Nonlinear Partial Differential Equations of Mixed Hyperbolic-Elliptic Type

Gui-Qiang G. Chen

Oxford Centre for Nonlinear PDE
Mathematical Institute, University of Oxford
Email: Gui-Qiang.Chen@maths.ox.ac.uk

Myoungjean Bae	Northwestern University
Mikhail Feldman	University of Wisconsin-Madison
Marshall Slemrod	University of Wisconsin-Madison
Dehua Wang	University of Pittsburgh

WIMCS-LMS Workshop on Calculus of Variations and Nonlinear PDEs
Mathematics Department, Swansea University, UK

Linear Partial Differential Equations

- Two of the Basic Types: Representatives

Hyperbolic: $\partial_{tt} u - \Delta_{\mathbf{x}} u = 0$ (Wave Equation)

Elliptic: $\Delta_{\mathbf{x}} u = 0$ (Laplace's Equation)

$$(\mathbf{x} = (x_1, \cdots, x_n), \qquad \Delta_{\mathbf{x}} = \textstyle\sum_{j=1}^{n} \frac{\partial^2}{\partial^2 x_j})$$

- Two of the Basic Types: Representatives

Hyperbolic: $\partial_{tt} u - \Delta_{\mathbf{x}} u = 0$ (Wave Equation)

Elliptic: $\Delta_{\mathbf{x}} u = 0$ (Laplace's Equation)

$$(\mathbf{x} = (x_1, \cdots, x_n), \qquad \Delta_{\mathbf{x}} = \textstyle\sum_{j=1}^{n} \frac{\partial^2}{\partial^2 x_j})$$

- Mixed Hyperbolic-Elliptic Type

Lavrentyev-Bitsadze Equation: $\partial_{xx} u + \mathrm{sign}(x) \partial_{yy} u = 0$

Tricomi Equation: $\partial_{xx} u + x \partial_{yy} u = 0$ (hyperbolic degeneracy at $x = 0$)

Keldysh Equation: $x \partial_{xx} u + \partial_{yy} u = 0$ (parabolic degeneracy at $x = 0$)

Outline

1. Nonlinear PDEs of Mixed Hyperbolic-Elliptic Type in Fluid Mechanics

2. Nonlinear PDEs of Mixed Hyperbolic-Elliptic Type in Differential Geometry

3. Nonlinear PDEs of No Type in Differential Geometry

Bow Shock in Space generated by a Solar Explosion

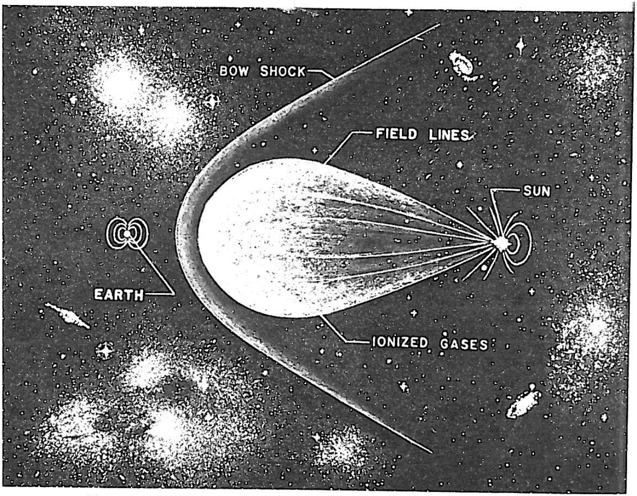


FIG. 50: SOLAR EXPLOSION

A shock wave in space generated by a solar eruption. The sketch shows the fully ionized nucleons attached to the solar magnetic field lines acting as the driving piston for the shock wave. (Courtesy: UTIAS, after Gold, 1962).

Blast Wave from a TNT Surface Explosion

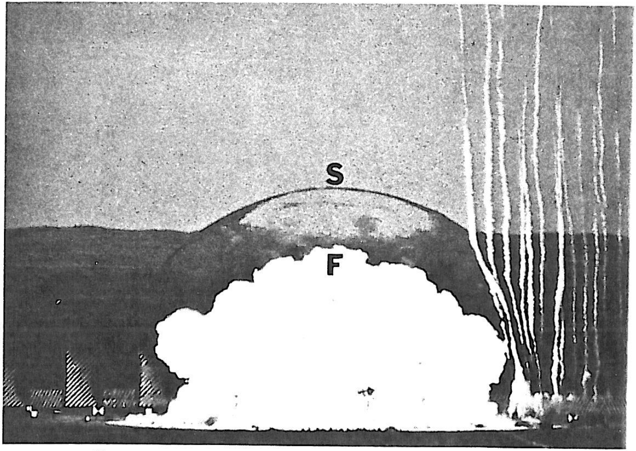
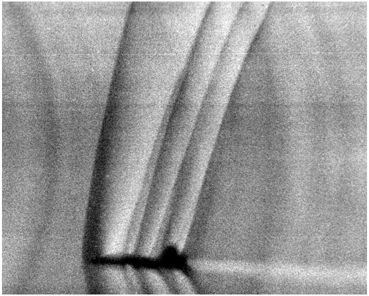
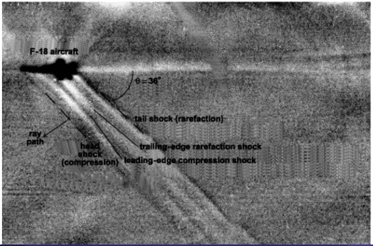


FIG. 22: EXPLOSION FROM A 20-TON HEMISPHERE OF TNT

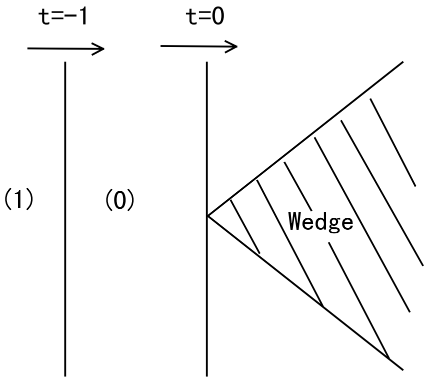
The blast wave S, and fireball F, from a 20-ton TNT surface explosion are clearly shown. The backdrops are 50 feet by 30 feet and in conjunction with the rocket smoke trails, it is possible to distinguish shock waves and particle paths and to measure their velocities. Owing to unusual daylight conditions, the hemispherical shock wave became visible. (Courtesy: Defence Research Board of Canada).

