THE CAUCHY PROBLEM FOR THIN FILM AND OTHER NONLINEAR PARABOLIC PDEs
(An Introduction to Higher-Order Diffusion Theory)

Summer School on Nonlinear Parabolic Equations and Applications

Math. Dept., Swansea Univ., 7-11 July 2008

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Asymptotic Theory for Parabolic PDEs: Rules

Main Goal:

not only Existence-Uniqueness (of key Importance!), but also to Describe the Actual Evolution Properties of the Solutions of the **Cauchy Problem** *(***Key** *for Many Applications)...* **.**

Asymptotic Theory for Parabolic PDEs: Rules

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General Rules for Discussions:

- ALL types of questions are encouraged! We have enough time, **5 hours!**, to discuss all we want.
- Stop the lecturer if he moves too fast, ANY TIME... .
- The audience is ALWAYS RIGHT!

Plan: Asymptotic Theory for Parabolic PDEs

Lecture 1

The **Cauchy Problem** for the Classic 1D **Heat Equation** (a canonical PDE of Math. Phys.)

$$u_t = u_{xx} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+,$$

with given bounded integrable initial data $u_0(x)$.

Back to C. Sturm (1836).

SHARP Asymptotic Theory:

(i) as $t \to +\infty$, large-time behaviour, and
(ii) blow-up behaviour, as $t \to T^- < \infty$.

Hermitian Spectral Theory of Self-Adjoint Operators (Sturm, 1836).

Lecture 2

The Bi-Harmonic Equation, the fourth-order parabolic equation:

$$u_t = -u_{xxxx} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+,$$

again, the Cauchy Problem with bounded data $u_0(x)$.

Twenty-First Century Theory ($2004 - 1836 = 168 \approx 170$**).**

SHARP Asymptotic Theory:

(i) as $t \to +\infty$, large-time behaviour, and
(ii) blow-up behaviour, as $t \to T^- < \infty$.

Hermitian Spectral Theory of **Non** Self-Adjoint Operators (2004).

Third-order PDEs: Similarities with Parabolic Ones

An Important Digress to *Third-Order Linear and Semilinear Dispersion Equations*, as other canonical PDEs:

$$u_t = u_{xxx} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+,$$

Tutorial, by Ray Fernandes, Bath

Third-order PDEs: Similarities with Parabolic Ones

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$$u_t = u_{xxx} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+,$$

Semilinear Equation with Absorption:

$$u_t = u_{xxx} - |u|^{p-1}u \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+,$$

again, the Cauchy Problem, where $p > 1$ is a constant.
(i) Self-Similar Solutions: Theory and
(ii) *MatLab* Experiments: How to Fight Strong Oscillatory Tails
(a Problem not Available for Parabolic PDEs)...

Lecture 3

The Fourth-Order Porous Medium Equation (the PME-4)

$$u_t = -(|u|^n u)_{xxxx} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+,$$

with given bounded integrable initial data $u_0(x)$,

where $n > 0$ is a fixed constant.

(i) Existence-Uniqueness Theory in Sobolev Spaces (1960s),

(ii) Nonlinear Eigenfunction Theory, Behaviour as $t \to +\infty$.

(iii) Homotopy Approach,

$$\boxed{n \to 0^+} \Longrightarrow \text{convergence to the bi-harm. eq.} \qquad (1)$$

(iv) Numerical Evidences by **MatLab**, as Unavoidable Tools of PDE Theory of the XXI Century... .

Lectures 4 and 5

The Fourth-Order Thin Film Equation (the TFE-4)

$$u_t = -(|u|^n u_{xxx})_x \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+,$$

where $u_0(x)$ is compactly supported and $n > 0$ is a constant.

(i) Self-Similar Solutions, Oscillatory Sign-Changing Behaviour, Nonlinear Eigenfunction Theory.

