Nonlinear Diffusion. Porous Medium and Fast Diffusion. From Analysis to Physics and Geometry

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  + M. Crandall, L. Evans, A. Friedman, C. Kenig,...

I. Diffusion

Populations diffuse, substances (like particles in a solvent) diffuse, heat propagates, electrons and ions diffuse, the momentum of a viscous (Newtonian) fluid diffuses (linearly), there is diffusion in the markets, ...

- *what is diffusion anyway?*
- *how to explain it with mathematics?*
- *A main question is: how much of it can be explained with linear models, how much is essentially nonlinear?*
- *The stationary states of diffusion belong to an important world, elliptic equations. Elliptic equations, linear and nonlinear, have many relatives: diffusion, fluid mechanics, waves of all types, quantum mechanics, ...*

The heat equation origins

- We begin our presentation with the Heat Equation $\boxed{u_t = \Delta u}$ and the analysis proposed by Fourier, 1807, 1822 (Fourier decomposition, spectrum). The mathematical models of heat propagation and diffusion have made great progress both in theory and application. They have had a strong influence on the 5 areas of Mathematics already mentioned.

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- The heat flow analysis is based on two main techniques: integral representation (convolution with a Gaussian kernel) and mode separation:

  $$u(x,t) = \sum T_i(t) X_i(x)$$

  where the $X_i(x)$ form the spectral sequence

  $$-\Delta X_i = \lambda_i \, X_i.$$

  This is the famous linear eigenvalue problem

Linear heat flows

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  (i) Fourier analysis decomposition of functions (and set theory),
  (ii) development of other linear equations
  $\Longrightarrow$ Theory of Parabolic Equations

$$\boxed{u_t = \sum a_{ij}\partial_i\partial_j u + \sum b_i\partial_i u + cu + f}$$

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- Main inventions in Parabolic Theory:
  (1) a_{ij}, b_i, c, f regular $\Rightarrow$ Maximum Principles, Schauder estimates, Harnack inequalities; C^α spaces (Hölder); potential theory; generation of semigroups.
  (2) coefficients only continuous or bounded $\Rightarrow W^{2,p}$ estimates, Calderón-Zygmund theory, weak solutions; Sobolev spaces.

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- The probabilistic approach: Diffusion as an stochastic process: Bachelier, Einstein, Smoluchowski, Wiener, Levy, Ito,...

$$\boxed{dX = b\,dt + \sigma dW}$$

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  I will present an overview and recent results in the theory mathematically called Nonlinear Parabolic PDEs. General formula

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- Typical nonlinear diffusion: $u_t = \Delta u^m$

  Typical reaction diffusion: $u_t = \Delta u + u^p$

The Nonlinear Diffusion Models

- The Stefan Problem (Lamé and Clapeyron, 1833; Stefan 1880)

$$SE : \begin{cases} u_t = k_1 \Delta u & \text{for } u > 0, \\ u_t = k_2 \Delta u & \text{for } u < 0. \end{cases} \quad TC : \begin{cases} u = 0, \\ \mathbf{v} = L(k_1 \nabla u_1 - k_2 \nabla u_2). \end{cases}$$

Main feature: the free boundary or moving boundary where $u = 0$. TC= Transmission conditions at $u = 0$.

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- The Hele-Shaw cell (Hele-Shaw, 1898; Saffman-Taylor, 1958)

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- The p-Laplacian Equation, $u_t = \text{div}\,(|\nabla u|^{p-2} \nabla u)$.

The Reaction Diffusion Models

- The Standard Blow-Up model (Kaplan, 1963; Fujita, 1966)

$$\boxed{u_t = \Delta u + u^p}$$

Main feature: If $p > 1$ the norm $\|u(\cdot,t)\|_\infty$ of the solutions goes to infinity in finite time. Hint: Integrate $u_t = u^p$.

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- The geometrical models: the Ricci flow: $\partial_t g_{ij} = -R_{ij}$.

An opinion of John Nash, 1958:

The open problems in the area of nonlinear p.d.e. are very relevant to applied mathematics and science as a whole, perhaps more so that the open problems in any other area of mathematics, and the field seems poised for rapid development. It seems clear, however, that fresh methods must be employed...

Little is known about the existence, uniqueness and smoothness of solutions of the general equations of flow for a viscous, compressible, and heat conducting fluid...

