d\zeta, \quad (z \geq 0). \end{aligned}$$

Then $F \in C^1[0,+\infty)$, $F(0) = 0$, and G is strictly convex, $G \in C^2[0,+\infty)$ with $G'(0) \geq 0 = G(0)$.
Multiplying the equation with $(u^t - u)$ we have

$$\begin{aligned} I &:= \int_{B_R} g(|\nabla u|)|\nabla u|^{-1}\nabla u \cdot \nabla(u^t - u)\,dx \\ &= \int_{B_R} f(u)(u^t - u)\,dx =: J. \end{aligned}$$

Further, by Cavalieri's principle and the continuity of f and the mapping $t \longmapsto u^t$ we obtain

$$\begin{aligned}
0 &= \int_{B_R} \big(F(u^t) - F(u)\big)\, dx \\
&= \int_{B_R} \int_0^1 f(u + \theta(u^t - u))(u^t - u)\, d\theta\, dx \\
&= \int_{B_R} \int_0^1 \big(f(u + \theta(u^t - u)) - f(u)\big)\, (u^t - u)\, d\theta\, dx \\
&\quad + \int_B f(u)(u^t - u)\, dx \\
&= o(t) + J. \qquad\qquad (*)
\end{aligned}$$

Finally, the convexity of G and Polya-Szegö's principle give

$$I \leq \int_{B_R} \big(G(|\nabla u^t|) - G(|\nabla u|) \big)\, dx \leq 0. \qquad (**)$$

Now (*) and (**) yield

$$\lim_{t \to 0} \frac{1}{t} \int_{B_R} \big(G(|\nabla u^t|) - G(|\nabla u|) \big)\, dx = 0,$$

and the assertion follows from Theorem 3.4. $\square$

Observe that if u is a solution of **(Q)** and is locally symmetric in every direction, then the Maximum Principle shows that $f(u)$ must be zero on the boundaries of the annuli A_k, ($k = 1, \ldots, m$), in the representation **(S)**. Using the Strong Maximum Principle for degenerate elliptic operators and assuming that f satisfies appropriate growth conditions near its zeros, one can then exclude the occurence of such annuli alltogether. In turn, this implies that the solution must be radial.
We give an example for the p-Laplace operator.

Theorem 3.6. *Let $f \in C[0, +\infty)$, $p \in (1, +\infty)$, and let $u \in C^1(\overline{B_R})$ be a non-negative distributional solution of*

$$(\mathbf{Q})_{\mathbf{p}} \begin{cases} -\Delta_p u \equiv -\nabla\left(|\nabla u|^{p-2}\nabla u\right) = f(u) \ \text{ in } B_R, \\ u = 0 \ \text{ on } \partial B_R. \end{cases}$$

Furthermore, assume that f satisfies the following growth condition near its zeros:

$$(***) \qquad \text{If } f(V) = 0, \text{ for some } V \geq 0, \text{ then there is some } C > 0 \text{ such that } |f(v)| \leq C|v - V|^{p-1} \ \ \forall v \in [0, +\infty).$$

Then u is radially symmetric and radially decreasing.

Remark 3.7.

1. The growth condition (***) is satisfied for instance, if $1 < p \leq 2$ and if $f \in C^{0,p-1}[0, +\infty)$.

2. Note that we did not require f to be Lipschitz continuous. Hence Theorem 3.6. is new even in the Laplacian case $p = 2$.

The CStS can also be used to show the symmetry of solutions to some overdetermined boundary value problems.

Theorem 3.8. *Let Ω be a bounded domain, and let $u \in C^1(\overline{\Omega})$ be a solution of the following overdetermined boundary value problem:*

$$(\mathbf{O}) \begin{cases} -\nabla\left(g(|\nabla u|)|\nabla u|^{-1}\nabla u\right) = f(u), \quad u > 0 \ \textit{ in } \Omega, \\ u = 0, \ |\nabla u| = \lambda \ \textit{ on } \partial\Omega, \end{cases}$$

where $\lambda > 0$, and $f \in C[0, +\infty)$. Then Ω is a ball and u is locally symmetric in every direction.

Finally we mention that the results of this section can be appropriately extended to nonlinearities of the form $f = f(|x|, u)$, where f is non-increasing in the first variable.

