Limit theorems for prices of options written on semi-Markov processes

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Pseudo-Differential Operators and Markov Processes
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Overview

Overview

A generic model for tick-by-tick financial data

$\{J_i\}_{i=1}^{\infty}$: a sequence of positive random variables with the meaning of inter-trade durations.

$\{Y_i\}_{i=1}^{\infty}$: a sequence of random variables with the meaning of tick-by-tick log-returns.

Define:

$$T_n = \sum_{i=1}^{n} J_i,$$

$$X_n = \sum_{i=1}^{n} Y_i,$$

$$N(t) = \max\{n : T_n \leq t\},$$

$$Y(t) = \sum_{i=1}^{N(t)} Y_i.$$

Scaling and limit theorems

Consider the rescaled variables $J_i^{(r)} = rJ_i$ and $Y_i^{(h)} = hY_i$. Define

$$T_n^{(r)} = \sum_{i=1}^{n} J_i^{(r)},$$

and

$$N^{(r)}(t) = \max\{n : T_n^{(r)} \leq t\}.$$

The general problem we are dealing with is the functional limit of

$$X^{(h,r)}(t) = \sum_{i=1}^{N^{(r)}(t)} Y_i^{(h)},$$

when $r \to 0$ and $h \to 0$, under an appropriate scaling. We will stay in the i.i.d. paradise!

Overview

The problem I

Let us consider the following problem. Assume that an option position is opened where the underlying asset is a share in a regulated equity market at an instant of time t after the beginning of continuous trading and with maturity T within the very same trading day before the end of continuous trading. What is the price of that option? In order to answer, we need to be more specific. The answer depends on several assumptions. In particular, it depends on the *kind of option*, on the *specific model for the price fluctuations* of the underlying asset and on the *option pricing method*.

The problem II

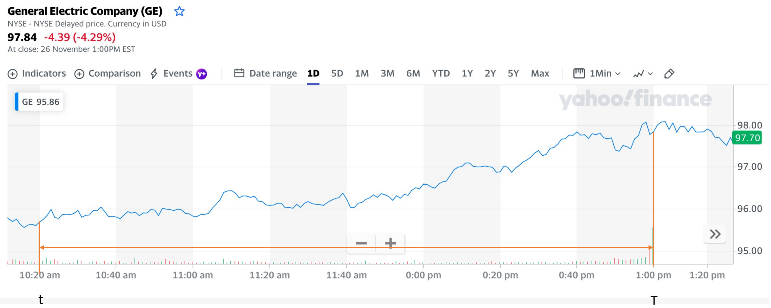

Figure: GE prices on 26 November 2021

The price model

We consider a price process in discrete time given by

$$S_n = S_0 \prod_{i=1}^{n} \mathrm{e}^{Y_i}, \qquad n \in \mathbb{N} \cup \{0\}, \tag{1}$$

where $\{Y_i\}_{i=1}^{\infty}$ is now a sequence of *i.i.d.* normal random variables with expected value 0 and variance σ^2 with the meaning of tick-by-tick log-returns and S_0 is the initial price. We use the convention $\prod_{i=1}^{0} \cdot = 1$. We now consider a sequence of positive *i.i.d.* random variables $\{J_i\}_{i=1}^{\infty}$ with the meaning of inter-trade durations, we introduce the corresponding counting renewal process $N(t) = \max\{n : \sum_{i=1}^{n} J_i \leq t\}$ and we define the price as

$$S(t) = S_0 \prod_{i=1}^{N(t)} \mathrm{e}^{Y_i}. \tag{2}$$

The option and the option price

For the sake of simplicity, we consider a *European plain-vanilla option* with payoff

$$\widetilde{C}(S(T)) = (S(T) - K)^+, \tag{3}$$

where T is the maturity and K is the strike price. We use an equivalent martingale measure given by

$$\mathcal{F}_T \ni A \mapsto \widetilde{\mathbb{P}}(A) := \mathbb{E}\mathbb{1}_A \prod_{i=1}^{N(T)} e^{-\frac{Y_i}{2} - \frac{\sigma^2}{8}} \tag{4}$$