Shock Waves generated by Aircrafts





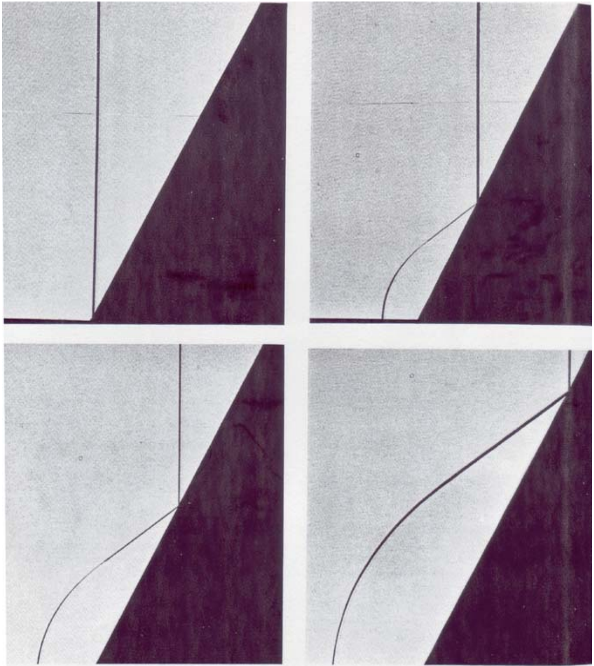
Shock Reflection-Diffraction Problem

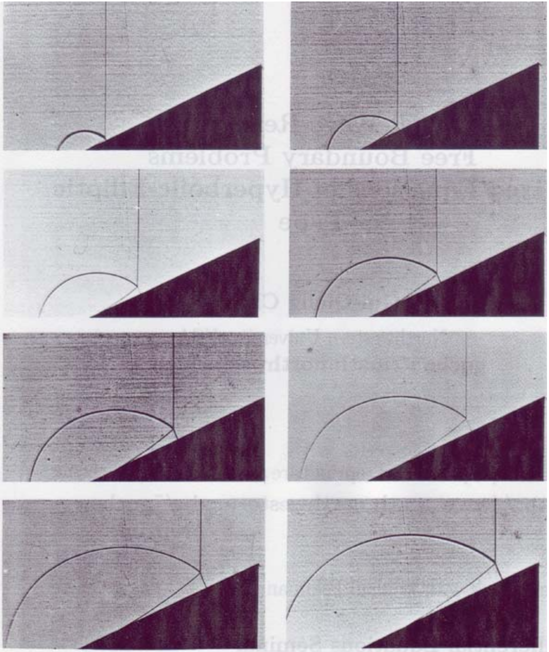


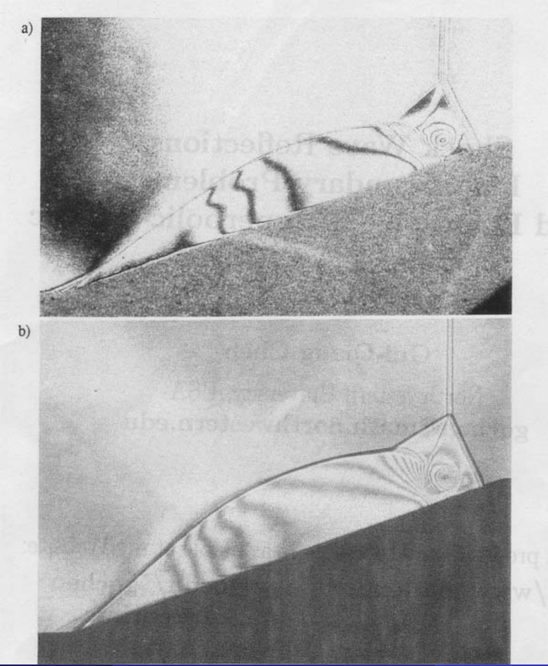
? Shock Wave Patterns Around a Wedge (airfoils, inclined ramps, $\cdots$)

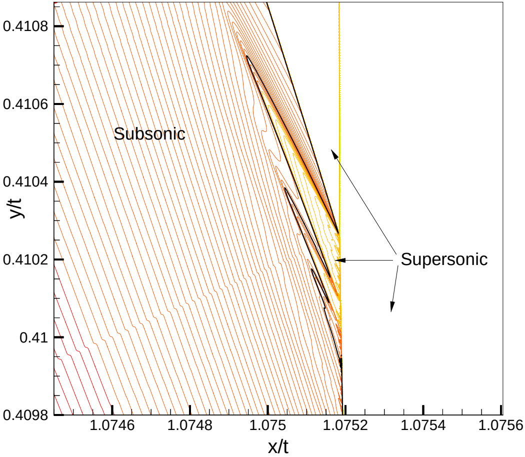
Complexity of Reflection Configurations First Reported: Ernst Mach 1879

Experimental Analysis: 1940s$\Longrightarrow$: von Neumann, Bleakney, Bazhenova
Glass, Takyama, Henderson, $\cdots$






a)
b)



A New Mach Reflection-Diffraction Pattern:

A. M. Tesdall and J. K. Hunter: TSD, 2002

A. M. Tesdall, R. Sanders, and B. L. Keyfitz: NWE, 2006; Full Euler, 2008

B. Skews and J. Ashworth: J. Fluid Mech. 542 (2005), 105-114

Shock Reflection-Diffraction Patterns

- **Gabi Ben-Dor** Shock Wave Reflection Phenomena
  Springer-Verlag: New York, 307 pages, 1992.

  Experimental results before 1991
  Various proposals for transition criteria

- **Peter O. K. Krehl** History of Shock Waves, Explosions and Impact
  A Chronological and Biographical Reference
  2009, XXII, 1288 p. 1200 illus., 300 in color.

- **Milton Van Dyke** An Album of Fluid Motion
  The parabolic Press: Stanford, 176 pages, 1982.

  Various photographs of shock wave reflection phenomena

- **Richard Courant & Kurt Otto Friedrichs**
  Supersonic Flow and Shock Waves
  Springer-Verlag: New York, 1948.

Scientific Issues

- **Structure of the Shock Reflection-Diffraction Patterns**
- **Transition Criteria among the Patterns**
- **Dependence of the Patterns on the Parameters**
  wedge angle θ_w, **adiabatic exponent $\gamma \geq 1$**
  incident-shock-wave Mach number M_s
- ······

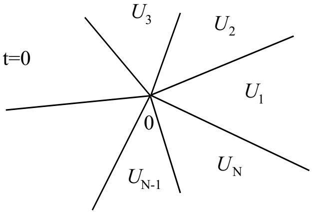
Interdisciplinary Approaches:

- **Experimental Data and Photographs**
- **Large or Small Scale Computing**
  Colella, Berger, Deschambault, Glass, Glaz, Woodward,....
  Anderson, Hindman, Kutler, Schneyer, Shankar, ...
  Yu. Dem'yanov, Panasenko, ...
- **Asymptotic Analysis**
  Lighthill, Keller, Majda, Hunter, Rosales, Tabak, Gamba, Harabetian...
  Morawetz: CPAM 1994
- **Rigorous Mathematical Analysis??** **(Global Solutions)**
  Existence, Stability, Regularity, Bifurcation, ······

2-D Riemann Problem for Hyperbolic Conservation Laws

$$\partial_t U + \nabla_{\mathbf{x}} \cdot \mathbf{F}(U) = 0, \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$$

or
$$\partial_t \mathbf{A}(U, U_t, \nabla_{\mathbf{x}} U) + \nabla_{\mathbf{x}} \cdot \mathbf{B}(U, U_t, \nabla_{\mathbf{x}} U) = 0$$



Books and Survey Articles:
Chang-Hsiao 1989, Glimm-Majda 1991, Li-Zhang-Yang 1998, Zheng 2001
Chen-Wang 2002, Serre 2005, Chen 2005, Dafermos 2010, $\cdots$

Numerical Solutions: Glimm-Klingenberg-McBryan-Plohr-Sharp-Yaniv 1985
Lax-Liu 1998, Schulz-Rinne-Collins-Glaz 1993, Chang-Chen-Yang 1995, 2000
Kurganov-Tadmor 2002, $\cdots$

Theoretical Roles: Asymptotic States and Attractors
Local Structure and Building Blocks...