“Continuity of solutions of elliptic and parabolic equations”,

paper published in Amer. J. Math, 80, no 4 (1958), 931-954

II. Porous Medium Diffusion

$$u_t = \Delta u^m = \nabla \cdot (c(u)\nabla u)$$

density-dependent diffusivity

$$c(u) = mu^{m-1} [= m|u|^{m-1}]$$

degenerates at $u = 0$ if $m > 1$

Applied motivation for the PME

- Flow of gas in a porous medium (Leibenzon, 1930; Muskat 1933) $m = 1 + \gamma \geq 2$

$$\begin{cases} \rho_t + \text{div}\,(\rho \mathbf{v}) = 0, \\ \mathbf{v} = -\frac{k}{\mu} \nabla p, \quad p = p(\rho). \end{cases}$$

Second line left is the Darcy law for flows in porous media (Darcy, 1856). *Porous media flows are potential flows due to averaging of Navier-Stokes on the pore scales.*

To the right, put $p = p_o\, \rho^\gamma$, with $\gamma = 1$ (isothermal), $\gamma > 1$ (adiabatic flow).

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- Underground water infiltration (Boussinesq, 1903) $m = 2$

Applied motivation II

- Plasma radiation $m \geq 4$ (Zeldovich-Raizer, < 1950)

  Experimental fact: diffusivity at high temperatures is not constant as in Fourier's law, due to radiation.

$$\frac{d}{dt}\int_{\Omega} c\rho T\,dx = \int_{\partial\Omega} k(T)\nabla T \cdot \mathbf{n} dS.$$

Put $k(T) = k_o T^n$, apply Gauss law and you get

$$c\rho\frac{\partial T}{\partial t} = \mathrm{div}(k(T)\nabla T) = c_1 \Delta T^{n+1}.$$

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- Many more (boundary layers, geometry).

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- No big problem when $m > 1$, $m \neq 2$. The pressure transformation gives:

$$\boxed{v_t = (m-1)v\Delta v + |\nabla v|^2}$$

where $\boxed{v = cu^{m-1}}$ is the pressure; normalization $c = m/(m-1)$.

This separates $m > 1$ PME - from - $m < 1$ FDE

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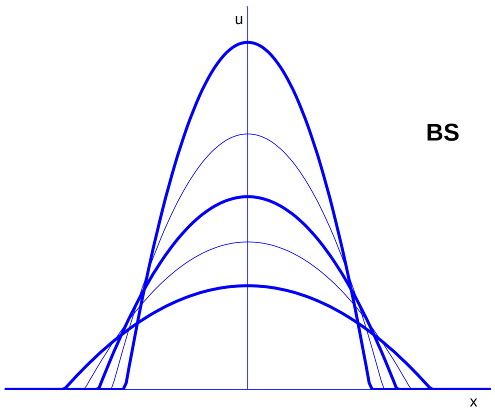

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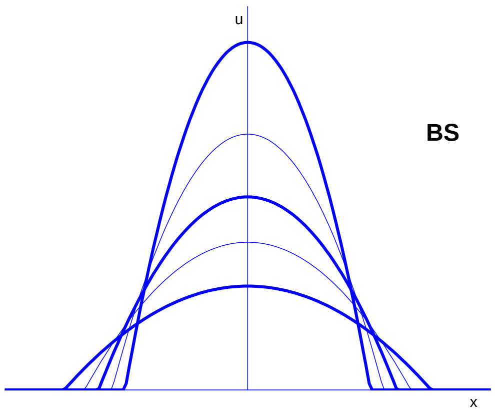

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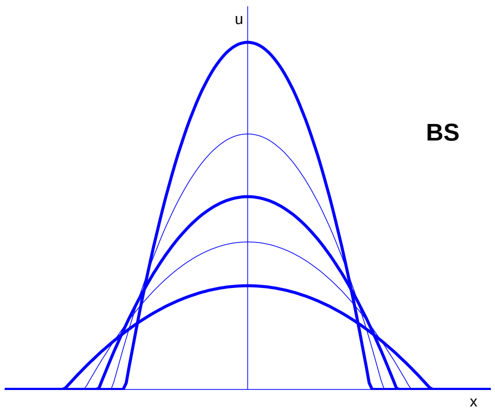