under which the e_i^Y have expectation 1 and $S(t)$, $t \in [0, T]$ is a martingale with respect to its natural filtration. We can safely assume $r_F = 0$ for the risk-free interest rate. For the option price at time $t < T$, we use the conditional expectation with respect to the martingale measure defined in (4)

$$C(t) := \mathbb{E}_{\widetilde{\mathbb{P}}}[\widetilde{C}(S(T))|\mathcal{F}_t] = \int_0^\infty \widetilde{C}(u) dF_{\widetilde{S}(T)}(u), \tag{5}$$

where

$$\widetilde{S}(t) = S_0 \prod_{i=1}^{N(t)} \mathrm{e}^{(Y_i - \sigma^2/2)}. \tag{6}$$

Markovian case I

In this case, we have that $N(t)$ is the Poisson process, so that $F_{\widetilde{S}(T)}(u)$ is given by

$$F_{\widetilde{S}(T)}(u) = \mathrm{e}^{-\lambda(T-t)} \sum_{n=0}^{\infty} \frac{[\lambda(T-t)]^n}{n!} G_n(u) \tag{7}$$

with $G_n(u)$ given by the n-fold Mellin convolution of the distribution function of exactly n terms of (6):

$$G_n(u) = F^{\mathcal{M}_n}_{\widetilde{S}(T)}(u). \tag{8}$$

Equation (7) can be derived by probabilistic arguments. In particular, the price can move from S_0 to $\widetilde{S}(T)$ in $n \geq 0$ steps in a mutually exclusive and exhaustive way. By infinite additivity (7) follows.

Markovian case II

Denote

$$C(t,x) := \mathbb{E}\left[\left(\widetilde{S}(T)-K\right)^{+} \mid \widetilde{S}(t)=x\right] = \mathbb{E}^{x}\left[\left(\widetilde{S}^{\star}(T-t)-K\right)^{+}\right].$$

If we plug (7) into (5), we get by the monotone convergence theorem and Lemma 7.25 in Wheeden and Zygmund, 2015

$$C^{\star}(t,x) = \mathrm{e}^{-\lambda(T-t)} \sum_{n=0}^{\infty} \frac{[\lambda(T-t)]^{n}}{n!} \int_{0}^{\infty} \widetilde{C}(u)dG_{n}(u), \tag{9}$$

and thus

$$C^{\star}(t,x) = \mathrm{e}^{-\lambda(T-t)} \sum_{n=0}^{\infty} \frac{[\lambda(T-t)]^{n}}{n!} C_{n}(S_{0}=x, K, r_{F}=0, \sigma^{2}), \tag{10}$$

where we set

$$C_{n}(S_{0}=x, K, r_{F}=0, \sigma^{2}) = \int_{0}^{\infty} \widetilde{C}(u)dG_{n}(u).$$

Markovian case III

One can write

$$C_n(S_0 = x, K, r_F = 0, \sigma^2) = \int_0^\infty \widetilde{C}(u) dG_n(u) = \mathcal{N}(d_{1,n})x - \mathcal{N}(d_{2,n})K, \tag{11}$$

where

$$\mathcal{N}(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} dv\, \mathrm{e}^{-v^2/2} \tag{12}$$

is the standard normal cumulative distribution function and

$$d_{1,n} = \frac{\log(x/K) + n(\sigma^2/2)}{\sigma\sqrt{n}}, \tag{13}$$

$$d_{2,n} = d_{1,n} - \sigma\sqrt{n}. \tag{14}$$

Equation (10) coincides with the result of equation (16) in Merton's 1976 paper when the diffusion part is suppressed and the risk-free interest rate is $r_F = 0$.