4. Weighted rearrangements and isoperimetric inequalities

A **manifold with density** is a manifold endowed with a positive function, the *density*, which weights both the volume and the perimeter. A natural issue then is to find **isoperimetric sets**, that is, sets that minimize the perimeter among all sets of a given fixed volume. The problem becomes even more challenging when perimeter and volume carry two different weights.

The related bibliography on this topic is very wide and, in this lecture, it is impossible to give an exhaustive account of it. We would like to refer the interested reader to the following sources and the references cited therein.

F. MORGAN, A. PRATELLI, *Existence of isoperimetric regions in* $\mathbb{R}^n$ *with density*, Annals of Global Analysis and Geometry **43** (2013), 331–365.

F. MORGAN, *Geometric Measure Theory. A Beginner's Guide.* 5th edition. Elsevier/Academic Press, Amsterdam, 2016. viii+263 pp.

F. MORGAN, *The log-convex density conjecture.* Frank Morgan's blog.

In this chapter we present some weighted isoperimetric inequalities, together with corresponding rearrangements. We will show how these results can be used to find best constants for imbedding inequalities between weighted function spaces, as well as sharp a-priori bounds for solutions to some boundary value problems involving weighted elliptic operators.

Some of this work has been done in collaboration with colleagues from the University of Napoli 'Federico II', see e.g.

A. ALVINO, F.B., F. CHIACCHIO, A. MERCALDO, M.R. POSTERARO, *Some isoperimetric inequalities on* $\mathbb{R}^N$ *with respect to weights* $|x|^\alpha$. J. Math. Anal. Appl. **451** (2017), no. 1, 280–318,

where also many further references can be found.

4.1. A weighted isoperimetric problem

Let f, g be two positive functions on $\mathbb{R}^N$ with g being locally integrable and f lower semi-continuous. For any measurable set $M \subset \mathbb{R}^N$ we define its weighted measure and perimeter by

$$\begin{aligned} |M|_g &:= \int_M g(x)\,dx, \quad \text{and} \\ P_f(M) &:= \int_{\partial M} f(x)\,\mathcal{H}_{N-1}(dx). \end{aligned}$$

We consider the isoperimetric problem

(P) Find $I_{f,g}(d) := \inf \Big\{ P_f(M) : \; M$ has locally finite perimeter and $|M|_g = d \Big\}, \quad (d > 0).$

Here and throughout, ∂M and $\mathcal{H}_{N-1}$ denote the essential boundary of M and $(N-1)$-dimensional Hausdorff-measure, respectively.

The function $I_{f,g}$ is also called the **isoperimetric profile** of problem **(P)**.
A set M with locally finite perimeter is called **isoperimetric** if $I_{f,g}(d) = P_f(M)$ and $|M|_g = d$.

Furthermore, we call a set $\Omega \subset \mathbb{R}^N$ a C^n-set, ($n \in \mathbb{N}$), if for every $x^0 \in \partial\Omega \cap \mathbb{R}^N$, there is a number $r > 0$ such that $B_r(x^0) \cap \Omega$ has exactly one connected component and $B_r(x^0) \cap \partial\Omega$ is the graph of a C^n-function on an open set in $\mathbb{R}^{N-1}$.

Consider a one-parameter family $\{\varphi_t\}_t$ of C^n-variations

$$\mathbb{R}^N \times (-\varepsilon, \varepsilon) \ni (x,t) \longmapsto \varphi(x,t) \equiv \varphi_t(x) \in \mathbb{R}^N,$$

with $\varphi(x,0) = x$, for any $x \in \mathbb{R}^N$.

The measure and perimeter functions of the variation are $m(t) := |\varphi_t(\Omega)|_g$ and $p(t) := P_f(\varphi_t(\Omega))$, respectively.
We say that the variation $\{\varphi_t\}_t$ of Ω is **measure-preserving** if $m(t)$ is constant for any small t.
We say that a C^1-set Ω is **stationary** if $p'(0) = 0$ for any measure-preserving C^1-variation.
Finally, we call a C^2-set Ω **stable** if it is stationary and $p''(0) \geq 0$ for any measure-preserving C^2-variation of Ω.