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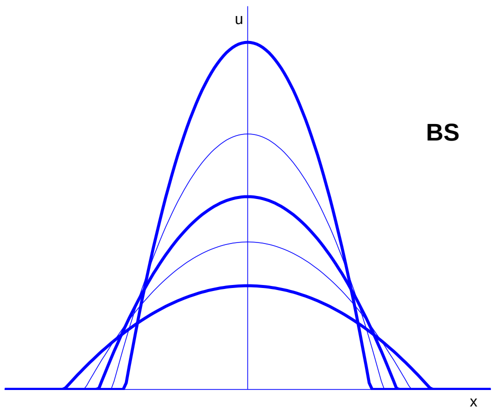

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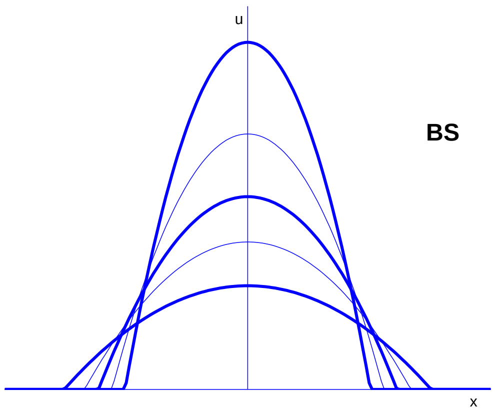
Barenblatt profiles (ZKB)

- These profiles are the alternative to the Gaussian profiles.
- They are source solutions. *Source means that* $u(x,t) \to M\,\delta(x)$ *as* $t \to 0$.
- Explicit formulas (1950): $\left(\alpha = \frac{n}{2+n(m-1)},\ \beta = \frac{1}{2+n(m-1)} < 1/2\right)$

$$\mathbf{B}(x,t;M) = t^{-\alpha}\mathbf{F}(x/t^{\beta}), \quad \mathbf{F}(\xi) = \left(C - k\xi^2\right)_+^{1/(m-1)}$$



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FDE profiles

- We again have explicit formulas for $1 > m > (n-2)/n$:

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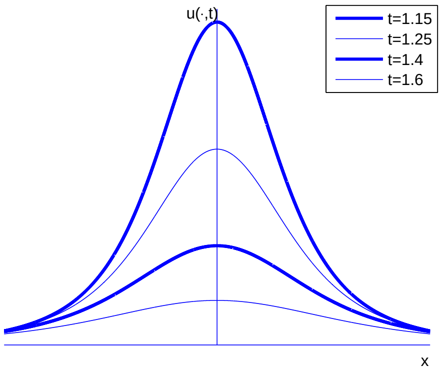

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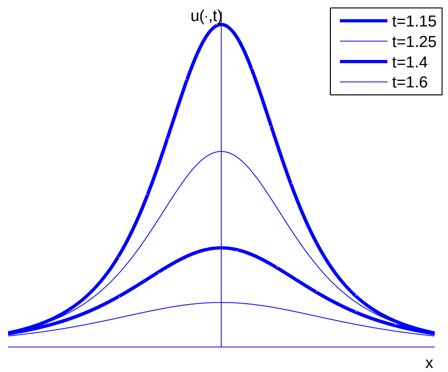

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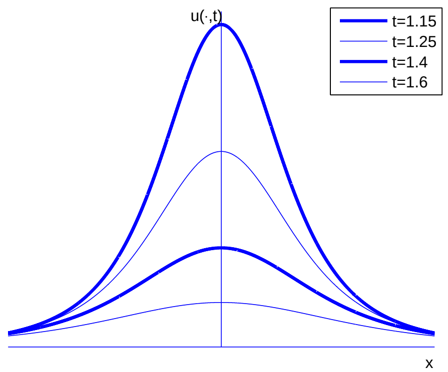

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- Limit solution: physical, but depends on the approximation (?).
- Weak solution Test against smooth functions and eliminate derivatives on the unknown function; it is the mainstream; (Oleinik, 1958)

$$\int\int (u\,\eta_t - \nabla u^m \cdot \nabla \eta)\,dxdt + \int u_0(x)\,\eta(x,0)\,dx = 0.$$

Very weak

$$\int\int (u\,\eta_t + u^m\,\Delta \eta)\,dxdt + \int u_0(x)\,\eta(x,0)\,dx = 0.$$

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Solutions of more complicated equations need new concepts:

- Viscosity solution Two ideas: (1) add artificial viscosity and pass to the limit; (2) viscosity concept of Crandall-Evans-Lions (1984); adapted to PME by Caffarelli-Vazquez (1999).