Limit in the Markovian case I

In the limit of rapid jumps with vanishing length (at suitable velocity) the process defined in (6) converges in distribution to the process $S_0 \mathrm{e}^{B(t)}$ under appropriate scaling, where $B(t)$ is standard Brownian motion. In detail, the limit is as follows. Suppose that the parameter of the exponential r.v.'s is λ_m and that the variance of the i.i.d. r.v.'s Y_i is σ_m^2. Further suppose that, as $m \to \infty$, $\lambda_m \uparrow \infty$ (frequent jumps) and $\sigma_m^2 \downarrow 0$ (vanishing length) in such a way that $\lambda_m \sigma_m^2 \to 1$ as $m \to \infty$. One can see that under the measure

$$\mathcal{F}_T \ni A \mapsto \widetilde{\mathbb{P}}(A) := \mathbb{E}\mathbb{1} \prod_{i=1}^{N(T)} \mathrm{e}^{-\frac{Y_i}{2} - \frac{\sigma_m^2}{8}}, \tag{15}$$

where $\mathcal{F}_t$, $t \in [0, T]$, is the natural filtration of $S(t)$, $t \in [0, T]$, the process (2) is still convergent to $\mathrm{e}^{B(t)}$ in the sense that $\widetilde{\mathbb{P}}\left(S(t) \in \cdot\right) \to \widetilde{\mathbb{P}}_\infty\left(\mathrm{e}^{B(t)} \in \cdot\right)$ where $B(t)$, $t \in [0, T]$, is a standard Brownian motion on $(\Omega, \mathcal{A}, \mathbb{P})$, with natural filtration $\mathcal{F}_t^B$, and

$$\mathcal{F}_T^B \ni A \mapsto \widetilde{\mathbb{P}}_\infty(A) := \mathbb{E}\mathbb{1}_A \mathrm{e}^{-\frac{B(T)}{2} - \frac{T}{8}}, \tag{16}$$

is Girsanov's measure under which $B(t) + \frac{1}{2}t$ is a Brownian motion and $e^{B(t)}$, $t \in [0, T]$, is a martingale.

Limit in the Markovian case II

The option price (5) converges to the option price obtained using the Black and Scholes formula (with $r_F = 0$), $C_{\mathrm{BS}}(t)$, i.e.,

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\left[(S(T)-K)^+ \mid \mathcal{F}_t\right] = C(t) \to$$
$$C_{\mathrm{BS}}(t) = \mathbb{E}_{\widetilde{\mathbb{P}}_\infty}\left[\left(S_0 \mathrm{e}^{B(T)} - K\right)^+ \mid \mathcal{F}_t^B\right], \quad (17)$$

where $\mathcal{F}_t$ denotes the natural filtration of $S(t)$ while $\mathcal{F}_t^B$ the natural filtration of Brownian motion. The convergence in (17) is pointwise convergence for $S(t)$ and $B(t)$ fixed.

Limit in the Markovian case: A formal proposition

Proposition

It is true that, for any fixed $x > 0$ and $t \in [0, T]$ and $T > 0$, as $m \to \infty$,

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\left[(S(T) - K)^+ \mid S^\star(t) = x\right] \to \mathbb{E}_{\widetilde{\mathbb{P}}_\infty}\left[\left(S_0 \mathrm{e}^{B(T)} - K\right)^+ \mid S_0 \mathrm{e}^{B(t)} = x\right], \quad (18)$$

where $S(t)$ is defined in (2) with $N(t)$ is a Poisson process of rate λ_m, the Y_i are i.i.d. Gaussian random variables of mean 0 and variance σ_m^2, with $\lambda_m \uparrow \infty$ and $\sigma_m^2 \downarrow 0$ so that $\lambda_m \sigma_m^2 \to 1$ for $m \to \infty$. The measure $\widetilde{\mathbb{P}}_\infty$ is defined in (15).