The classical isoperimetric inequality says that in the unweighted case, $f = g \equiv 1$, the infimum for problem **(P)** is given by any ball having Lebesgue measure d. This result has been known since ancient times, although proofs were found only during the last 150 years. See

G. Talenti, *The standard isoperimetric problem.* Handbook of convex geometry, Vol. A, B, 73–123, North-Holland, Amsterdam, 1993.

4.2. The case of radial weights with $f = g$

In this section we give a list of isoperimetric inequalities with equal radial weights for perimeter and volume, that is,

$$f(x) = g(x) = h(|x|), \quad (x \in \mathbb{R}^N).$$

For simplicity we will write I_f for the isoperimetric profile $I_{f,f}$.

1. The Gauss space is a probability space endowed with the weight function

$$h(|x|) = (2\pi)^{-N/2} e^{-|x|^2/2} =: \gamma_N(x).$$

Then all halfspaces are isoperimetric. This **isoperimetric inequality in Gauss space** is due to Sudakov/Tsirelson, 1974.

Note that the only finite radial measures with $f = g$ for which all half-spaces are stable are given by

$$h(r) = ce^{-dr^2}, \quad (r \geq 0),$$

for some numbers $c, d > 0$.

2. Let $c > 0$ and

$$h(r) = e^{cr^2}, \quad (r \geq 0; \text{'Anti-Gaussian measure'}).$$

Then all balls B_R, ($R > 0$), are isoperimetric, see C. Borell, 1984.

3. Now assume, more generally, that $h : [0, +\infty) \to (0, +\infty)$ is **log-convex** - that is,

$$h(r) = e^{k(r)}, \quad (r \geq 0), \text{ with } k \text{ being convex.}$$

Then all balls B_R, ($R > 0$), are stable.

G.R. Chambers, proved in 2016 (see arxiv.org:1311.4012) the famous **Log-convex-Theorem** which says that, if in addition $f \in C^1$ - equivalently, if $k'(0) = 0$ - then these balls are also isoperimetric.

4. Let

$$h(r) = r^p, \quad (r \geq 0),$$

for some $p > 0$. Then all balls B with $0 \in \partial B$ are isoperimetric, see Boyer/Brown/Chambers/Loving/ Tammen, 2016.

Remark 4.1. Interesting are also isoperimetric problems where the admissible sets M are required to be subsets of a **given, fixed domain** Ω. In such a case, only parts of ∂M that lie inside Ω count for the perimeter. The resulting problem is called a **relative isoperimetric problem**, or, in reference to a popular ancient legend, **Dido's problem**:

$(\mathbf{P})_{\boldsymbol{\Omega}}$ Find $\inf\Big\{P_f(M,\Omega):\ M\subset\Omega,\ M$ has locally finite perimeter and $|M|_g = d\Big\},\quad (d>0).$

Here, $P_f(M,\Omega)$ is the **perimeter relative to Ω** and is defined by

$$P_f(M,\Omega) := \int_{\partial M\cap\Omega} f(x)\,\mathcal{H}_{N-1}(dx).$$

Let us give one example in the case that Ω is a half-space,

$$\Omega = \mathbb{R}^N_+ := \{x \in \mathbb{R}^N : \, x_N > 0\}.$$

Assume that

$$f(x) = g(x) = e^{c|x|^2}(x_N)^\alpha, \quad (x \in \mathbb{R}^N_+),$$

where $c \geq 0$ and $\alpha \geq 0$. Then half-balls $B_R \cap \mathbb{R}^N_+$, ($R > 0$), are isoperimetric. See B./Chiacchio/Mercaldo, 2012, and also Maderna/Salsa, 1981, for the case $N = 2$, $c = 0$.

4.3. Isoperimetric inequality w.r.t. weights $|x|^\alpha$

In this section we discuss problem **(P)** when

$$f(x) = |x|^k \text{ and } g(x) = |x|^\ell,$$

for some numbers $k, \ell \in \mathbb{R}^N$.