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$$\int u_t\, h(w)\,dx = \int \Delta w\, h(w)\,dx = -\int h'(w)|\nabla w|^2\,dx \le 0.$$

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- Contraction is also true in H^{-1} and in the Wasserstein W_2 space

The standard solutions

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- For every $u_0 \in L^1(\Omega)$, $u_0 \geq 0$, there exists a weak solution such that $u, u^m \in L^2_{x,t}$ and $\nabla u^m \in L^2_{x,t}$.

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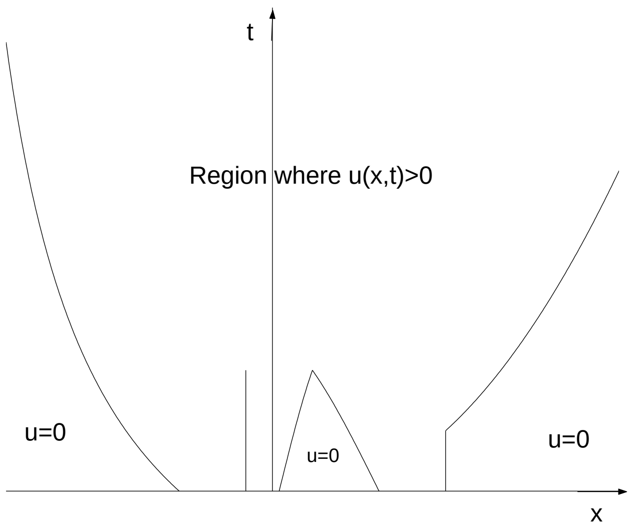
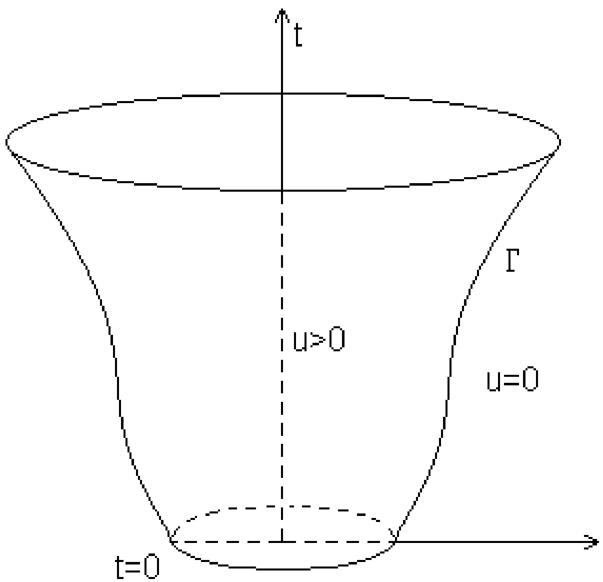
Free Boundaries in several dimensions





A complex free boundary in 1-D A regular free boundary in n-D

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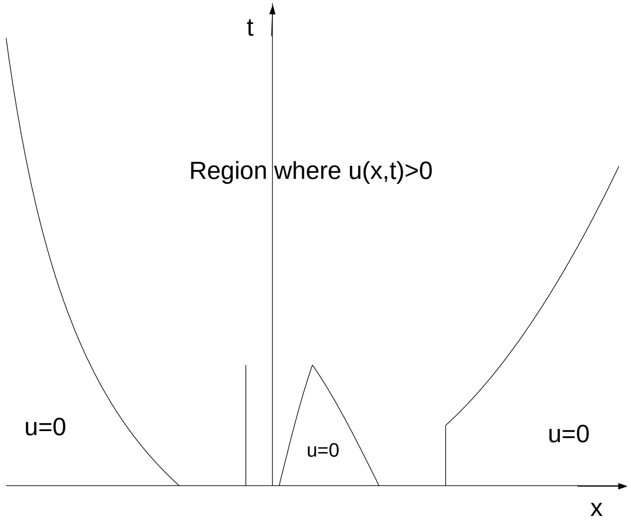