Euler Equations for Potential Flow: $\mathbf{v} = \nabla\Phi$, $\gamma > 1$

$$\begin{cases} \partial_t\rho + \nabla_{\mathbf{x}} \cdot (\rho\nabla_{\mathbf{x}}\Phi) = 0, & \text{(conservation of mass)} \\ \partial_t\Phi + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2 + \frac{\rho^{\gamma-1}}{\gamma-1} = \frac{\rho_0^{\gamma-1}}{\gamma-1}, & \text{(Bernoulli's law)} \end{cases}$$

or, equivalently,

$$\partial_t\rho(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi, \rho_0) + \nabla_{\mathbf{x}} \cdot \big(\rho(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi, \rho_0)\nabla_{\mathbf{x}}\Phi\big) = 0,$$

with

$$\rho(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi, \rho_0) = \big(\rho_0^{\gamma-1} - (\gamma-1)(\partial_t\Phi + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2)\big)^{\frac{1}{\gamma-1}}.$$

Celebrated steady potential flow equation of aerodynamics:

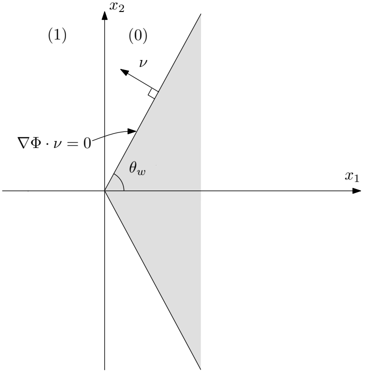
$$\nabla_{\mathbf{x}} \cdot (\rho(\nabla_{\mathbf{x}}\Phi, \rho_0)\nabla_{\mathbf{x}}\Phi) = 0.$$

J. Hadamard: Leçons sur la Propagation des Ondes, Hermann: Paris 1903
Majda-Thomann: CPDE 1987, Morawetz: CPAM 1994, $\cdots$

Initial-Boundary Value Problem: $0 < \rho_0 < \rho_1,\ u_1 > 0$

Initial condition: $(\rho, \Phi)|_{t=0} = \begin{cases} (\rho_0, 0), & |x_2| > x_1 \tan\theta_w, x_1 > 0, \\ (\rho_1, u_1 x_1), & x_1 < 0; \end{cases}$

Slip boundary condition on the wedge boundary: $\nabla\Phi \cdot \nu = 0$



Invariant under the Self-Similar Scaling:

$$(t, \mathbf{x}) \longrightarrow (\alpha t, \alpha \mathbf{x}), \quad (\rho, \Phi) \longrightarrow (\rho, \Phi/\alpha) \qquad \alpha \neq 0$$

$$\rho(t, \mathbf{x}) = \rho(\xi, \eta), \quad \Phi(t, \mathbf{x}) = t\big(\varphi(\xi, \eta) + \frac{1}{2}(\xi^2 + \eta^2)\big)$$

$$\nabla \cdot (\rho(\nabla\varphi, \varphi, \rho_0)\nabla\varphi) + 2\rho(\nabla\varphi, \varphi, \rho_0) = 0$$

$$\rho(t, \mathbf{x}) = \rho(\xi, \eta), \quad \Phi(t, \mathbf{x}) = t\big(\varphi(\xi, \eta) + \frac{1}{2}(\xi^2 + \eta^2)\big)$$

$$\nabla \cdot (\rho(\nabla\varphi, \varphi, \rho_0)\nabla\varphi) + 2\rho(\nabla\varphi, \varphi, \rho_0) = 0$$

- Incompressible: $\rho = const. \implies \Delta\varphi + 2 = 0$

$$\rho(t, \mathbf{x}) = \rho(\xi, \eta), \quad \Phi(t, \mathbf{x}) = t\big(\varphi(\xi, \eta) + \tfrac{1}{2}(\xi^2 + \eta^2)\big)$$

$$\nabla \cdot (\rho(\nabla\varphi, \varphi, \rho_0)\nabla\varphi) + 2\rho(\nabla\varphi, \varphi, \rho_0) = 0$$

- Incompressible: $\rho = const. \Longrightarrow \Delta\varphi + 2 = 0$
- Elliptic: $|\nabla\varphi| < c_*(\varphi, \rho_0) := \sqrt{\frac{2}{\gamma+1}(\rho_0^{\gamma-1} - (\gamma-1)\varphi)}$
- Hyperbolic: $|\nabla\varphi| > c_*(\varphi, \rho_0) := \sqrt{\frac{2}{\gamma+1}(\rho_0^{\gamma-1} - (\gamma-1)\varphi)}$

Second-order nonlinear equations of mixed hyperbolic-elliptic type

Simpler Nonlinear Models and Equations of Mixed Type

- **Transonic Small Disturbance Equation**:

$$\Big((u-x)u_x+\frac{u}{2}\Big)_x+u_{yy}=0$$

or, for $v=u-x$,

$$v\,v_{xx}+v_{yy}+\text{l.o.t.}=0$$

Morawetz, Hunter, Canic-Keyfitz-Lieberman-Kim, $\cdots$

- **Steady Potential Flow Equation of Aerodynamics**

$$\nabla\cdot(\rho(\nabla\varphi,\rho_0)\nabla\varphi)=0$$

  - **Elliptic:** $|\nabla\varphi|<c_*(\rho_0):=\sqrt{\frac{2}{\gamma+1}\rho_0^{\gamma-1}}$
  - **Hyperbolic:** $|\nabla\varphi|>c_*(\rho_0)$

$$\nabla \cdot (\rho(\nabla\varphi, \rho_0)\nabla\varphi) = 0$$

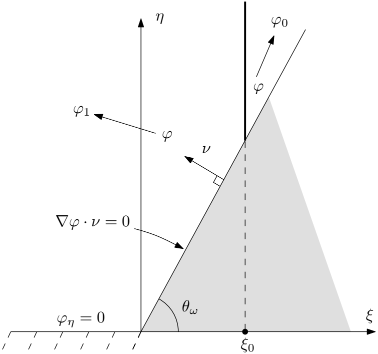
- **Pure Elliptic Case: Subsonic Flow past an Obstacle**
  Shiffman, L. Bers, Finn-Gilbarg, G. Dong, $\cdots$
- **Degenerate Elliptic Case: Subsonic-Sonic Flows**
  Shiffman, Chen-Dafermos-Slemrod-Wang, Elling-Liu, $\cdots$
- **Pure Hyperbolic Case (even Full Euler Eqs.)**:
  Gu, Li, Schaeffer, S. Chen, Xin-Yin, Y. Zheng, $\cdots$
  T.-P. Liu-Lien, S. Chen-Zhang-Wang, Chen-Zhang-Zhu, $\cdots$
- **Elliptic-Hyperbolic Mixed Case**
  Transonic Nozzles: Chen-Feldman, S. Chen, J. Chen, Yuan, Xin-Yin,...
  Wedge or Conical Body: S. Chen, B. Fang, Chen-Fang, $\cdots$
  Transonic Flow past an Obstacle: Morawetz, Chen-Slemrod-Wang,...