4.3.1. Introduction to the problem

Caffarelli, Kohn and Nirenberg proved in 1984 the following:

$\exists C > 0$, such that $\forall u \in C_0^\infty(\mathbb{R}^N)$,

$$C\Big\||x|^b u\Big\|_q \leq \Big\||x|^a |\nabla u|\Big\|_p^\lambda \cdot \Big\||x|^c u\Big\|_r^{1-\lambda},$$

where: $$\begin{aligned} & p,q,r \geq 1, \quad 0 < \lambda \leq 1, \\ & \frac{1}{p}+\frac{a}{N},\ \frac{1}{q}+\frac{b}{N},\ \frac{1}{r}+\frac{c}{N} > 0, \\ & \frac{1}{q}+\frac{b}{N} = \lambda\left(\frac{1}{p}+\frac{a-1}{N}\right)+(1-\lambda)\left(\frac{1}{r}+\frac{c}{N}\right), \end{aligned}$$

$+$ some further conditions.

In the case $\lambda = 1$, we obtain the inequality

$$(\mathbf{CKN}) \qquad C\left\| |x|^b u \right\|_q \leq \left\| |x|^a |\nabla u| \right\|_p$$

where: $p, q \geq 1$,

$$\frac{1}{p} + \frac{a-1}{N} = \frac{1}{q} + \frac{b}{N} > 0$$

and $0 \leq a - b \leq 1$.

What is the best constant C in the inequality?

For $p = 1$ this becomes

$$C\left(\int_{\mathbb{R}^N} |x|^{bq}|u|^q \, dx\right)^{1/q} \leq \int_{\mathbb{R}^N} |x|^a |\nabla u| \, dx.$$

It can be shown that this is equivalent to the following weighted isoperimetric inequality:

$$C\left(\int_M |x|^{bq}\,dx\right)^{1/q} \le \int_{\partial M} |x|^a\,\mathcal{H}_{N-1}(dx)$$

for all sets $M \subset \mathbb{R}^N$ with locally finite perimeter, and we wish to find the largest possible constant C in this inequality.

Let us reformulate this last problem.
Let $N \in \mathbb{N}$, $k, \ell \in \mathbb{R}$, assume $\ell + N > 0$, and define

$$\mathcal{R}(M) \quad := \quad \frac{\displaystyle\int_{\partial M} |x|^k\,\mathcal{H}_{N-1}(dx)}{\left(\displaystyle\int_M |x|^\ell\,dx\right)^{\frac{k+N-1}{\ell+N}}},$$

where M has locally finite perimeter and $|M| > 0$.

Note that $\mathcal{R}(M) = \mathcal{R}(tM)$ for all $t > 0$.
Now our isoperimetric problem becomes

$$(\mathbf{R}) \qquad \text{Find } \; C := \inf \Big\{ \mathcal{R}(M) : \; |M| > 0, \\ M \text{ has locally finite perimeter} \Big\}.$$

4.3.2. Necessary conditions

Lemma 4.2. *A necessary condition for $C > 0$ is*

$$\ell \, \frac{N-1}{N} \leq k.$$

Idea of the proof: Choose $M = B_1(t, 0, \ldots, 0)$ and send $t \to +\infty$. $\square$

Next we obtain a necessary condition for radiality of the isoperimetric sets. We define

$$C^{rad} := \mathcal{R}(B_1).$$

Lemma 4.3. *If*

$$\ell > \ell^*(k) := k - 1 + \frac{N-1}{k+N-1}, \qquad (*)$$

then $C^{rad} > C$.

Idea of the proof: The ball B_1 is not stable for the isoperimetric problem **(R)** iff (*) holds. $\square$

4.3.3. Sufficient conditions for radiality

Theorem 4.4. *Let $k, \ell \in \mathbb{R}$ with $\ell + N > 0$. We have $C = C^{rad}$ in each one of the following cases:*

(i) $N \geq 1$ *and* $\ell + 1 \leq k$.

(ii) $N \geq 2$, $k \leq \ell + 1$ *and* $\ell \frac{N-1}{N} \leq k \leq 0$.

(iii) $N \geq 3$, $0 \leq k \leq \ell + 1$ *and*

$$\ell \leq \ell_1(k) := \frac{(k+N-1)^3}{(k+N-1)^2 - (N-1)^2/N} - N.$$

(iv) $N = 2$, $k \leq l + 1$, *and*

$$\ell \leq \ell_2(k) := \begin{cases} 0 & \text{if } 0 \leq k \leq \frac{1}{3} \\ \frac{(k+1)^3}{(k+1)^2 - \frac{16}{27}} - 2 & \text{if } k \geq \frac{1}{3} \end{cases}.$$

Ideas of the proof:
to **(i)**: See Howe, 2015. Use Gauss' Divergence Theorem.

to **(ii)**: See Chiba, Horiuchi (2015). Reduce to the case $k = 0$ by means of a transformation of variables, $x \longmapsto y := x|x|^{k/(N-1)}$.

to **(iii)** and **(iv)**: See Alvino, B., Chiacchio, Mercaldo, Posteraro, 2017. Use the same transformation of variables and some interpolation and rearrangement arguments.