Semi-Markov case I

Now, $N(t)$ in (2) is a generic counting renewal process. The r.v. J_1 has c.d.f. $F_{J_1}(u)$, p.d.f. $f_{J_1}(u)$ and c.c.d.f. $\overline{F}_{J_1}(u) = 1 - F_{J_1}(u)$. The process $S(t)$ is no longer Markovian, but belongs to the class of *semi-Markov processes* by construction. If we are sitting at a generic time t, the probability that this is a renewal epoch $T_n := \sum_{i=1}^{n} J_i$ is zero, in fact T_n, $n \in \mathbb{N}$, are absolutely continuous r.v.'s with zero measure on the real line. We assume that we know the past of the process and, in particular, the value of the previous renewal epoch $T_{N(t)}$ that we identify with the instant at which the previous transaction is recorded. Therefore, at time t, the *age* $\gamma(t) := t - T_{N(t)}$ is also known, whereas the *residual life-time* $\mathcal{J}(t) := T_{N(t)+1} - t$ is unknown. Formally, if $\mathcal{F}_t$, $t \in [0, T]$ denotes the natural filtration generated by the process $S(t)$, $t \in [0, T]$, the r.v. $\gamma(t)$ is measurable with respect to $\mathcal{F}_t$ while $\mathcal{J}(t)$ is not. Since the waiting times between transactions are not exponential r.v.'s the quantity $\gamma(t)$ (which is known at time t) is relevant in order to compute the probability of events in the future and therefore for the option pricing formula. In other words: the process $(S(t), \gamma(t))$ is a homogeneous Markov process, while $S(t)$ is not. The same holds for $\widetilde{S}(t)$ which is defined as in the Markov case.

Semi-Markov case II

We have

$$\mathbb{E}\left[\left(\widetilde{S}(T) - K\right)^+ \mid \mathcal{F}_t\right] = \mathbb{E}\left[\left(\widetilde{S}(T) - K\right)^+ \mid \widetilde{S}(t), \gamma(t)\right]$$
$$= \mathbb{E}^{\widetilde{S}(t), \gamma(t)}\left[\left(\widetilde{S}(T-t) - K\right)^+\right] \quad (19)$$

where we used the classical notation of Markov processes

$$\mathbb{P}^{x,s}(\cdot) := \mathbb{P}\left(\cdot \mid S_0 = x, \gamma(0) = s\right) \quad (20)$$

and $\mathbb{E}^{x,s}$ for the corresponding expectation. Let us denote by $\{N(T) - N(t) = k\}$ the event corresponding to the fact that there are k transactions between time t and the maturity T.

Semi-Markov case III

One can derive the option price

$$\widetilde{C}(x, s, T-t) := \mathbb{E}^{x,s}\left[\left(\widetilde{S}(T-t) - K\right)^{+}\right] \quad (21)$$

as

$$\widetilde{C}(x, s, T-t) = (x-K)^{+}\frac{\overline{F}_J(s+T-t)}{\overline{F}_J(s)} +$$
$$\sum_{n=1}^{\infty}\left[\int_0^{T-t}\mathbb{P}(N(T) - N((t+w)) = n-1)dF^{s}_{\mathcal{J}_t}(w)\right] C_n(\widetilde{S}(t) = x, K, r_F = 0, \sigma^2). \quad (22)$$

where $\mathcal{J}_t = T_{N(t)+1} - t$ is the residual lifetime and

$$F^{s}_{\mathcal{J}_t}(w) := \mathbb{P}\left(\mathcal{J}_t \leq w \mid \gamma(t) = s\right). \quad (23)$$

More explicitly, one has

$$F^{s}_{\mathcal{J}_t}(w) = \frac{F_J(s+w) - F_J(s)}{1 - F_J(s)}. \quad (24)$$

Semi-Markov case IV

The transition probabilities for (1) are for any $n \in \mathbb{N} \cup \{0\}$ and Borel set B,

$$h(x,B) = \mathbb{P}\left(S_{n+1} \in B \mid S_n = x\right) = \mathbb{P}\left(S_1 \in B \mid S_0 = x\right) = \mathbb{P}\left(xe^{Y_1} \in B\right) \tag{25}$$

Theorem

For the option price $\mathbb{E}_{\widetilde{\mathbb{P}}}\left((S(T)-K)^+ \mid \mathcal{F}_t\right)$ *it is true that, for* $z := (T-t)$,