Slip boundary condition on the wedge boundary:

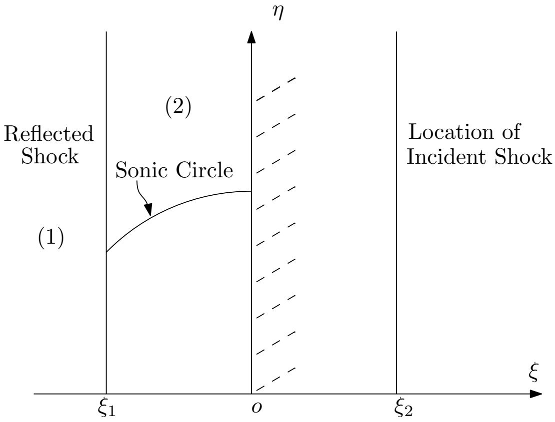
$$\nabla\varphi \cdot \nu = 0 \qquad \text{on} \quad \partial D$$

Asymptotic boundary condition as $\xi^2 + \eta^2 \to \infty$:

$$\varphi \sim \bar{\varphi} := \begin{cases} \varphi_0 = -\frac{1}{2}(\xi^2 + \eta^2), & \xi > \xi_0, \eta > \xi \tan\theta_w, \\ \varphi_1 = -\frac{1}{2}(\xi^2 + \eta^2) + u_1(\xi - \xi_0), & \xi < \xi_0, \eta > 0. \end{cases}$$



When $\theta_w = \pi/2$, the reflection becomes the normal reflection, for which the incident shock normally reflects and the reflected shock is also a plane.



von Neumann Criterion & Conjecture (1943)

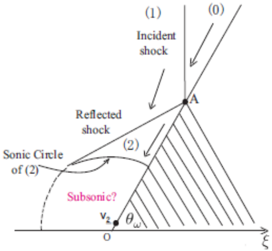
Local Theory for Regular Reflection

$\exists\, \theta_c = \theta_c(M_s, \gamma) \in (0, \frac{\pi}{2})$ such that, when $\theta_W \in (\theta_c, \frac{\pi}{2})$, there exist two states $\varphi_2^a(\xi, \eta)$ and $\varphi_2^b(\xi, \eta)$ such that $|\nabla\varphi_2^a| > |\nabla\varphi_2^b|$ and $|\nabla\varphi_2^b)| < c_2^b$.

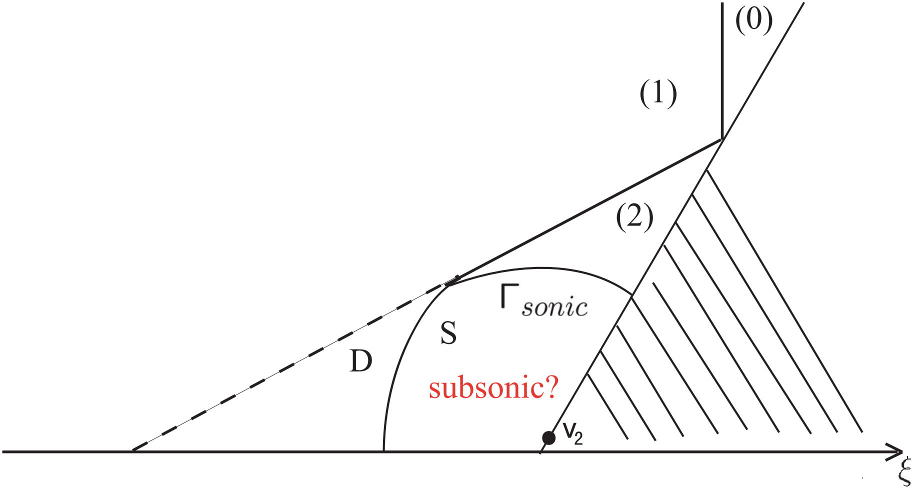
Stability as $\theta_W \to \frac{\pi}{2}$ (Chen-Feldman 2005): Choose $\varphi_2^a(\xi, \eta)$.

von Neumann's Sonic Conjecture (1943):

There exists a global Regular Reflection Configuration when the wedge angle $\theta_W \in (\theta_s, \frac{\pi}{2})$, for θ_s such that $|\nabla\varphi_2^a(\xi, \eta)| > c_2$ near A.



Global Theory?



Setup of the Problem for $\psi := \varphi - \varphi_2$ in Ω

- $\operatorname{div}\left(\rho(\nabla\psi, \psi, \xi, \eta, \rho_0)(\nabla\psi - (\xi - u_2, \eta - v_2))\right) + l.o.t. = 0 \qquad (*)$
- $\nabla\psi \cdot \nu|_{wedge} = 0$
- $\psi|_{\Gamma_{sonic}} = 0 \Longrightarrow \psi_\nu|_{\Gamma_{sonic}} = 0$
- Rankine-Hugoniot Conditions on Shock S:

$$[\psi]_S = 0$$

$$[\rho(\nabla\psi, \psi, \xi, \eta, \rho_0)(\nabla\psi - (\xi - u_2, \eta - v_2)) \cdot \nu]_S = 0 \qquad (**)$$

Free Boundary Problem

Setup of the Problem for $\psi := \varphi - \varphi_2$ in Ω

- $\operatorname{div}\left(\rho(\nabla\psi, \psi, \xi, \eta, \rho_0)(\nabla\psi - (\xi - u_2, \eta - v_2))\right) + l.o.t. = 0 \qquad (*)$
- $\nabla\psi \cdot \nu|_{wedge} = 0$
- $\psi|_{\Gamma_{sonic}} = 0 \implies \psi_\nu|_{\Gamma_{sonic}} = 0$
- Rankine-Hugoniot Conditions on Shock S:

$$[\psi]_S = 0$$

$$[\rho(\nabla\psi, \psi, \xi, \eta, \rho_0)(\nabla\psi - (\xi - u_2, \eta - v_2)) \cdot \nu]_S = 0 \qquad (**)$$

Free Boundary Problem

- $\exists\, S = \{\xi = f(\eta)\}$ such that $f \in C^{1,\alpha}, f'(0) = 0$ and

$$\Omega_+ = \{\xi > f(\eta)\} \cap D = \{\psi < \varphi_1 - \varphi_2\} \cap D,$$

$$S = \{\psi = \varphi_1 - \varphi_2\} \cap D \qquad \text{(free boundary as a level set)}$$

Setup of the Problem for $\psi := \varphi - \varphi_2$ in Ω

- $\operatorname{div}\left(\rho(\nabla\psi, \psi, \xi, \eta, \rho_0)(\nabla\psi - (\xi - u_2, \eta - v_2))\right) + l.o.t. = 0 \qquad (*)$
- $\nabla\psi \cdot \nu|_{wedge} = 0$
- $\psi|_{\Gamma_{sonic}} = 0 \Longrightarrow \psi_\nu|_{\Gamma_{sonic}} = 0$
- Rankine-Hugoniot Conditions on Shock S:

$$[\psi]_S = 0$$

$$[\rho(\nabla\psi, \psi, \xi, \eta, \rho_0)(\nabla\psi - (\xi - u_2, \eta - v_2)) \cdot \nu]_S = 0 \qquad (**)$$

Free Boundary Problem

- $\exists\, S = \{\xi = f(\eta)\}$ such that $f \in C^{1,\alpha}$, $f'(0) = 0$ and

$$\Omega_+ = \{\xi > f(\eta)\} \cap D = \{\psi < \varphi_1 - \varphi_2\} \cap D,$$

$$S = \{\psi = \varphi_1 - \varphi_2\} \cap D \qquad \text{(free boundary as a level set)}$$