(ii) Finite Interfaces, Homoclinic Bifurcation Parameter

$$\boxed{n_h = 1.758665...\,.}$$

(iii) Existence-Uniqueness Concepts by Homotopy:

$$\boxed{n \to 0^+} \Longrightarrow \text{convergence to } u_t = -u_{xxxx}. \tag{2}$$

(iv) ALL Supported by Numerical Evidences by MatLab... .

Lecture 1: The Classic Heat Equation

The Cauchy problem for the heat equation

$$u_t = u_{xx} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+,$$

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Models various conductivity and viscosity phenomena, the most well-known canonical PDE.

The Fundamental (Similarity) Solution

The 1D Heat Equation

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The Fundamental Solution

$$b(x,t) = \frac{1}{\sqrt{t}} F(y), \quad y = \frac{x}{\sqrt{t}}.$$

$$\mathbf{B}F \equiv F'' + \frac{1}{2}(Fy)' = 0, \quad \int_{\mathbb{R}} F = 1.$$

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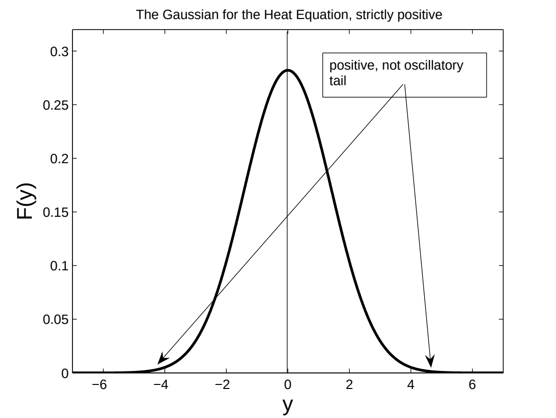
Hence, F is the Gaussian:

$$F(y) = \frac{1}{2\sqrt{\pi}} \mathrm{e}^{-y^2/4} > 0.$$

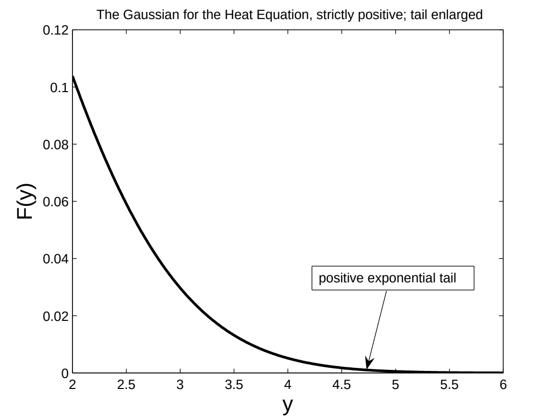
The Positive Gaussian for the HE

Rescaled Kernel of the Fundamental Solution to $u_t = u_{xx}$



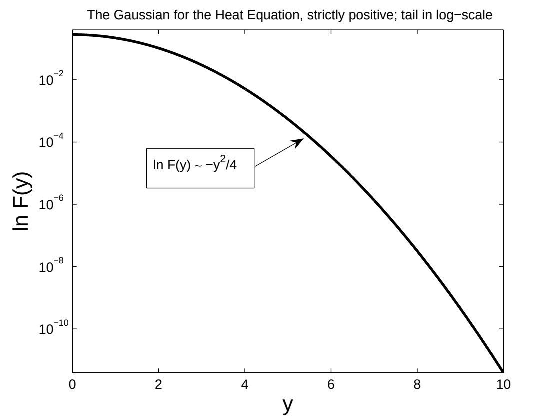
The Positive Gaussian for the HE: exp. tail enlarged

Rescaled Kernel of the Fundamental Solution to $u_t = u_{xx}$



The Positive Gaussian for the HE: tail in log-scale

Rescaled Kernel of the Fundamental Solution to $u_t = u_{xx}$



By Convolution Theorem for Fourier Transforms

For bounded L^1 data, $\exists$! solution

$$u(x,t) = b(t) * u_0 \equiv \frac{1}{2\sqrt{\pi}} \frac{1}{\sqrt{t}} \int_{\mathbb{R}} \mathrm{e}^{-(x-z)^2/4t}\, u_0(z)\, \mathrm{d}z.$$

