Steady Water Waves with Prescribed Distribution of Vorticity

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I. Nothing You Didn't Know Already

The Euler equations for the velocity field of an incompressible fluid in a conservative force field $-\nabla\Phi$ is

$$\rho\left(\frac{\partial}{\partial t}+\vec{u}\cdot\nabla\right)\vec{u}+\nabla\Phi+\nabla p=0, \qquad \nabla\cdot\vec{u}=0$$

and in 2D the vorticity, $\vec{\zeta}=\nabla\times\vec{u}=\zeta\vec{k}$ satisfies,

$$\vec{\zeta}\times\vec{u}=\vec{0} \text{ and the transport equation } \zeta_t+\vec{u}\cdot\nabla\zeta=0$$

It follows, as Lord Kelvin knew, that

$$\int_{\Omega(t)}\lambda(\zeta(t))\,dx_1dx_2=\int_{\Omega(0)}\lambda(\zeta(0))\,dx_1dx_2$$

for any reasonable function λ

The vorticity distribution function is conserved under smooth evolution of the Euler equation

Stream function formulation

in a simply connected domain in $\mathbb{R}^2$

$\nabla \cdot \vec{u} = 0$ implies that there is a stream function ψ with

$$\nabla^{\perp}\psi = -\vec{u}$$

where $\nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1})$ and the Euler equations become

$$\zeta_t - (\nabla^{\perp}\psi) \cdot \nabla\zeta = 0, \qquad -\Delta\psi = \zeta$$

For a steady flow $\zeta_t = 0$

$$\boxed{(\nabla^{\perp}\psi) \cdot \nabla\zeta = 0, \qquad -\Delta\psi = \zeta}$$

vorticity is constant on level sets of ψ

II. Rearrangements and Distribution Functions

Definition: Suppose Ω_1 and Ω_2 have the same finite measure
Then $\zeta_1 : \Omega_1 \to \mathbb{R}$ and $\zeta_2 : \Omega_2 \to \mathbb{R}$ are rearrangements of one another $\zeta_1 \sim \zeta_2$ if

$$Z_1(a) := \text{meas}\,\{\zeta_1 > a\} = \text{meas}\,\{\zeta_2 > a\} =: Z_2(a) \quad \text{for all } a \in \mathbb{R}$$

Equivalently the distribution functions are equal: $Z_1 = Z_2$

If $\zeta_1 \sim \zeta_2$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is Borel, the $\varphi \circ \zeta_1 \sim \varphi \circ \zeta_2$
Definition A measurable function ζ on Ω has a decreasing rearrangement $\zeta_{®}$ on the interval $[0, |\Omega|)$

$\zeta_{®}$ is unique except for its values at points where it jumps

Rearrangements in $L^2(\Omega)$

For a fixed $\zeta^* \in L^2(\Omega)$ let $\mathcal{R}(\Omega) \subset L^2(\Omega)$ be the set of ζ on Ω with $\zeta \sim \zeta^*$

Let $\overline{\mathcal{R}(\Omega)}^w$ denote the weak closure of $\mathcal{R}(\Omega)$ in $L^2(\Omega)$

- $\overline{\mathcal{R}(\Omega)}^w$ coincides with the closed convex hull of $\mathcal{R}(\Omega)$
- $\overline{\mathcal{R}(\Omega)}^w$ is weakly sequentially compact
- $\mathcal{R}(\Omega)$ is the set of extreme points of $\overline{\mathcal{R}(\Omega)}^w$
- $\overline{\mathcal{R}(\Omega)}^w = \Big\{\zeta \in L^1(\Omega) : \int_0^s \zeta_{Ⓡ} \leqslant \int_0^s \zeta^*_{Ⓡ},\ s \in (0, |\Omega|),$ and $\int_0^{|\Omega|} \zeta_{Ⓡ} = \int_0^{|\Omega|} \zeta^*_{Ⓡ}\Big\}$
- A rearrangement of an element of $\overline{\mathcal{R}(\Omega)}^w$ is again in $\overline{\mathcal{R}(\Omega)}^w$
- for all $\zeta \in \mathcal{R}(\Omega)$ and $\psi \in L^2(\Omega)$, $\displaystyle\int_\Omega \zeta\,\psi \leqslant \int_0^{|\Omega|} \zeta^*_{Ⓡ}\psi_{Ⓡ}$