- $\psi \in C^{1,\alpha}(\overline{\Omega_+}) \cap C^2(\Omega_+) \begin{cases} \text{solve } (*) \text{ in } \Omega_+, \\ \text{is subsonic in } \Omega_+ \end{cases}$

with $(\psi, \psi_\nu)|_{\Gamma_{sonic}} = 0, \quad \nabla\psi \cdot \nu|_{wedge} = 0$

Setup of the Problem for $\psi := \varphi - \varphi_2$ in Ω

- $\operatorname{div}\left(\rho(\nabla\psi, \psi, \xi, \eta, \rho_0)(\nabla\psi - (\xi - u_2, \eta - v_2))\right) + l.o.t. = 0 \qquad (*)$
- $\nabla\psi \cdot \nu|_{wedge} = 0$
- $\psi|_{\Gamma_{sonic}} = 0 \implies \psi_\nu|_{\Gamma_{sonic}} = 0$
- Rankine-Hugoniot Conditions on Shock S:

  $[\psi]_S = 0$

  $[\rho(\nabla\psi, \psi, \xi, \eta, \rho_0)(\nabla\psi - (\xi - u_2, \eta - v_2)) \cdot \nu]_S = 0 \qquad (**)$

Free Boundary Problem

- $\exists\, S = \{\xi = f(\eta)\}$ such that $f \in C^{1,\alpha}$, $f'(0) = 0$ and

$$\Omega_+ = \{\xi > f(\eta)\} \cap D = \{\psi < \varphi_1 - \varphi_2\} \cap D,$$

$$S = \{\psi = \varphi_1 - \varphi_2\} \cap D \qquad \text{(free boundary as a level set)}$$

- $\psi \in C^{1,\alpha}(\overline{\Omega_+}) \cap C^2(\Omega_+) \begin{cases} \text{solve } (*) \text{ in } \Omega_+, \\ \text{is subsonic in } \Omega_+ \end{cases}$

  with $(\psi, \psi_\nu)|_{\Gamma_{sonic}} = 0, \quad \nabla\psi \cdot \nu|_{wedge} = 0$
- (ψ, f) satisfy the R-H Condition: Free Boundary Condition (**)

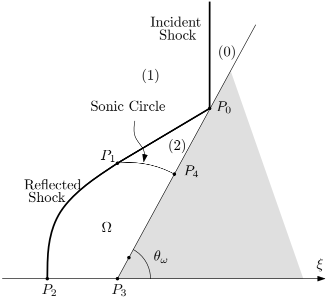
Global Theory for Regular Shock Reflection-Diffraction

(Chen-Feldman: Proc. Nat. Acad. Sci. USA, 2005; Annals of Math. 2010)

$\exists\ \theta_c = \theta_c(\rho_0, \rho_1, \gamma) \in (0, \frac{\pi}{2})$ such that, when $\theta_W \in (\theta_c, \frac{\pi}{2})$, there exist (φ, f) satisfying

- $\varphi \in C^\infty(\Omega) \cap C^{1,\alpha}(\bar{\Omega})$ and $f \in C^\infty(P_1P_2) \cap C^2(\{P_1\})$;
- $\varphi \in C^{1,1}$ across the sonic circle P_1P_4
- $\varphi \longrightarrow \varphi_{NR}$ in $W^{1,1}_{loc}$ as $\theta_W \to \frac{\pi}{2}$.

$\Rightarrow$ $\Phi(t, \mathbf{x}) = t\varphi(\frac{\mathbf{x}}{t}) + \frac{|\mathbf{x}|^2}{2t}$, $\rho(t, \mathbf{x}) = \left(\rho_0^{\gamma-1} - (\gamma-1)(\Phi_t + \frac{1}{2}|\nabla\Phi|^2)\right)^{\frac{1}{\gamma-1}}$ form a solution of the IBVP.



Optimal Regularity and Sonic Conjecture

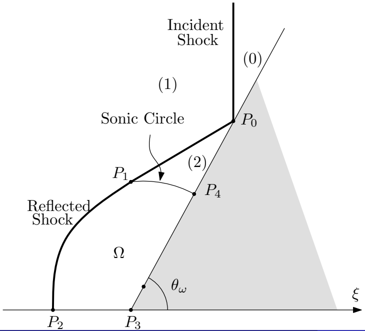
Theorem (Optimal Regularity; Bae-Chen-Feldman: Invent. Math. 2009):

$\varphi \in C^{1,1}$ but NOT in C^2 across P_1P_4;

$\varphi \in C^\infty(\bar{\Omega} \setminus (\{P_1\} \cup \{P_3\})) \cap C^{1,1}(\{P_1\}) \cup C^{1,\alpha}(\{P_3\})$;

$f \in C^\infty(P_1P_2) \cap C^2(\{P_1\})$.

$\Longrightarrow$ **C-Feldman 2011**: The global existence and the optimal regularity hold up to the sonic wedge-angle θ_s for any $\gamma \geq 1$ for $u_1 < c_1$; $u_1 \geq c_1$. (the von Neumann's sonic conjecture)



Approach for the Large-Angle Case

- **Cutoff Techniques by Shiffmanization**
  $\Rightarrow$ Elliptic Free-Boundary Problem with Elliptic Degeneracy on $\boldsymbol{\Gamma}_{sonic}$
- **Iteration Scheme for the Free Boundary Problem**
  Chen-Feldman: J. Amer. Math. Soc. 2003
- **Domain Decomposition**
  Near $\boldsymbol{\Gamma}_{sonic}$; Away from $\boldsymbol{\Gamma}_{sonic}$
- $C^{1,1}$ **Parabolic Estimates near the Degenerate Elliptic Curve** $\boldsymbol{\Gamma}_{sonic}$;
- **Corner Singularity Estimates**
  In particular, when the Elliptic Degenerate Curve $\boldsymbol{\Gamma}_{sonic}$ Meets the Free Boundary, i.e., the Transonic Shock
- **Removal of the Cutoff**
  Require the Elliptic-Parabolic Estimates and the Large-Angle

Large Angles $\Longrightarrow$ **Sonic Angle** θ_{sonic}**??**

Approach: Apriori Estimates + Separation + Compactness + $\cdots\cdots$
$\cdots\cdots$ + Continuity Method/Degree Theory

PDE Near Γ_{sonic}: Mixed Elliptic-Hyperbolic Type

$$\begin{cases} (2x - (\gamma+1)\psi_x)\,\psi_{xx} + \frac{1}{c_2^2}\psi_{yy} - \psi_x \sim o(x^2) \\ \psi|_{x=0} = 0 \end{cases}$$

Ellipticity: $\psi_x \leq \frac{2x}{\gamma+1}$

Apriori Estimate: $|\psi_x| \leq \frac{4x}{3(\gamma+1)}$

$$\psi \sim \frac{x^2}{2(\gamma+1)} + h.o.t. \qquad \text{when } x \approx 0$$

Mach Reflection: Self-Similar Solutions for the Full Euler Equations $(u, v, p, \rho)(t, \mathbf{x}) = (u, v, p, \rho)(\xi, \eta)$, $(\xi, \eta) = (\frac{x_1}{t}, \frac{x_2}{t})$

$$
\begin{cases}
(\rho U)_\xi + (\rho V)_\eta + 2\rho = 0, \\
(\rho U^2 + p)_\xi + (\rho UV)_\eta + 3\rho U = 0, \\
(\rho UV)_\xi + (\rho V^2 + p)_\eta + 3\rho V = 0, \\
(U(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}))_\xi + (V(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}))_\eta + 2(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}) = 0,
\end{cases}
$$

where $q = \sqrt{U^2 + V^2}$ and $(U, V) = (u - \xi, v - \eta)$ is the pseudo-velocity.