Note that the statements **(iii)** and **(iv)** cover the important range

$$\ell = 0 \leq k \leq 1.$$

However, the numbers $l_1(k)$ and $l_2(k)$ in these cases are less than $l^*(k)$, which is the threshold number for radiality, see Lemma 4.3.

We have the following

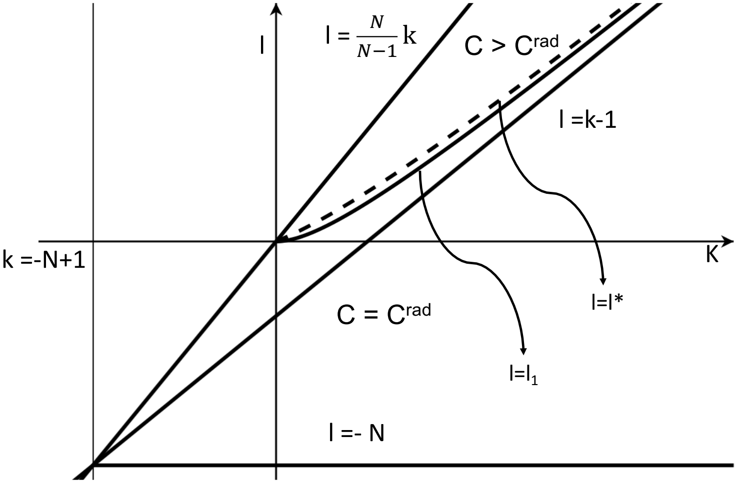
Conjecture 4.5. *Assume that* $\ell + N > 0$, $k \geq 0$ *and*

$$\ell \leq k - 1 + \frac{N-1}{k+N-1}.$$

Then $C^{rad} = C$.

di Giosia, Habib, Kenigsberg, Pittman, Zhu have published a proof of this conjecture on arXiv, in 2016. However, as they communicated in March 2019 - see arXiv:1610.05830 - that proof contained a mistake. They had *assumed* convexity of the generating curve of isoperimetric sets, a fact that still needs to be proved...

The following picture shows the established and the conjectured regions of radiality in the (k, ℓ)-plane, for $N \geq 3$.



l
l = $\frac{N}{N-1}$ k
C > C^{rad}
l =k-1
k =-N+1
K
l=l*
C = C^{rad}
l=l$_1$
l =- N

4.3.4. Weighted symmetrization

Let g be measurable and positive. If $|M|_g < \infty$ then let $M^\star$ be the ball B_R such that $|B_R|_g = |M|_g$. We define

$$\mathscr{S}_{+,g} := \{u : \mathbb{R}^N \to \mathbb{R}_0^+, \text{ measurable}, \ |\{u > t\}|_g < +\infty \ \forall t > 0\}.$$

If Ω is a domain in $\mathbb{R}^N$ and $p \in [1, +\infty)$, then let $L_g^p(\Omega)$ be the weighted Hölder space

$$\{u : \Omega \to \mathbb{R}, \text{ measurable, with } \|u\|_{p,g} < +\infty\},$$

where

$$\|u\|_{p,g} := \left(\int_\Omega |u|^p g \, dx \right)^{1/p}.$$

By $W_{0,g}^{1,p}(\Omega)$ we denote the completion of $C_0^\infty(\Omega)$ under the norm $\|\nabla u\|_{p,g}$.

If $u \in \mathscr{S}_{+,g}$, then we define its **g-symmetrization**, $u^\star$, by the layer-cake formula,

$$u^\star(x) := \int_0^\infty \chi\left(\{u > t\}^\star\right)(x)\,dt, \quad (x \in \mathbb{R}^N).$$

Then $u^\star$ is radial and radially non-increasing, and appropriate modifications of the properties of section 1.3 follow for this type of rearrangement, too. For instance, there hold versions of **Cavalieri's principle** and the **Hardy-Littlewood inequality**.