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Tikhonov-Täklind Uniqueness Theorem (1937)

$\exists$! : $\quad |u(x,t)| \le C\mathrm{e}^{cx^2}, \quad$ Tikhonov,

Täklind: $\quad |u(x,t)| \le \mathrm{e}^{|x| h(|x|)}, \quad \int^{\infty} \frac{\mathrm{d}s}{h(s)} = \infty.$

By Convolution Theorem for Fourier Transforms

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Relation:

$h(s) = s \Longrightarrow$ Tikhonov, $\quad h(s) = s \ln s \ldots$.

Positivity of the Gaussian rescaled kernel:

Implies various key properties of the flow:

Maximum Principle,
Comparison Principle,
Barrier Techniques (key anal. tools),
symmetry of rescaled operators, etc.

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Positivity and Order-Preserving Properties:

are key for many nonlinear
second-order parabolic PDEs,
such as the PME, p-Laplacian and others.

Precise Asymptotic Behaviour as $t \to +\infty$

The Heat Equation

$$u_t = u_{xx}, \quad x \in \mathbb{R}, \quad t > 0.$$

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Rescaled Variables, $t \mapsto 1 + t$, Shifting

$$u(x,t) = \frac{1}{\sqrt{1+t}} v(y,\tau), \quad y = \frac{x}{\sqrt{1+t}}, \quad \tau = \ln(1+t)$$

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The Rescaled Equation

$$v_\tau = \mathbf{B}v \equiv v_{yy} + \tfrac{1}{2} y v_y + \tfrac{1}{2} v, \quad y \in \mathbb{R}, \; \tau > 0.$$

$\mathbf{B}$ is the Hermite Classic Self-Adjoint Operator

Exponential Weight

$$\mathbf{B}v = v_{yy} + \tfrac{1}{2}\,yv_y + \tfrac{1}{2}\,v \equiv \tfrac{1}{\rho}\,(\rho v_y)_y + \tfrac{1}{2}\,v,$$
$$\rho(y) = \mathrm{e}^{y^2/4} = \frac{1}{2\sqrt{\pi}\,F(y)}.$$

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Discrete Spectrum

$$\sigma(\mathbf{B}) = \left\{\lambda_k = -\tfrac{k}{2}, \quad k = 0, 1, 2, \dots\right\},$$
$$\psi_k(y) = \frac{(-1)^k}{\sqrt{k!}}\, D_y^k F(y) \equiv H_k(y)\, F(y).$$

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Hermite Polynomials (up to norm. factors)

$$H_1(y) = 1, \quad H_2(y) \sim 1 - \tfrac{y^2}{2},$$
$$H_3(y) \sim -\tfrac{3y}{2} + \tfrac{y^3}{4}, \;\; H_k(y) \sim y^k + \ldots.$$

Completeness and Closure of $\Phi = \{\psi_k\}$

Metrics:

$$\langle \cdot, \cdot \rangle : \quad \text{in} \quad L^2,$$

$$\langle \cdot, \cdot \rangle_\rho : \quad \text{in} \quad L^2_\rho, \quad \text{so:}$$

$$\langle v, w \rangle_\rho = \int_{\mathbb{R}} \mathrm{e}^{y^2/4} \, v(y) w(y) \, \mathrm{d}y$$

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Completeness (by the Riesz-Fischer theorem):

$$\langle g, \psi_k\rangle_\rho = 0 \;\forall\; k \geq 0 \Longrightarrow g = 0. \tag{3}$$

Closure of $\Phi = \{\psi_k\}$

Closure in L^2_ρ: for $v \in L^2_\rho$,

$$v = \sum_{(k)} c_k \psi_k, \quad c_k = \langle v, H_k \rangle \equiv \langle v, \psi_k \rangle_\rho,$$

converging in the mean (in L^2_ρ), a classic theory of linear self-adjoint operators... .