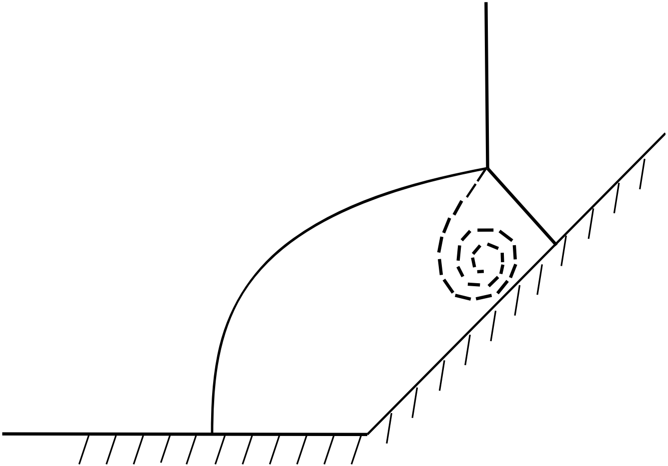
Eigenvalues: $\lambda_0 = \frac{V}{U}$(repeated), $\lambda_\pm = \frac{UV \pm c\sqrt{q^2 - c^2}}{U^2 - c^2}$,

where $c = \sqrt{\gamma p / \rho}$ is the **sonic speed**

When the flow is pseudo-subsonic: $q < c$, the system consists of

- 2-transport equations: Compressible vortex sheets
- 2-nonlinear equations of mixed hyperbolic-elliptic type: Two kinds of transonic flow: Transonic shocks and sonic curves

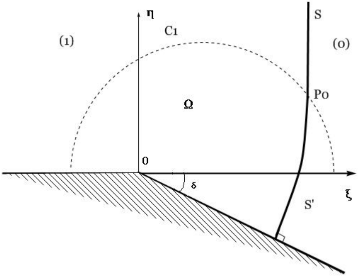
Mach Reflection: Full Euler Equations

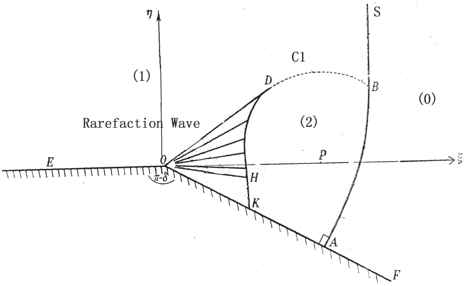


? Right space for vorticity ω?

? Chord-arc $z(s) = z_0 + \int_0^s e^{ib(s)} ds$, $b \in BMO$?

Further Problem: Shock Diffraction by the Wedge Corner



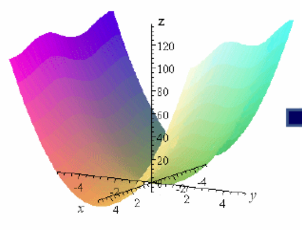


Nonlinear model equations: C-Deng-Xiang 2011
More general equations: In progress ...

When the angle δ increases, the situation becomes more complicated, which involves vorticity waves, etc.

Outline

Isometric Embedding Problems



g_{ij} ⟷ **Metric**

(First Fundamental Form: $\mathrm{I} = \sum g_{ij} dx^i dx^j$)

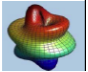
h_{ij} ⟷ **Curvatures**

(Second Fundamental Form: $\mathrm{II} = \sum h_{ij} dx^i dx^j$)

Given a metric g_{ij} and certain curvatures

Inverse Problem: CAN we find a surface in our real world with this metric g_{ij} and corresponding curvatures?

Realization Question?



Question: CAN *we produce even more sophisticated surfaces or thin sheets?*

Fundamental

- Mathematics: Differential Geometry, Topology, ……
- Understanding evolution of sophisticated shapes of surfaces or thin sheets in nature, including
  - --Elasticity, Materials Science, ……
  - --Biology and Algorithmic Origami: Protein Folding, ……
    - *US DARPA's 10th question of the 23 Challenge Questions in the Sciences [US Defense Advanced Research Project Agency]:
    - Build a stronger mathematical theory for isometric and rigid embedding that can give insight into protein folding.
- Design, Visual Arts, ……

History: *Schlaefli (1873), Darboux (1894), Hilbert (1901), Weyl (1916),*

Janet (1926-27), Cartan (1926-27), Lewy (1936), Nash (1954-56), Kuiper (1955), Yau (1980's, 1990's), Gromov (1970, 1986), Günther (1989), Poznyak (1973), Levi (1908), Heinz (1962), Alexandroff (1938, 1942), Pogorelov (late 1940's, 1972), Nirenberg (1953, 1963), Efimov (1963), Bryant-Griffiths-Yan (1983), Lin (1985-86), Hong (1991,1993), ……

Nash Isometric Embedding Theorem

(C^k embedding theorem, $k \geq 3$)

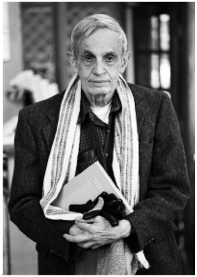
Every n-Dimensional Riemannian manifold (analytic or $C^k, k \geq 3$) can be C^k isometrically imbedded in the Euclidean space $\mathbb{R}^N$:

Compact Case: $N = 3s_n + 4n$

Noncompact Case: $N = (n+1)(3s_n + 4n)$

Gromov (1986): $N = s_n + 2n + 3$

Günther (1989): $N = \max\{s_n + 2n, s_n + n + 5\}$



the Subject of the Hollywood Movie *A Beautiful Mind*

Open Problems Important for Applications

Optimal Dimension? Janet-D: $N = s_n = \frac{n(n+1)}{2}$?

$C^{1,1}$ Isometric Embedding? What about $BV(C^1)$?

Current Research Activities, ...

Efimov's Example (1966): **No C^2** Isometric Embedding when $n = 2, s_n = 3$.

Gauss-Codazzi System: Compatibility/Constraint

Fundamental Theorem in Differential Geometry:
There exists a surface in $\mathbb{R}^3$ with 1st and 2nd fundamental form coefficients $\{g_{ij}\}$ and $\{h_{ij}\}$, $\{g_{ij}\}$ being positive definite, provided that the coefficients satisfy the Gauss-Codazzi system.