4.3.5. Polya-Szegö's principle

For some radial weights which are powers of the distance to the origin, we also have a Polya-Szegö principle:

Theorem 4.6. *Let* $k, \ell \in \mathbb{R}$*,* $\ell + N > 0$ *and* $p \in [1, +\infty)$*. Assume that* k, ℓ *satisfy one of the conditions* **(i)–(iv)** *of Theorem 4.4, and let*

$$\begin{aligned} f(x) &:= |x|^{kp+\ell(1-p)} \quad \textit{and} \\ g(x) &:= |x|^{\ell}, \quad (x \in \mathbb{R}^N). \end{aligned}$$

Then

$$\|\nabla u\|_{p,f} \geq \|\nabla u^\star\|_{p,f} \quad \forall u \in W^{1,p}_{0,f}(\mathbb{R}^N),$$

where $u^\star$ *is the* g*-symmetrization of* u*.*

Sketch of the proof in the special case $p = 2$, $\ell = 0$, $k \in [0, 1]$:

We employ the so-called **method of level sets** which has been developed by G.Talenti and other Italian Mathematicians since the seventieth of last century. Let

$$
\begin{aligned}
I &:= \int_{\mathbb{R}^N} |\nabla u|^2 |x|^{2k}\, dx, \quad (u \in C_{0+}^{\infty}(\mathbb{R}^N)), \quad \text{and} \\
I^\star &:= \text{same for } u^\star.
\end{aligned}
$$

By the **co-area formula**,

$$
\begin{aligned}
I &= \int_0^\infty \left(\int_{u=t} |\nabla u| |x|^{2k}\, \mathcal{H}_{N-1}(dx) \right) dt, \\
I^\star &= \text{same for } u^\star.
\end{aligned}
$$

Further, the Cauchy-Schwarz inequality gives

$$\left(\int_{u=t}|x|^k\right)^2 \leq \left(\int_{u=t}|\nabla u||x|^{2k}\right)\cdot\left(\int_{u=t}|\nabla u|^{-1}\right),$$

for a.e. $t \in (0, \sup u)$, where equality holds if u is replaced by $u^\star$. Hence,

$$\begin{aligned} I &\geq \int_0^\infty \left(\int_{u=t}|x|^k\right)^2 \cdot \left(\int_{u=t}|\nabla u|^{-1}\right)^{-1} dt, \\ I^\star &= \int_0^\infty \left(\int_{u^\star=t}|x|^k\right)^2 \cdot \left(\int_{u^\star=t}|\nabla u^\star|^{-1}\right)^{-1} dt. \end{aligned}$$

Moreover, since $|\{u > t\}| = |\{u^\star > t\}|$, **Fleming-Rishel's formula** gives

$$\begin{aligned}\int_{u=t} |\nabla u|^{-1} &= -\frac{d}{dt}\Big|\{u > t\}\Big| \\ &= -\frac{d}{dt}\Big|\{u^\star > t\}\Big| = \int_{u^\star=t} |\nabla u^\star|^{-1}.\end{aligned}$$

Finally, the weighted isoperimetric inequality tells us that

$$\int_{u=t} |x|^k \geq \int_{u^\star=t} |x|^k \quad \text{for a.e. } t > 0.$$

Putting together the above properties, we obtain $I \geq I^\star$.
$\square$

Remark 4.7.

The best constants in the Caffarelli-Kohn-Nirenberg inequalities **(CKN)** of section 4.3.1. can be obtained by solving an associated variational problem. For some range of the parameters a, b, p, q, a reduction to **radial** functions u is possible, using Theorem 4.6. In turn, this leads to an ODE problem that can be solved explicitly. See Alvino,B., Chiacchio, Mercaldo, Posteraro, 2017.

4.3.6. A-priori estimates for PDE

It is well-known that isoperimetric inequalities can be used to obtain sharp a-priori estimates for some elliptic and parabolic problems. The next Theorem is an unpublished result for an elliptic problem associated to a weighted p-Laplace operator. Its proof again uses the method of level sets.