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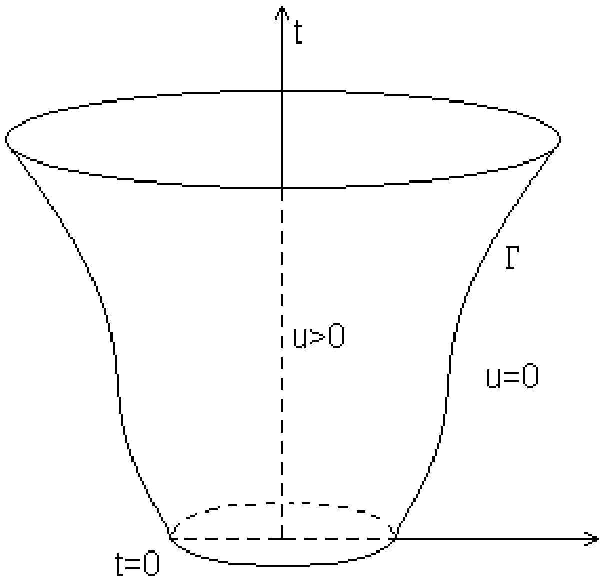

A regular free boundary in n-D

- (Caffarelli-Vazquez-Wolanski, 1987) If u_0 has compact support, then after some time T the interface and the solutions are $C^{1,\alpha}$.
- (Koch, thesis, 1997) If u_0 is transversal then FB is C^∞ after T. Pressure is “laterally" C^∞. *it is a broken profile always when it moves.*

Free Boundaries II. Holes

- A free boundary with a hole in 2D, 3D is the way of showing that focusing accelerates the viscous fluid so that the speed becomes infinite. This is blow-up for $\mathbf{v} \sim \nabla u^{m-1}$.
  The setup is a viscous fluid on a table occupying an annulus of radii r_1 and r_2. As time passes $r_2(t)$ grows and $r_1(t)$ goes to the origin. As $t \to T$, the time the hole disappears, the speed $r_1'(t) \to -\infty$.

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- There is a semi-explicit solution displaying that behaviour

$$u(x,t) = (T-t)^{\alpha} F(x(T-t)^{-\beta}).$$

The interface is then $r_1(t) = a(T-t)^{\beta}$. It is proved that $\beta < 1$. Aronson and Graveleau, 1993. Later Angenent, Aronson,..., Vazquez,

III. Asymptotics

Asymptotic behaviour

Nonlinear Central Limit Theorem

- Choice of domain: $I\!R^n$. Choice of data: $u_0(x) \in L^1(I\!R^n)$. We can write

$$u_t = \Delta(|u|^{m-1}u) + f$$

Let us put $f \in L^1_{x,t}$. Let $M = \int u_0(x)\,dx + \int\!\int f\,dxdt$.

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- Asymptotic Theorem [Kamin and Friedman, 1980; V. 2001] *Let $B(x,t;M)$ be the Barenblatt with the asymptotic mass M; u converges to B after renormalization*

$$\boxed{t^{\alpha}|u(x,t) - B(x,t)| \to 0}$$

For every $p \geq 1$ we have

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- Starting result by FK takes $u_0 \geq 0$, compact support and $f = 0$

Asymptotic behaviour. Picture

+ The rate cannot be improved without more information on u_0

+ m also less than 1 but supercritical ($\longrightarrow$ with even better convergence called relative error convergence)

$m < (n-2)/n$ has big surprises;

$m = 0 \ \to \ u_t = \Delta \log u \ \to$ Ricci flow with strange properties;

Proof works for p-Laplacian flow

Asymptotic behaviour. II

- The rates. Carrillo-Toscani 2000. Using entropy functional with entropy dissipation control you can prove decay rates when $\int u_0(x)|x|^2\,dx < \infty$ (finite variance):

$$\|u(t) - B(t)\|_1 = O(t^{-\delta}),$$

We would like to have $\delta = 1$. This problem is still open for $m > 2$. New results by JA Carrillo, McCann, Del Pino, Dolbeault, Vazquez et al. include $m < 1$.

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- We rescale the function as $u(x,t) = r(t)^n\,\rho(y\,r(t), s)$ where $r(t)$ is the Barenblatt radius at $t+1$, and “new time" is $s = \log(1+t)$. Equation becomes