*This theorem holds even when $h_{ij} \in L^p$ (Mardare 2003–05)

Given $\{g_{ij}\}$, $\{h_{ij}\}$ is determined by the Codazzi Eqs. (Compatibility):

$$\begin{cases} \partial_x M - \partial_y L = L\Gamma_{22}^{(2)} - 2M\Gamma_{12}^{(2)} + N\Gamma_{11}^{(2)}, \\ \partial_x N - \partial_y M = -L\Gamma_{22}^{(1)} + 2M\Gamma_{12}^{(1)} - N\Gamma_{11}^{(1)}, \end{cases}$$

satisfying the Gauss Equation (Constraint):

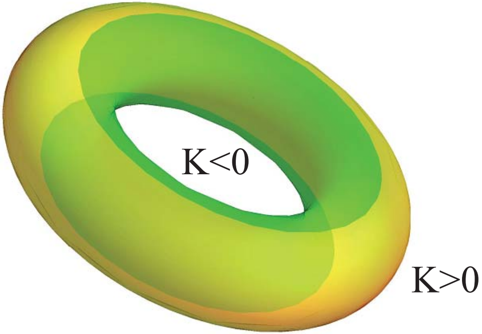
$$LN - M^2 = K,$$

where $L = \frac{h_{11}}{\sqrt{|g|}}$, $M = \frac{h_{12}}{\sqrt{|g|}}$, $N = \frac{h_{22}}{\sqrt{|g|}}$, $|g| = g_{11}g_{22} - g_{12}^2$

$\Gamma_{ij}^{(k)}$—Christoffel symbols, depending on g_{ij} up to their 1st derivatives
$K(x,y)$—Gauss curvature, determined by g_{ij} up to their 2nd derivatives

***Nonlinear PDEs of Mixed Elliptic-Hyperbolic Equations: Sign of K**

Surfaces with Gauss Curvature of Changing Sign



Gauss Curvature K on a Torus:
Toroidal Shell or Doughnut Surface

Fluid Dynamics Formalism for Isometric Embedding

Set $\quad L = \rho v^2 + p, \quad M = -\rho uv, \quad N = \rho u^2 + p, \quad q^2 = u^2 + v^2.$
Choose p as the Chaplygin type gas: $\qquad p = -1/\rho.$

The Codazzi Equations become the Momentum Equations:

$$\begin{cases} \partial_x(\rho uv) + \partial_y(\rho v^2 + p) = -(\rho v^2 + p)\Gamma_{22}^{(2)} - 2\rho uv\Gamma_{12}^{(2)} - (\rho u^2 + p)\Gamma_{11}^{(2)}, \\ \partial_x(\rho u^2 + p) + \partial_y(\rho uv) = -(\rho v^2 + p)\Gamma_{22}^{(1)} - 2\rho uv\Gamma_{12}^{(1)} - (\rho u^2 + p)\Gamma_{11}^{(1)}, \end{cases}$$

and the Gauss Equation becomes the Bernoulli Relation:

$$p = -\sqrt{q^2 + K}.$$

Define the sound speed: $c^2 = p'(\rho)$. Then $c^2 = 1/\rho^2 = q^2 + K$.

Fluid Dynamics Formalism for Isometric Embedding

Set $\quad L = \rho v^2 + p, \quad M = -\rho uv, \quad N = \rho u^2 + p, \quad q^2 = u^2 + v^2.$
Choose p as the Chaplygin type gas: $\quad p = -1/\rho.$

The Codazzi Equations become the Momentum Equations:

$$\begin{cases} \partial_x(\rho uv) + \partial_y(\rho v^2 + p) = -(\rho v^2 + p)\Gamma_{22}^{(2)} - 2\rho uv\Gamma_{12}^{(2)} - (\rho u^2 + p)\Gamma_{11}^{(2)}, \\ \partial_x(\rho u^2 + p) + \partial_y(\rho uv) = -(\rho v^2 + p)\Gamma_{22}^{(1)} - 2\rho uv\Gamma_{12}^{(1)} - (\rho u^2 + p)\Gamma_{11}^{(1)}, \end{cases}$$

and the Gauss Equation becomes the Bernoulli Relation:

$$p = -\sqrt{q^2 + K}.$$

Define the sound speed: $c^2 = p'(\rho)$. Then $c^2 = 1/\rho^2 = q^2 + K$.

$c^2 > q^2$ and the "flow" is subsonic when $K > 0$,
$c^2 < q^2$ and the "flow" is supersonic when $K < 0$,
$c^2 = q^2$ and the "flow" is sonic when $K = 0$.

Fluid Dynamics Formalism for Isometric Embedding

Set $\quad L = \rho v^2 + p, \quad M = -\rho uv, \quad N = \rho u^2 + p, \quad q^2 = u^2 + v^2.$
Choose p as the Chaplygin type gas: $\qquad p = -1/\rho.$

The Codazzi Equations become the Momentum Equations:

$$\begin{cases} \partial_x(\rho uv) + \partial_y(\rho v^2 + p) = -(\rho v^2 + p)\Gamma_{22}^{(2)} - 2\rho uv\Gamma_{12}^{(2)} - (\rho u^2 + p)\Gamma_{11}^{(2)}, \\ \partial_x(\rho u^2 + p) + \partial_y(\rho uv) = -(\rho v^2 + p)\Gamma_{22}^{(1)} - 2\rho uv\Gamma_{12}^{(1)} - (\rho u^2 + p)\Gamma_{11}^{(1)}, \end{cases}$$

and the Gauss Equation becomes the Bernoulli Relation:

$$p = -\sqrt{q^2 + K}.$$

Define the sound speed: $c^2 = p'(\rho)$. Then $c^2 = 1/\rho^2 = q^2 + K$.

$c^2 > q^2$ and the "flow" is subsonic when $K > 0$,
$c^2 < q^2$ and the "flow" is supersonic when $K < 0$,
$c^2 = q^2$ and the "flow" is sonic when $K = 0$.

?? **Existence/Continuity of Isometric Embedding**

$\Leftarrow$ **Weak Convergence Methods: Compensated Compactness**

Chen-Slemrod-Wang: Commun. Math. Phys. 2010

Outline

Gauss-Codazzi-Ricci System for Isometric Embedding of d-D Riemannian Manifolds into $\mathbb{R}^N$: $d \geq 3$

Gauss equations: $h^a_{ji} h^a_{kl} - h^a_{ki} h^a_{jl} = R_{ijkl}$

Codazzi equations:

$$\frac{\partial h^a_{lj}}{\partial x^k} - \frac{\partial h^a_{kj}}{\partial x^l} + \Gamma^m_{lj} h^a_{km} - \Gamma^m_{kj} h^a_{lm} + \kappa^a_{kb} h^b_{lj} - \kappa^a_{lb} h^b_{kj} = 0$$

Ricci equations:

$$\frac{\partial \kappa^a_{lb}}{\partial x^k} - \frac{\partial \kappa^a_{kb}}{\partial x^l} - g^{mn} \left(h^a_{ml} h^b_{kn} - h^a_{mk} h^b_{ln} \right) + \kappa^a_{kc} \kappa^c_{lb} - \kappa^a_{lc} \kappa^c_{kb} = 0$$

where R_{ijkl} is the Riemann curvature tensor, $\kappa^a_{kb} = -\kappa^b_{ka}$ is the coefficients of the connection form (torsion coefficients) on the normal bundle; the indices a, b, c run from 1 to N, and i, j, k, l, m, n run from 1 to $d \geq 3$.

***The Gauss-Codazzi-Ricci system has no type, neither purely hyperbolic nor purely elliptic for general Riemann curvature tensor R_{ijkl}**

*Bryant-Griffiths-Yang (**1983**): Duke Math. J., 102 pages.

*Chen-Clelland-Slemrod-Wang-Yang (**2011**): Connection with Entropy

Weak Continuity of the Gauss-Codazzi-Ricci System

Theorem (Chen-Slemrod-Wang: Proc. Amer. Math. Soc. 2010)

- **Let $(h_{ij}^{a,\varepsilon}, \kappa_{lb}^{a,\varepsilon})$ be a sequence of solutions to the Gauss-Codazzi-Ricci system, which is uniformly bounded in $L^p, p > 2$. Then the weak limit vector field $(h_{ij}^{a}, \kappa_{lb}^{a})$ of the sequence $(h_{ij}^{a,\varepsilon}, \kappa_{lb}^{a,\varepsilon})$ in L^p is still a solution to the Gauss-Codazzi-Ricci system.**

- **There exists a minimizer $(h_{ij}^{a}, \kappa_{lb}^{a})$ for the minimization problem:**

$$\min_S \|(h,\kappa)\|_{L^p(\Omega)}^p := \min_S \int_\Omega \sqrt{|g|}\left(|h_{ij}h_{ij}|^{\frac{p}{2}} + |\kappa_{lb}\kappa_{lb}|^{\frac{p}{2}}\right)dx,$$

where S is the set of weak solutions to the Gauss-Codazzi-Ricci system.