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\left((S(T)-K)^+ \mid \mathcal{F}_t\right) = \mathbb{E}_{\widetilde{\mathbb{P}}}^{S(t),\gamma(t)}\left((S(z)-K)^+\right). \tag{26}$$

Define $C(x,s,z) := \mathbb{E}_{\widetilde{\mathbb{P}}}^{x,s}\left((S(z)-K)^+\right)$. *Then* $C(x,s,z)$ *satisfies the renewal equation*

$$C(x,s,z) = (x-K)^+ \frac{\overline{F}_J(z+s)}{\overline{F}_J(s)} + \int_0^z \int_0^\infty C(y,0,z-\tau) \frac{f_J(s+\tau)}{\overline{F}_J(s)} \widetilde{h}(x,dy)d\tau, \tag{27}$$

where $\widetilde{h}(x,dy)$ *is the martingale modification of* (25).

Semi-Markov case V

Proposition

The right-hand side (rhs) of (22) *is a solution to equation* (27).

Remark

Note that when the waiting times are exponential, the price process is a function of the Markov chain $S(t)$, *i.e., for any* $s, s' \geq 0$,

$$C(x,z) = \mathbb{E}^{x,s}_{\widetilde{P}} (S(z) - K)^+ = \mathbb{E}^{x,s'}_{\widetilde{P}} (S(z) - K)^+ . \tag{28}$$

Therefore one can write a differential (Kolmogorov's) equation

$$\frac{d}{dz} C(x,z) = \lambda \int_0^\infty (C(y,z) - C(x,z)) \, \widetilde{h}(x, dy), \qquad C(0,x) = (x-K)^+, \tag{29}$$

since the operator appearing at the rhs of (29) *is the generator of* $S(z)$. *Note that this equation reduces to equation (14) in Merton (1976) in the case in which the diffusive part of the price process is absent.*

Semi-Markov case V (continued)

Remark

Equation (27) can be compared with the unnumbered equation immediately after equation (2) in the paper by Montero (2008). Our equation coincides with Montero's one as our risk-free interest rate is zero and we integrate on prices and not on returns assuming that prices are positive random variables.

Limit in the semi-Markov case I

Assume the semi-Markov $S(t)$ converges to some limit process (in law, at least). What is the limit of $C(x,s,z)$? In other words, is there an analogue of $C_{\mathrm{BS}}(t)$? To answer this question, at least in some cases, let us introduce *subordinators*. A subordinator $\sigma(t), t \geq 0$, is a strictly increasing Lévy process whose Lévy Laplace exponent is a Bernstein function; in other words, one has

$$\mathbb{E}e^{-\phi\sigma(t)} = e^{-tf(\phi)}, \tag{30}$$

where

$$f(\phi) = b\phi + \int_0^\infty \left(1 - e^{-\phi s}\right) \nu(ds), \tag{31}$$

for a non-negative constant $b \geq 0$ and a Lévy measure $\nu(\cdot)$, with tail denoted by $\bar{\nu}(\cdot)$, supported on $(0, +\infty)$ and satisfying

$$\int_0^\infty (s \wedge 1)\, \nu(ds) < \infty. \tag{32}$$

Then we define

$$L(t) := \inf\{s \geq 0 : \sigma(s) > t\} \tag{33}$$

as the inverse process of σ.

Limit in the semi-Markov case II

If the process $\sigma(t)$ is independent from the Poisson process $N^\star(t)$ and it is strictly increasing, we have the following result.

Theorem (Theorem 4.1 in Meerschaert et al. (2011))

If either $b > 0$ or $\nu(0, +\infty) = +\infty$, the time-changed Poisson process $N^\star(L(t))$ is a renewal process whose i.i.d. waiting times J_1 satisfy

$$\overline{F}_{J_1}(y) = \mathbb{P}\left(J_1 > t\right) = \mathbb{E}e^{-\lambda L(t)}. \tag{34}$$

We shall assume from now on that $N(t) = N^\star\left(L(t)\right)$ for some $L(t)$ inverse of a strictly increasing subordinator, $b = 0$ and $\nu(0, \infty) = \infty$.