Theorem 4.8. *Let Ω be a bounded domain in $\mathbb{R}^N$ and $p \in (1, +\infty)$. Further, let $k, \ell \in \mathbb{R}$ satisfy one of the conditions* **(i)**-**(iv)** *of Theorem 4.4.,*

$$\begin{aligned}
&\ell + N > p(\ell + 1 - k), \\
&g(x) = |x|^\ell, \\
&f(x) = |x|^{kp+\ell(1-p)}, \quad (x \in \mathbb{R}^N), \ \text{and} \\
&h \in L_{+,g}^{p(\ell+N)/(p[2\ell+1-k+N]-\ell-N)}(\Omega).
\end{aligned}$$

Finally let $u \in W^{1,p}_{0,f}(\Omega)$ and $v \in W^{1,p}_{0,f}(\Omega^\star)$ be distributional solutions of

$$-\nabla\left(|x|^{kp+\ell(1-p)}|\nabla u|^{p-2}\nabla u\right) = h|x|^\ell \quad \textit{in } \Omega, \textit{ and}$$
$$-\nabla\left(|x|^{kp+\ell(1-p)}|\nabla v|^{p-2}\nabla v\right) = h^\star|x|^\ell \quad \textit{in } \Omega^\star,$$

where $\Omega^\star$ and $h^\star$ are the g-symmetrizations of Ω and h, respectively. Then

$$u^\star \le v \quad \textit{in } \Omega^\star.$$

Sketch of the proof in the special case $p = 2$, $\ell = 0$, $k \in [0, 1]$*:*

We proceed similarly as in the proof of Theorem 4.6. Assume that $h \in C(\overline{\Omega})$ and $h > 0$. Then the sets $\{u > t\}$ and $\{v > s\}$ are C^1-sets for a.e. $t \in [0, \sup u)$, respectively $s \in [0, \sup v)$. Using the isoperimetric inequality and Fleming-Rishel's formula and using the fact that $u^\star$ is radial, we find that for a.e. $t \in [0, \sup u)$,

$$\begin{aligned} I(t) &:= \int_{\{u>t\}} h\,dx = -\int_{\{u>t\}} \nabla(|x|^{2k}\nabla u)\,dx \\ &= \int_{u=t} |x|^{2k}|\nabla u|\,\mathcal{H}_{N-1}(dx) \geq \frac{\left(\int_{u=t} |x|^k\,\mathcal{H}_{N-1}(dx)\right)^2}{\int_{u=t} |\nabla u|^{-1}\mathcal{H}_{N-1}(dx)} \\ &\geq \frac{\left(\int_{u^\star=t} |x|^k\,\mathcal{H}_{N-1}(dx)\right)^2}{\int_{u^\star=t} |\nabla u^\star|^{-1}\mathcal{H}_{N-1}(dx)} = \int_{u^\star=t} |x|^{2k}|\nabla u^\star|\,\mathcal{H}_{N-1}(dx). \end{aligned}$$

Similarly we have for a.e. $s \in [0, \sup v)$,

$$J(s) := \int_{\{v>s\}} h^\star \, dx = \int_{v=s} |x|^{2k} |\nabla v| \, \mathcal{H}_{N-1}(dx).$$

On the other hand, Hardy-Littlewood's inequality yields

$$\begin{aligned} I(t) &= \int_{\mathbb{R}^N} h\chi(\{u > t\}) \, dx \\ &\leq \int_{\mathbb{R}^N} h^\star \, \chi(\{u^\star > t\}) \, dx \\ &= \int_{u^\star > t} h^\star \, dx =: I^\star(t). \end{aligned}$$

Since $h > 0$, we have $|\nabla u^\star| \neq 0$ and $|\nabla v| \neq 0$ a.e. in $\Omega^\star =: B_R$. Hence we may find for every $r \in (0, R]$ values $s = s(r)$ and $t = t(r)$ such that $B_r = \{u^\star > t(r)\} = \{v > s(r)\}$.

The previous calulus then shows that

$$
\begin{aligned}
& \int_{\partial B_r} |x|^{2k} |\nabla u^\star| \, \mathcal{H}_{N-1}(dx) \leq I(t(r)) \\
\leq \quad & I^\star(t(r)) = J(s(r)) = \int_{\partial B_r} |x|^{2k} |\nabla v| \, \mathcal{H}_{N-1}(dx),
\end{aligned}
$$

which implies that $|\nabla u^\star| \leq |\nabla v|$ on ∂B_r. From this the assertion follows by integration from r to R. $\square$