Observations: Div-Curl Structure of the GCR System

$$\mathrm{div}\,(\underbrace{\overbrace{0,\cdots,0,h^{a,\varepsilon}_{lj}}^{k},0,\cdots,-h^{a,\varepsilon}_{kj}}_{l},0,\cdots,0)=R_1,\quad \mathrm{curl}\,(h^{a,\varepsilon}_{1j},h^{a,\varepsilon}_{2j},\cdots,h^{a,\varepsilon}_{dj})=R_2,$$

$$\mathrm{div}\,(\underbrace{\overbrace{0,\cdots,0,\kappa^{a,\varepsilon}_{lb}}^{k},0,\cdots,-\kappa^{a,\varepsilon}_{kb}}_{l},0,\cdots,0)=R_3,\quad \mathrm{curl}\,(\kappa^{a,\varepsilon}_{1b},\kappa^{a,\varepsilon}_{2b},\cdots,\kappa^{a,\varepsilon}_{db})=R_4,$$

$$\mathrm{div}\,(\underbrace{\overbrace{0,\cdots,0,h^{b,\varepsilon}_{li}}^{k},0,\cdots,-h^{b,\varepsilon}_{ki}}_{l},0,\cdots,0)=R_5,\quad \mathrm{curl}\,(h^{b,\varepsilon}_{1i},h^{b,\varepsilon}_{2i},\cdots,h^{b,\varepsilon}_{di})=R_6,$$

$$\mathrm{div}\,(\underbrace{\overbrace{0,\cdots,0,\kappa^{b,\varepsilon}_{lc}}^{k},0,\cdots,-\kappa^{b,\varepsilon}_{kc}}_{l},0,\cdots,0)=R_7,\quad \mathrm{curl}\,(\kappa^{b,\varepsilon}_{1c},\kappa^{b,\varepsilon}_{2c},\cdots,\kappa^{b,\varepsilon}_{dc})=R_8.$$

$$\mathrm{div}\,(\underbrace{\overbrace{0,\cdots,0,h^{a,\varepsilon}_{lj}}^{k},0,\cdots,-h^{a,\varepsilon}_{kj}}_{l},0,\cdots,0)=R_1,\quad \mathrm{curl}\,(h^{a,\varepsilon}_{1j},h^{a,\varepsilon}_{2j},\cdots,h^{a,\varepsilon}_{dj})=R_2,$$

$$\mathrm{div}\,(\underbrace{\overbrace{0,\cdots,0,\kappa^{a,\varepsilon}_{lb}}^{k},0,\cdots,-\kappa^{a,\varepsilon}_{kb}}_{l},0,\cdots,0)=R_3,\quad \mathrm{curl}\,(\kappa^{a,\varepsilon}_{1b},\kappa^{a,\varepsilon}_{2b},\cdots,\kappa^{a,\varepsilon}_{db})=R_4,$$

$$\mathrm{div}\,(\underbrace{\overbrace{0,\cdots,0,h^{b,\varepsilon}_{li}}^{k},0,\cdots,-h^{b,\varepsilon}_{ki}}_{l},0,\cdots,0)=R_5,\quad \mathrm{curl}\,(h^{b,\varepsilon}_{1i},h^{b,\varepsilon}_{2i},\cdots,h^{b,\varepsilon}_{di})=R_6,$$

$$\mathrm{div}\,(\underbrace{\overbrace{0,\cdots,0,\kappa^{b,\varepsilon}_{lc}}^{k},0,\cdots,-\kappa^{b,\varepsilon}_{kc}}_{l},0,\cdots,0)=R_7,\quad \mathrm{curl}\,(\kappa^{b,\varepsilon}_{1c},\kappa^{b,\varepsilon}_{2c},\cdots,\kappa^{b,\varepsilon}_{dc})=R_8.$$

Weak Convergence: Div-Curl $\Rightarrow$

$$\begin{aligned} h^{a,\varepsilon}_{lj}h^{b,\varepsilon}_{ki}-h^{a,\varepsilon}_{kj}h^{b,\varepsilon}_{li} &\rightharpoonup h^{a}_{lj}h^{b}_{ki}-h^{a}_{kj}h^{b}_{li},\\ \kappa^{a,\varepsilon}_{kb}\kappa^{b,\varepsilon}_{lc}-\kappa^{a,\varepsilon}_{lb}\kappa^{b,\varepsilon}_{kc} &\rightharpoonup \kappa^{a}_{kb}\kappa^{b}_{lc}-\kappa^{a}_{lb}\kappa^{b}_{kc},\\ \kappa^{a,\varepsilon}_{kb}h^{b,\varepsilon}_{li}-\kappa^{a,\varepsilon}_{lb}h^{b,\varepsilon}_{ki} &\rightharpoonup \kappa^{a}_{kb}h^{b}_{li}-\kappa^{a}_{lb}h^{b}_{ki} \end{aligned}$$

Weak Continuity of Nonlinear Functions/Functionals

- **Limiting Surfaces in Geometry**:

  The weak limit of isometrically embedded surfaces is still an isometrically embedded surface in $\mathbb{R}^d$ for any Riemann curvature tensor R_{ijkl} without restriction

- **Motivation/Connection: Theory of Polyconvexity in Nonlinear Elasticity by John Ball (1977):**

  Weak Continuity of Determinants $\cdots$

- **Stronger Compactness Framework for the Gauss-Codazzi-Ricci System (CSD 2010)**: **Given any sequence of approximate solutions to this system which is uniformly bounded in L^2 and has reasonable bounds on the errors made in the approximation (the errors are confined in a compact subset of $H^{-1}_{\mathbf{loc}}$), then the approximating sequence has a weakly convergent subsequence whose limit is still a solution of the Gauss-Codazzi-Ricci system.**

Concluding Remarks

Nonlinear Partial Differential Equations of Mixed Hyperbolic-Elliptic Type, or even No Type,

naturally arise in many fundamental problems in

Fluid Mechanics
Differential Geometry
Elasticity, Relativity, String Theory
Optimization, Dynamical Systems
......

The solution to these fundamental problems in the areas greatly requires a **deep understanding of**

Nonlinear Partial Differential Equations of Mixed Hyperbolic-Elliptic Type

Concluding Remarks: Conti.

- During the last half century, the two different types of nonlinear PDEs have been separately studied.
  Focuss: Mathematical tools to understand different properties of solutions; **Great progress has been made**.
- With these achievements, my message is that **it is the time**:
  To revisit/attack the nonlinear PDEs of mixed type;
  To explore possible unifying mathematical approaches ideas, and techniques to deal with such problems.
  In particular, I have presented several fundamental examples of such PDEs, which indicate that **some of the mixed-type problems have been ready to be tractable.**
- **Many important mixed-type problems are wide open and very challenging, which require further new ideas, approaches, techniques, ..., and deserve our special attention and true effort.**