Limit in the semi-Markov case III

We consider here the limit of the option price $C(x, s, z)$, as $\lambda_m \uparrow \infty$, $\sigma_m^2 \downarrow 0$ with $\lambda_m \sigma_m^2 \to 1$. The parameters λ_m represent the rate of the Poisson process, $N^\star(t)$, which is time-changed. To highlight the dependence of $C(x, s, z)$ on the above parameters, we can use the notation $C_m(x, s, z)$.

The functional limit of the process $S(t)$ is the time-changed geometric (standard, i.e., $B(0) = 0$) Brownian motion $e^{B(L(t))}$ (see Meerschaert and Straka (2014)). This limiting process is a martingale under an appropriate probability measure. Moreover, it is semi-Markov in the sense that it has the Markov property at its renewal points. Given that the process has intervals in which it is constant, induced by the (independent) time-change, as in

$$e^{B(L(t))} = e^{B(w)}, \qquad \sigma(w^-) \le t < \sigma(w), \tag{35}$$

the renewal points can be represented through the undershooting and overshooting process of the subordinator, namely the processes $\sigma(L(t)^-)$, $t \in [0, T]$, and $\sigma(L(t))$, $t \in [0, T]$. In other words the age of the process $e^{B(L(t))}$ is

$$\gamma^\infty(t) := t - \sigma(L(t)^-) \tag{36}$$

while the remaining lifetime is

$$\mathcal{J}^\infty(t) := \sigma\left(L(t)\right) - t. \tag{37}$$

Limit in the semi-Markov case IV

At time t, we assume we know the full history of the process, which is contained in the filtration $\mathcal{G}_t := \mathcal{F}^{\infty}_{L(t)-}$, $t \in [0, T]$, where $\mathcal{F}^{\infty}_{w}$ is the filtration generated by the process $\left(S_0 e^{B_w}, \sigma_w\right)$. Note that the age $\gamma^{\infty}(t)$ is measurable with respect to $\mathcal{G}_t$, because the process $e^{B(L(t))}$ is constant between the renewal point $\sigma(L(t)^-)$ and t, while the remaining lifetime $\mathcal{J}^{\infty}(t)$ is not measurable with respect to $\mathcal{G}_t$. In what follows, we use the process

$$S_0 e^{B(L(t))}, \qquad t \in [0, T], \tag{38}$$

where S_0 is a L^1 positive r.v., and the option price

$$\mathbb{E}_{\widetilde{\mathbb{P}}_{\infty}} \left[\left(S_0 e^{B(L(T))} - K \right)^+ \mid \mathcal{G}_t \right], \qquad t \in [0, T], \tag{39}$$

where

$$\mathcal{G}_T \ni A \mapsto \widetilde{\mathbb{P}}_{\infty}(A) := \mathbb{E} \mathbb{1}_A e^{-\frac{B(L(T))}{2} - \frac{L(T)}{8}}, \tag{40}$$

is a probability measure under which $\sigma(t)$ is again a subordinator with Laplace exponent $f(\phi)$ and $e^{B(L(t))}$, $t \in [0, T]$ is a martingale.

Limit in the semi-Markov case V

We use the Markov property of the process $(\mathrm{e}^{B(L(t))}, \gamma^{\infty}(t))$ with respect to $\mathcal{G}_t$ to prove

Theorem

We have that, for $z = T - t$, $t \in [0, T]$,

$$\mathbb{E}_{\widetilde{\mathbb{P}}_\infty}\left[\left(S_0 e^{B(L(T))} - K\right)^+ \mid \mathcal{G}_t\right] = \mathbb{E}_{\widetilde{\mathbb{P}}_\infty}^{e^{B(L(t))}, \gamma^\infty(t)}\left(S_0 e^{B(L(z))} - K\right)^+. \tag{41}$$

Moreover, let $q'(x, y, z) := \mathbb{E}_{\widetilde{\mathbb{P}}_\infty}^{x,y}\left(S_0 e^{B(L(z))} - K\right)^+$. Then, for any $0 < y < z$,

$$q'(x, y, z) = (x - K)^+ \frac{\bar{\nu}(y+z)}{\bar{\nu}(y)} + \int_0^z q'(x, 0, z - \tau)\frac{\nu(y+\tau)}{\bar{\nu}(y)} d\tau. \tag{42}$$