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(Bi)-Orthonormality Property:

$$\langle \psi_k, \psi_l \rangle_\rho \equiv \langle \psi_k, H_l \rangle = \delta_{kl}.$$

C. Sturm, 1836; PDE Blow-up Theory

Hermitian Spectral Theory by Sturm

All eigenfunctions of $\mathbf{B}$ and all Hermite polynomials were obtained by C. Sturm in 1836, in his classic analysis of formation and collapse of multiple zeros (in x) of solutions of the HE. This was the first finite-time blow-up theory!

Sturm's PDE Paper:
C. Sturm, *Mémoire sur une classe d'équations à différences partielles,* J. Math. Pures Appl., **1** (1836), 373–444.

Not confuse with **Sturm's ODE paper** (the basics of Sturm–Liouville Theory)
C. Sturm, *Mémoire sur les équations différences du second ordre,* J. Math. Pures Appl., **1** (1836), 106–186.

C. Sturm, 1836; PDE Blow-up Theory

Completeness: described ALL types of multiple zeros

Sturm also used a kind of completeness of the eigenfunction subset Φ of Hermite polynomials, and introduced other key concepts... .

All Possible Types of Asymptotics as $t \to +\infty$

For $u_0 \in L^2_\rho$:

$$v(y,\tau) = \sum_{(k)} c_k \mathrm{e}^{-k\tau/2} \psi_k(y), \quad c_k = \langle u_0, H_k \rangle$$
$$\equiv \langle u_0, \psi_k \rangle_\rho .$$

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By Completeness/Closure: a Countable Set of Patterns:

For any $u_0 \in L^2_\rho$, $u_0 \neq 0$, there exists a finite $l \geq 0$ (the first non-zero Fourier coefficient!) such that

$$u(x,t) = c_l \mathrm{e}^{-(1+l)\,t/2} \left[\psi_l(x/\sqrt{t}) + o(1) \right], \quad t \to +\infty.$$

Precise Asymptotic Behaviour as $t \longrightarrow 0^-$

Multiple zero at $(0, 0^-)$

This is the actual Sturmian Theory (1836), written using spectral language...
Let $u(x,t)$ be defined in $\mathbb{R} \times [-1, 0)$, and at the point $(0, 0^-)$ (here $T = 0$ is the actual *blow-up time*)

$$u(0,0) = 0, \quad \text{a multiple zero in } x \text{ occurs.} \tag{4}$$

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Sturm's Problem:

Describe evolution (blow-up) formation of ALL possible types of multiple zeros that can occur at $(0, 0^-)$.
Meaning micro-scale ("turbulent") structure of a PDE... .

Adjoint Operator $\mathbf{B}^*$: Blow-up Behaviour

Blow-up Rescaled Variables

The Heat Equation

$$u_t = u_{xx}, \quad , x \in \mathbb{R}, \quad -1 < t < 0.$$

$$u(x,t) = v(y,\tau), \quad y = \frac{x}{\sqrt{-t}}, \quad \tau = -\ln(-t) \to +\infty$$

as $t \to 0^-$.

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Exponential Weight $\rho^* = 1/\rho$

$$\mathbf{B}^* v = v_{yy} - \tfrac{1}{2}\, y v_y \equiv \tfrac{1}{\rho^*}\,(\rho^* v_y)_y,$$

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$\mathbf{B}^*$ is Adjoint to $\mathbf{B}$ in metric L^2

One can see that

$$\langle \mathbf{B} v, w \rangle = \langle v, \mathbf{B}^* w \rangle, \quad v \in L^2_\rho, \;\; w \in L^2_{\rho^*}; \tag{5}$$

by integration by parts, $v, w, \in C_0^\infty$, and by closure... .