Suppose $\psi \in L^2(\Omega)$ and $\varphi \circ \psi =: \zeta \in \mathcal{R}(\Omega)$ for an increasing function φ

- $\displaystyle\int_\Omega \zeta\,\psi = \int_0^{|\Omega|} \zeta^*_{®}\psi_{®}$

  and
- $\zeta \in \mathcal{R}(\Omega)$ is the <u>unique maximizer on $\overline{\mathcal{R}(\Omega)}^w$</u> of $\displaystyle\int_\Omega \zeta\,\psi$

Theorem: Suppose that $-\Delta\psi > O$ a.e. on Ω and that $\zeta \in \overline{\mathcal{R}(\Omega)}^w$ maximizes $\displaystyle\int_\Omega \zeta\psi$ on $\overline{\mathcal{R}(\Omega)}^w$.
Then $\zeta \in \mathcal{R}(\Omega)$

III. Linear Elliptic Boundary-Value Problem

- a 2π-periodic Jordan curve $\mathscr{S}$ in the open upper half plane
- the region $\mathbf{\Omega}$ between $\mathscr{S}$ and the x_1-axis
- Ω one period of $\mathbf{\Omega}$; $\mathcal{S}$ one period of $\mathscr{S}$
- $\mu \in \mathbb{R}$ and $\zeta \in L^2_{\text{loc}}(\mathbf{\Omega})$ and 2π-periodic in x_1

- $\psi \in W^{1,2}(\Omega)$ the weak solution of

$$\left\{ \begin{array}{c} -\Delta\psi = \zeta \text{ on } \mathbf{\Omega} \\ \psi(x_1, 0) = 0 \\ \psi \equiv C(\psi), \text{ a constant, on } \mathscr{S} \\ \psi \text{ is } 2\pi\text{-periodic in } x_1 \\ \int_{\mathcal{S}} \nabla\psi \cdot n dS = \mu \end{array} \right\} \tag{$\star$}$$

Free-boundary problem for water waves

To find curves $\mathscr{S}$ such that the unique solutions of elliptic problem

$$\left\{ \begin{array}{c} -\Delta\psi = \zeta \text{ on } \mathbf{\Omega} \\ \psi(x_1, 0) = 0 \\ \psi \equiv C(\psi),\ \text{a constant, on } \mathscr{S} \\ \psi \text{ is } 2\pi\text{-periodic in } x_1 \\ \int_{\mathcal{S}} \nabla\psi \cdot n dS = \mu \end{array} \right\} \qquad (\star)$$

have the additional properties that

the vorticity ζ is constant on level sets of ψ

the pressure $-\frac{1}{2}|\nabla\psi(x_1, x_2)|^2 - g\,x_2$ is constant on $\mathcal{S}$

This – the Bernoulli boundary condition – is really a Neumann boundary condition because ψ is constant on $\mathcal{S}$

The constant $C(\psi)$ and the function λ are <u>not</u> prescribed they are to be found

Surface Tension and Hydroelastic Waves

In the presence of simple surface tension with coefficient T, the Bernoulli boundary condition is

$$\frac{1}{2}|\nabla\psi(x_1,x_2)|^2 + g\,x_2 - \boxed{T\sigma(x_1,x_2)} = \text{ constant on } \mathcal{S},$$

where $\sigma(x_1,x_2)$ is the $\boxed{\textit{curvature}}$ of $\mathcal{S}$ at (x_1,x_2)
More generally if an elastic membrane that nonlinearly resists stretching and bending is on the surface the Bernoulli condition at points of $\mathcal{S}$ is

$$\frac{1}{2}|\nabla\psi(x_1,x_2)|^2 + g\,x_2 + E\left(\sigma'' + \frac{1}{2}\sigma^3\right) - \beta T(\ell(\mathcal{S}) - 2\pi)^{\beta-1}\sigma = \text{ constant}$$

where $'$ denotes differentiation with respect to arc length

$\ell(\mathcal{S}) =$ length of $\mathcal{S}$

$E \geqslant 0$ is a coefficient of bending resistance

$T \geqslant 0$ and $\beta \geqslant 1$ measure nonlinear resistance to stretching and compression