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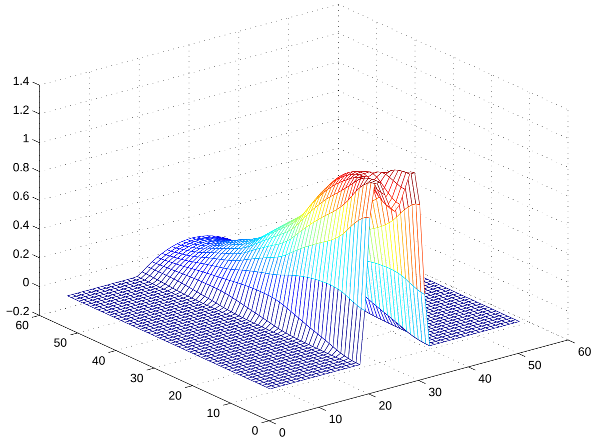
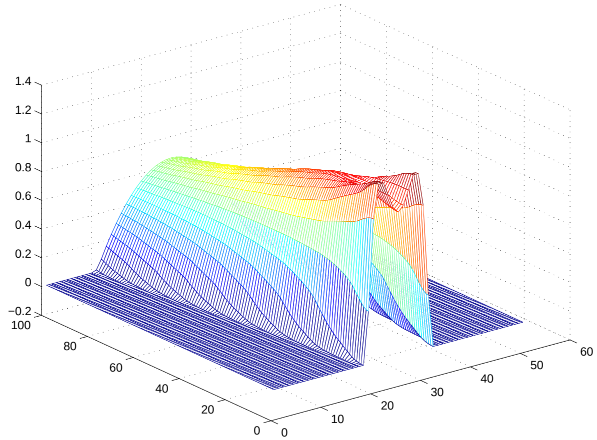
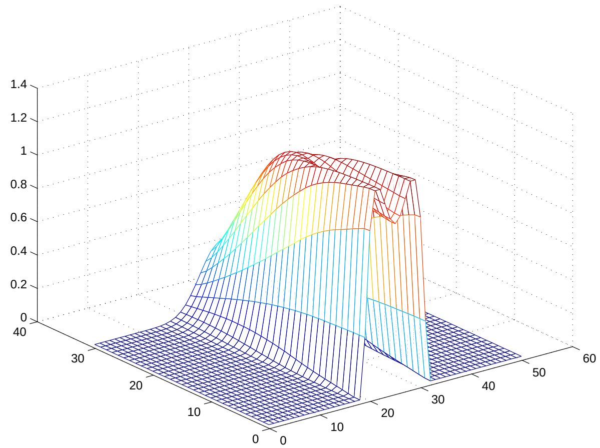
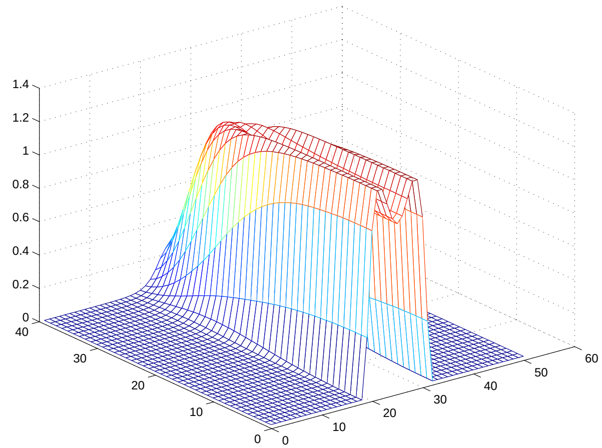



Eventual concavity for PME in 3D and in 1D

Eventual concavity for HE   Eventual concavity for FDE

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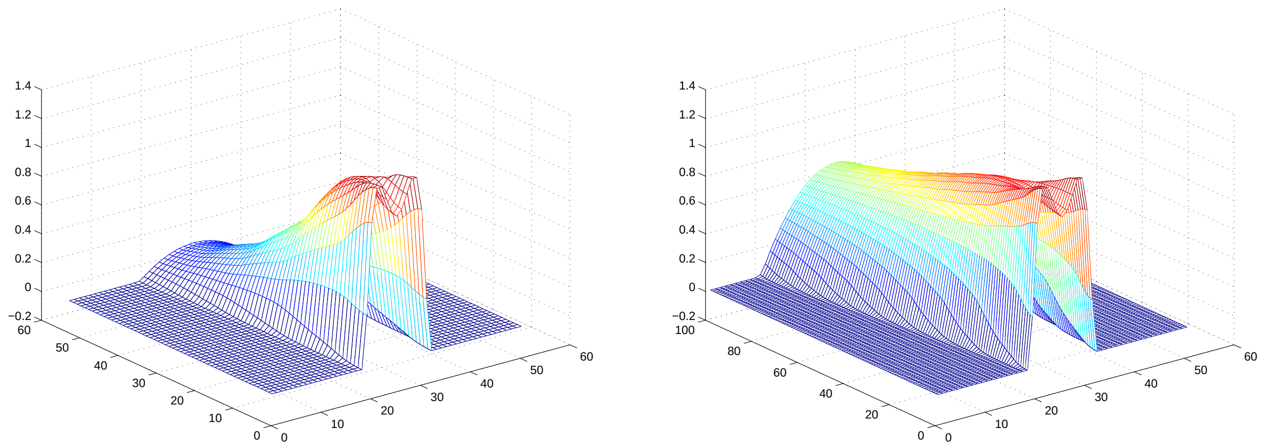