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Self-Adjoint Operator: $\mathbf{B} = \mathbf{B}^*$ in the Metric of L^2_ρ

$$\langle \mathbf{B} v, w\rangle_\rho = \langle v, \mathbf{B} w\rangle_\rho, \quad v, w \in H^2_\rho. \tag{6}$$

Polynomial Eigenfunctions of $\mathbf{B}^*$

Discrete Spectrum of $\mathbf{B}^*$

Hence, classic self-adjoint theory applies. In particular, in $L^2_{\rho^*}$ (not identifying $\mathbf{B}$ and $\mathbf{B}^*$ in L^2_ρ)

$$\sigma(\mathbf{B}^*) = \left\{\lambda_k = -\tfrac{k}{2}, \quad k = 0, 1, 2, ...\right\},$$
$$\psi_k^*(y) = H_k(y),$$

the Hermite polynomials.

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Sturm's First Theorem: Full Classification of Multiple Zeros

Asymptotic Blow-up Patterns

Returning to the rescaled blow-up equation

$$v_\tau = \mathbf{B}^* v = v_{yy} - \tfrac{1}{2}\, y v_y, \quad v\big|_{\tau=0} = u_0(y),$$

by completeness/closure of the eigenfunction subset $\Phi^* = \{\psi_k^*\}$ of Hermite polynomials:

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Countable Set of Blow-up Patterns

There exists a finite $l \geq 1$ such that

$$v(y,\tau) = c_l \big(\mathrm{e}^{-l\tau/2} \psi_l^*(y) + o(1) \big) \quad \text{as} \quad \tau \to +\infty. \tag{7}$$

Sturm's First Theorem: Full Classification of Multiple Zeros

Blow-up Asymptotics

In terms of the original function $u(x,t)$ this means:

$$u(x,t) = c_l(-t)^{\frac{l}{2}}\left[\psi_l^*\left(\frac{x}{\sqrt{-t}}\right) + o(1)\right] \tag{8}$$

as $t \to 0^-$ (uniformly on compact subsets in y...).
Each $\psi_l^*(y)$ has l transversal zeros $\{y_j\}$ (Sturm, 1836!), so

$$\exists \ l \text{ zero curves,} \quad x_j(t) = y_j\sqrt{-t} \to 0, \ \ j = 1, ..., l.$$

Sturm's Second Theorem: the Number of Zeros of Solutions does not Increase with Time

On Zero Curves

Since all the zeros of Hermite polynomials are transversal (proved by Sturm), each multiple zero is produced by blow-up focusing of a finite number of zero curves generated by a Hermite polynomial.

Naturally extending the solution for $t > 0$:

(i) after blow-up at $t = 0$!

(ii) this extension is not related to any spectral theory, the extensions are governed polynomials $\hat{H}_k(y)$, which are Hermite ones, in which all negative coefficients are replaced by POSITIVE.

Sturm showed:

Sturm's Second Theorem: the Number of Zeros of Solutions does not Increase with Time

Sturm's Second Theorem

The number of zeros (sign changes) of solution $u(x, t)$ **does not increase in time**.

This is the so-called "Sturm's Lost Theorem" (1836), which was almost completely forgotten for almost 150 years.

It seems first Hurwitz in 1903 was the first who mentioned this Sturm's result. Sometimes, his lower bound on zeros of Fourier series is referred to as the Hurwitz Theorem, which, possibly, was better known than the Sturm's PDE Theorem. This **Sturm-Hurwitz Theorem** is the origin of many striking results, ideas and conjectures in topology of curves and symplectic geometry.

Sturm's Second Theorem: the Number of Zeros of Solutions does not Increase with Time

Sturm's Second Theorem, cont.

The number of zeros (sign changes) of solution $u(x, t)$ does not increase in time.

A detailed history of Sturm's discoveries from 19th century to 21st one is available in Chapter 1 (with a number of the *original* Sturm's computations in his variables; a nice history!) of

V.A. Galaktionov, Geometric Sturmian Theory of Nonlinear Parabolic Equations and Applications, Chapman and Hall/CRC, Boca Raton, Florida, 2004.