Traditional approach

From pioneering work in the 1930s by Marie-Louise Dubreil-Jacotin to global bifurcation theory by Adrian Constantin & Walter Strauss in this century, the common practice was to <u>prescribe</u> dependence of vorticity on the stream function

$$\zeta = \lambda(\psi) \text{ for a given function } \lambda$$

This yields a free-boundary problem for solutions of a semi-linear elliptic equation with fixed nonlinearity λ and over-determined boundary conditions

- $-\Delta\psi = \lambda(\psi)$ on Ω
- $\psi(x_1, 0) = 0$
- $\psi \equiv C(\psi)$, a constant, on $\mathscr{S}$
- ψ is 2π-periodic in x_1,
- $\int_{\mathcal{S}} \nabla\psi \cdot n dS = \mu$
- $\frac{1}{2}|\nabla\psi(x_1, x_2)|^2 + g x_2$ constant on $\mathscr{S}$

$\zeta = \lambda \circ \psi$ ensures vorticity is constant on streamlines

Problems with λ and the semi-linear approach

- λ is chosen to facilitate existence-theory - not to reflect physical reality
- λ has no rôle in the initial-value problem - in initial data there may be no relation between vorticity and stream function
- Only quantities conserved by the initial-value problem should be prescribed
- Different solutions ψ_1 and ψ_2 of the semi-linear problem with the <u>same</u> λ are not related if $\lambda(\psi_1)$ and $\lambda(\psi_2)$ are in different rearrangement classes – they cannot exchange stability by the Euler equations

For periodic steady waves, conserved quantities are

- cross-sectional area of one period
- circulation per period on free streamlines
- at positive time vorticity is a rearrangement of the initial vorticity

$\mu \in \mathbb{R}, \quad Q > 0, \quad \Omega_* = (-\pi, \pi) \times (0, Q), \quad \zeta^* \in L_2(\Omega_*)$

We seek solutions for which

- the area-per-period, $\text{meas}\,(\Omega) = 2\pi Q$
- the circulation-per-period: $\displaystyle\int_{\mathcal{S}} \nabla\psi \cdot n \, dS = \mu$
- the vorticity $\zeta \in \mathcal{R}(\Omega)$, the set of rearrangements on Ω of ζ^*

there is no λ in this formulation of the problem

Warning: for given ζ^* it is easy find a rearrangement ζ that is independent of x_1 and using ODE methods to construct ψ which also is a function of x_2 only. The existence of λ is then obvious.
These are called parallel-flow solutions – they are not of interest

Non-trivial solutions are not functions of x_2 only.

Energies: Let Ω denote one period of $\mathbf{\Omega}$

The energy in one period of a wave is $KE + PE + SE$ where

- the kinetic energy is
$$KE = \frac{1}{2}\int_{\Omega} |\nabla\psi|^2 dx_1 dx_2 \quad \text{where } \psi \text{ satisfies } (\star)$$
- the gravitational potential energy is
$$PE = g\int_{\Omega} x_2 \; dx_1 dx_2,$$
- and the surface energy is
$$SE = E\int_{\mathcal{S}} |\sigma|^2 \, ds + T(\ell(\mathcal{S}) - 2\pi)^{\beta} =: \mathcal{E}(\mathcal{S}),$$
  where $\ell(\mathcal{S})$ is the length of one period of $\mathscr{S}$, σ is curvature and $\beta \geqslant 1$.
- $E = 0$ and $\beta = 1$ corresponds to simple surface tension. More generally $\mathcal{E}$ is surface energy due to bending and stretching