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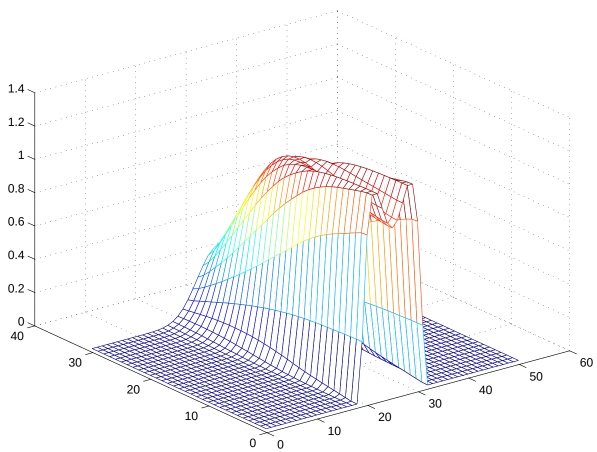

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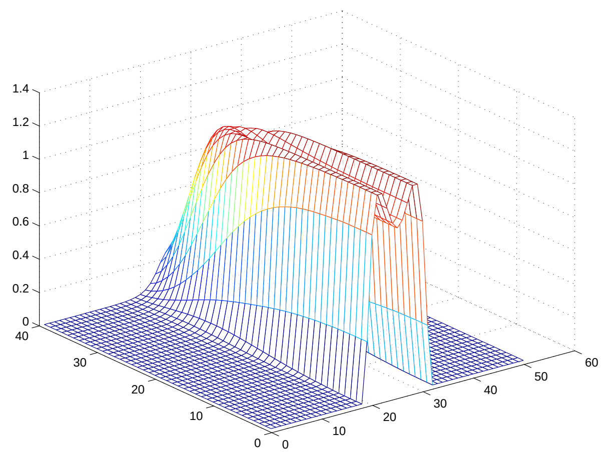

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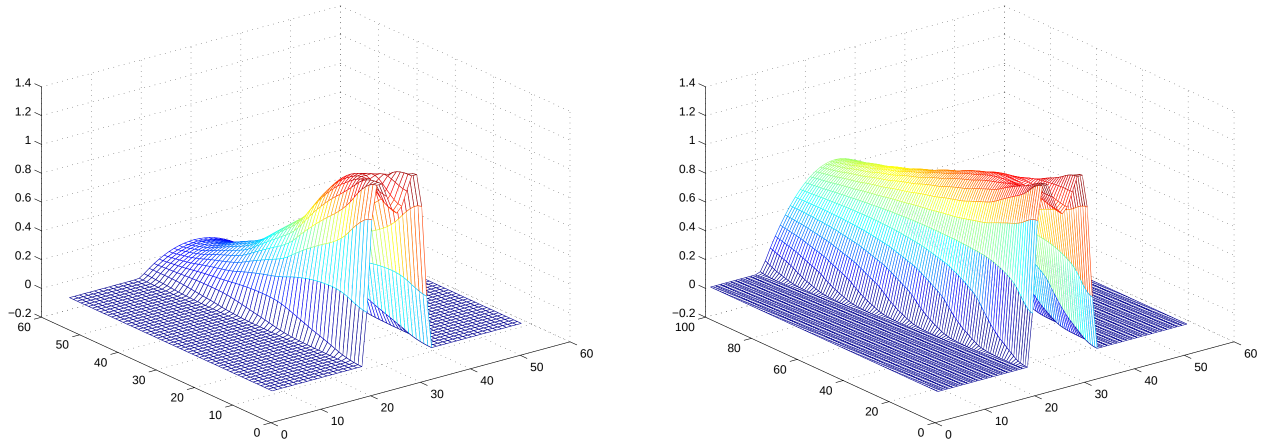