Limit in the semi-Markov case: Heuristic derivation I

Remember that for $z = T - t$ from Theorem 2 we have

$$C_m(x,y,z) = (x-K)^+ \frac{\overline{F}_J(y+z)}{\overline{F}_J(y)} + \int_0^z \int_0^\infty C_m\left(y,0,z-\tau\right) \widetilde{h}^{\sigma^2_m}(x,dy) \frac{f_J(y+\tau)}{\overline{F}_J(y)} d\tau. \tag{43}$$

with the following limits

$$\frac{\overline{F}_J(y+z)}{\overline{F}_J(y)} \overset{m\to\infty}{\to} \frac{\bar{\nu}(y+z)}{\bar{\nu}(y)}, \tag{44}$$

$$C_m(y,0,z) = \int_0^\infty C^\star_m(y,0,s)\mathbb{P}\left(L(z) \in ds\right) \tag{45}$$

where

$$C^\star_m(y,0,z) \overset{m\to\infty}{\to} C_{\mathrm{BS}}(y,0,z), \tag{46}$$

and

$$\widetilde{h}^{\sigma^2_m}(x,dy) \overset{m\to\infty}{\to} \delta_x(dy). \tag{47}$$

Limit in the semi-Markov case: Heuristic derivation II

Putting these pieces together, it follows that, moving limits inside the integrals in (43), one gets

$$\lim_{m\to\infty} C_m(x,y,z) = (x-K)^+ \frac{\bar{\nu}(y+z)}{\bar{\nu}(y)} + \int_0^z C_\infty(x,0,z-\tau)\frac{\nu(y+d\tau)}{\bar{\nu}(y)}, \tag{48}$$

where

$$C_\infty(x,0,z) = \int_0^\infty \mathbb{E}^x \left(S_0 \mathrm{e}^{B(s)-s/2} - K\right)^+ \mathbb{P}\left(L(z)\in ds\right) = \mathbb{E}^{x,0}_{\widetilde{\mathbb{P}}}\left(S_0 e^{B(L(z))-L(z)/2} - K\right)^+. \tag{49}$$

The above argument is not rigorous and needs refinement.

Limit in the semi-Markov case: A theorem

The conjecture of (48) can be proven when the Bernstein function is $f(\phi) = \phi^\alpha$, $\alpha \in (0, 1)$, in other words in the case when the subordinator is α-stable.

Theorem

Suppose that $f(\phi) = \phi^\alpha$, $\alpha \in (0, 1)$. With $z = T - t$, let $q(x, y, z) := \lim_{m\to\infty} C_m(x, y, z)$. We have that $q(x, y, z)$ exists and sastisfies, for any $z > 0$ and $0 < y < z$, the renewal type equation

$$q(x, y, z) = (x - K)^+ \frac{y^\alpha}{(z + y)^\alpha} + \int_0^z q(w, 0, z - \tau) \frac{\alpha y^\alpha}{(y + \tau)^{\alpha+1}} d\tau \tag{50}$$

and further $q(x, y, z) = q'(x, y, z)$ for any $(x, y, z) \in \mathbb{R} \times (0, z) \times [0, T]$ and $T > 0$.

Overview

Summary and Outlook

- We derived a limit theorem for the option price in the semi-Markov case, when the time-change is the fractional Poisson process (the inverse stable subordinator).
- We derived fractional-type BS equations when the underlying price follows a time-changed geometric Brownian motion (not shown here).
- Numerical work forthcoming.
- Stay tuned for future developments.

Joint work with Bruno Toaldo

Enrico Scalas, Bruno Toaldo
Limit theorems for prices of options written on semi-Markov processes
To appear in: Theory of Probability and Mathematical Statistics
Preprint `https://arxiv.org/abs/2104.04817`.

Thank you for your attention!

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