If ψ and ζ satisfy the linear boundary-value problem $(\star)$

$$\left\{ \begin{array}{c} -\Delta\psi = \zeta \text{ on } \mathbf{\Omega} \\ \psi(x_1, 0) = 0 \\ \psi \equiv C(\psi), \text{ a constant, on } \mathscr{S} \\ \psi \text{ is } 2\pi\text{-periodic in } x_1 \\ \int_{\mathcal{S}} \nabla\psi \cdot n dS = \mu \end{array} \right\} \qquad (\star)$$

then

$$\begin{aligned} KE = \frac{1}{2}\int_\Omega |\nabla\psi|^2 dx_1 dx_2 \\ = \int_\Omega \zeta\psi dx_1 dx_2 - \frac{1}{2}\int_\Omega |\nabla\psi|^2 dx_1 dx_2 + \mu\psi\Big|_{\mathcal{S}} \end{aligned}$$

Minimax principle

Therefore we seek extrema of a Lagrangian

$$\mathcal{L}(\Omega, \psi, \zeta) = \boldsymbol{E} \int_{\mathcal{S}} |\sigma|^2 \, \boldsymbol{ds} + \boldsymbol{T}(\ell(\mathcal{S}) - \boldsymbol{2}\pi)^{\beta} + \boldsymbol{g} \int_{\Omega} \boldsymbol{x_2} \, \boldsymbol{dx_1} \boldsymbol{dx_2}$$
$$- \frac{\boldsymbol{1}}{\boldsymbol{2}} \int_{\Omega} |\nabla \psi|^2 \boldsymbol{dx_1} \boldsymbol{dx_2} + \int_{\Omega} \zeta \psi \boldsymbol{dx_1} \boldsymbol{dx_2} + \mu \boldsymbol{C}(\psi)$$

by examining a variational problem

$$\min_{\Omega \in \mathfrak{O}, \, \zeta \in \mathcal{R}(\Omega)} \max_{\psi \in \mathcal{A}(\Omega)} \mathcal{L}(\Omega, \psi, \zeta),$$

where

- $\mathcal{R}(\Omega)$ is the rearrangements on Ω of ζ^*,
- $\mathcal{A}(\Omega)$ stream functions satisfying boundary conditions:
  zero on bottom, constant on top
- $\mathfrak{O}$ is an admissible class of domains with area $2\pi Q$

VI. Results

Although this is not a smooth problem a first-variation condition for its extremals yields

- The weak form $-\Delta\psi = \zeta$ which says that vorticity is the curl of the velocity.
- A decreasing function λ with $\zeta = \lambda(\psi)$ the infinite-dimensional Lagrange multiplier corresponding to having prescribed the vorticity's distribution function
- A weak form of the Bernoulli boundary conditions involving parameters g, E, T and β.

If $|\mu|$ is sufficiently large the minimax solution is not parallel

There is no distinction between the theory of square-integrable vorticity and irrotational theory with zero vorticity.

Remarks

This weak formulation of the free-boundary problem, independent of the existence question, is valid for all values of T, $E \geqslant 0$ and $\beta \geqslant 1$ and any prescribed square-integrable function .

For existence theory we use the convexity of the weak closure $\overline{\mathcal{R}(\Omega)}^{w}$ of the constraint set $\mathcal{R}(\Omega)$ and that every rearrangement on Ω of a function in $\overline{\mathcal{R}(\Omega)}^{w}$ also belongs to $\overline{\mathcal{R}(\Omega)}^{w}$

We illustrate it in the simplest case, when vorticity is essentially one-signed and E, $T > 0$ and $\beta > 1$ have values that ensure coercivity which leads to existence of weak solutions of via the direct method of the calculus of variations.

Interesting consequence

Suppose $\zeta^* \geqslant 0$ and support of $\zeta^* \geqslant 0$ has measure in $(0, 2\pi Q)$
For example ζ^* might be a simple function taking two values, one of which is zero, on sets of positive measure.

Suppose also that $\mu > 0$.

By the maximum principle $\psi \geqslant 0$ on $\boldsymbol{\Omega}$ and $\psi > 0$ on $\mathscr{S}$.