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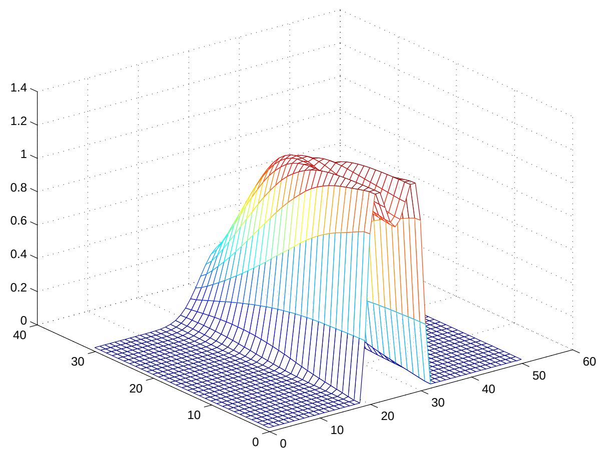

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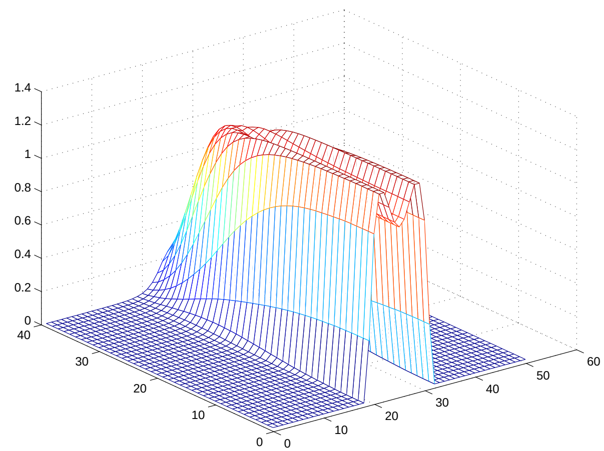

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References

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- J. L. Vázquez, "The Porous Medium Equation. Mathematical Theory", Oxford Univ. Press, 2006 in press. approx. 600 pages

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Probabilities. Wasserstein

- Definition of Wasserstein distance.

  Let $\mathcal{P}(I\!R^n)$ be the set of probability measures. Let $p > 0$. μ_1, μ_2 probability measures.

$$(d_p(\mu_1, \mu_2))^p = \inf_{\pi \in \Pi} \int_{I\!R^n \times I\!R^n} |x - y|^p \, d\pi(x, y),$$

  $\Pi = \Pi(\mu_1, \mu_2)$ is the set of all transport plans that move the measure μ_1 into μ_2. This is a distance.
  Technically, this means that π is a probability measure on the product space $I\!R^n \times I\!R^n$ that has marginals μ_1 and μ_2. It can be proved that we may use transport functions $y = T(x)$ instead of transport plans (this is Monge's version of the transportation problem).

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Wasserstein II

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- The metric d_∞ plays an important role in controlling the location of free boundaries. Definition $d_\infty(\mu_1, \mu_2) = \inf_{\pi \in \Pi} d_{\pi,\infty}(\mu_1, \mu_2)$, with

$$d_{\pi,\infty}(\mu_1, \mu_2) = \sup\{|x - y| : (x, y) \in \text{support}(\pi)\}.$$

In other words, $d_{\pi,\infty}(\mu_1, \mu_2)$ is the maximal distance incurred by the transport plan π, i.e., the supremum of the distances $|x - y|$ such that $\pi(A) > 0$ on all small neighbourhoods A of (x, y). We call this metric the maximal transport distance.

Wasserstein III

- The contraction properties in $n = 1$

Theorem (Vazquez, 1983, 2004) *Let μ_1 and μ_2 be finite nonnegative Radon measures on the line and assume that $\mu_1(\mathbf{R}) = \mu_2(\mathbf{R})$ and $d_\infty(\mu_1, \mu_2)$ is finite. Let $u_i(x,t)$ the continuous weak solution of the PME with initial data μ_i. Then, for every $t_2 > t_1 > 0$*

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  Theorem (Vazquez, 2004) *For the PME, contraction does not hold in d_∞ for any $n > 1$. It does not in d_p for $p \ge p(n) > 2$.*

New fields

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Geometrical applications: Yamabe flow, $m = (n-2)/n$. Extinction.

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- Nonlinear diffusion in image processing. Gradient dependent diffusion. Work on total variation models.

  Andreu, Caselles, Mazón, ...