The function λ given by the minimax principle is *decreasing*

Hence the region of non-zero vorticity is next to the bottom

the flow is irrotational where the stream function is near its maximum

This is a satisfactory state of affairs since vorticity is commonly associated with a boundary layer near to the bottom

VI. The Variational Set-Up

admissible vector fields, diffeomorphisms and domains

A vector field ω on $\mathbb{R}^2$ is admissible if $\omega = \nabla^\perp \Phi$ where Φ is smooth, 2π-periodic and zero on the line $\{x_2 = 0\}$.

Thus ω is solenoidal (divergence free) and induces an area preserving flow as follows:

For t in a maximal interval of existence, let $\tau(t) : \mathbb{R}^2 \to \mathbb{R}^2$ be the diffeomorphism defined by

$$\tau(t)(x_1, x_2) = (X_1(t), X_2(t))$$

where, with $\dot{}$ denoting differentiation with respect to t,

$$(\dot{X}_1(t), \dot{X}_2(t)) = \omega(X_1(t), X_2(t)), \quad (X_1(0), X_2(0)) = (x_1, x_2).$$

$\tau(t)$ is area-preserving and leaves the line $\{x_2 = 0\}$ invariant

Admissible domains

Let $\mathbb{R}^2_+$ denote the open upper half plane. A class $\mathfrak{O}$ of open sets $\mathbf{\Omega} \subset \mathbb{R}^2_+$ will be called admissible if it has the following properties:

- $\mathbf{\Omega}_* \in \mathfrak{O}$;
- $(2\pi, 0) + \mathbf{\Omega} = \mathbf{\Omega}$, $\mathbf{\Omega} \in \mathfrak{O}$;
- Ω is bounded and $\operatorname{meas} \Omega = 2\pi Q$;
- the boundary of $\mathbf{\Omega} \cup (\mathbb{R} \times (-\infty, 0])$ is connected and rectifiable. Let $\ell(\mathcal{S})$ denote the length of one period;
- if $\mathbf{\Omega} \in \mathfrak{O}$ and ω is an admissible vector field, then there exists $\epsilon > 0$ such that $\tau(t)\mathbf{\Omega} \in \mathfrak{O}$ for all $t \in (-\epsilon, \epsilon)$

Admissible functions

For $\Omega \in \mathfrak{O}$ let $\mathcal{A}(\Omega)$ be the space of $W^{1,2}_{\text{loc}}(\mathbf{\Omega})$-functions ψ with

$\psi = 0$ on $x_2 = 0$

$\psi = C(\psi)$, a constant, on $\mathcal{S}$ - the top boundary

By the Poincaré inequality,

$$\langle u, v\rangle = \int_\Omega \nabla u \cdot \nabla v \, dx_1 \, dx_2$$

defines an inner product on $\mathcal{A}(\Omega)$

Lagrangian and the minimax problem

$$\mathcal{L}(\Omega,\psi,\zeta) = E\int_{\mathcal{S}} |\sigma|^2\, ds + T(\ell(\mathcal{S})-2\pi)^\beta + g\int_\Omega x_2\; dx_1 dx_2$$
$$-\frac{1}{2}\int_\Omega |\nabla\psi|^2 dx_1 dx_2 + \int_\Omega \zeta\psi dx_1 dx_2 + \mu C(\psi)$$

For $\Omega \in \mathfrak{O}$ and $\zeta \in \mathcal{R}(\Omega)$ there exists $\boxed{\overline{\psi}(\Omega,\zeta) \in \mathcal{A}(\Omega)}$ which maximizes the blue part and therefore

$$0 = -\int_\Omega \nabla\overline{\psi}\cdot\nabla\phi\, dx_1 dx_2 + \int_\Omega \phi\zeta\, dx_1 dx_2 + \mu C(\phi) \text{ for } \phi \in \mathcal{A}(\Omega)$$

the weak form of the linear problem $(\star)$:

- $-\Delta\psi = \zeta$ on $\boldsymbol{\Omega}$
- $\psi(x_1, 0) = 0$
- $\psi \equiv C(\psi),$ a constant, on $\mathscr{S}$,
- ψ is 2π-periodic in x_1,
- $\int_{\mathcal{S}} \nabla\psi \cdot n dS = \mu$,

Minimization

after maximization

$\zeta \mapsto \overline{\psi}$ is affine and bounded from $L^2(\Omega) \to \mathcal{A}(\Omega)$,

$$\mathcal{L}(\Omega, \overline{\psi}(\Omega,\zeta),\zeta) = \mathcal{E}(\mathcal{S}) + g\int_\Omega x_2 \, dx_1 dx_2 + \frac{1}{2}\int_\Omega |\nabla\overline{\psi}(\Omega,\zeta)|^2 dx_1 dx_2$$

let $\overline{\mathcal{R}(\Omega)}^w$ denote the weak-$L^2(\Omega)$ closure of $\mathcal{R}(\Omega)$, $\Omega \in \mathfrak{O}$

$\overline{\mathcal{R}(\Omega)}^w$ is a convex set.

Suppose $\underline{\zeta} \in \overline{\mathcal{R}(\underline{\Omega})}^w$, $\underline{\Omega} \in \mathfrak{O}$, and

$$\mathcal{L}(\underline{\Omega}, \overline{\psi}(\underline{\Omega},\underline{\zeta}),\underline{\zeta}) \leqslant \mathcal{L}(\Omega, \overline{\psi}(\Omega,\zeta),\zeta),$$

for all domains $\Omega \in \mathfrak{O}$ and all rearrangements ζ of ζ^* on Ω.

Vorticity function as Lagrange multiplier

Lemma

Suppose that $\underline{\zeta} \in \overline{\mathcal{R}(\underline{\Omega})}^{w}$ *and*

$$\mathcal{L}(\underline{\Omega}, \overline{\psi}(\underline{\Omega}, \underline{\zeta}), \underline{\zeta}) \leqslant \mathcal{L}(\underline{\Omega}, \overline{\psi}(\underline{\Omega}, \zeta), \zeta) \text{ for all } \zeta \in \mathcal{R}(\underline{\Omega}).$$

Then there exists a decreasing function λ *such that*

$$\underline{\zeta} = \lambda \circ \overline{\psi}(\underline{\Omega}, \underline{\zeta}) \tag{0.1}$$

If ζ^* *is positive almost everywhere then* $\underline{\zeta} \in \mathcal{R}(\underline{\Omega})$.

Minimax principle gives weak solutions

Suppose $\mathfrak{O}$ is admissible and that

$$\min_{\Omega\in\mathfrak{O},\,\zeta\in\mathcal{R}(\Omega)} \max_{\psi\in\mathcal{A}(\Omega)} \mathcal{L}(\Omega,\psi,\zeta) = \mathcal{L}(\underline{\Omega},\overline{\psi}(\underline{\Omega},\underline{\zeta}),\underline{\zeta}).$$

Then $\underline{\Omega}\in\mathfrak{O}$, $\underline{\zeta}\in\mathcal{R}(\underline{\Omega})$ and $\overline{\psi}(\underline{\Omega},\underline{\zeta})\in\mathcal{A}(\underline{\Omega})$ is a weak solution.

Theorem

Suppose that $Q>0$, $\zeta^*\in L^2(\Omega_*)$ is essentially one-signed, $E>0$ and $\beta>1$. Then, for $|\mu|$ sufficiently large, determined by Q, E, $\|\zeta^*\|_{L^2(\Omega_*)}$ and g (but not T), and for T sufficiently large, determined by μ, Q, E, $\|\zeta^*\|_{L^2(\Omega_*)}$ and g, there exists a weak solution which is not a parallel flow.

Caveat

There is a long way from this theorem to the existence of classical solutions of in water-wave theory when surface elasticity is absent. The difficulties involves regularity questions for weak solutions, which are independent of the rôle of vorticity, when $E = T = 0$

What we have done is to formulate the classical water wave problem as a modern **shape optimization problem**

When $E = T = 0$ I don't believe that a minimax solution exists

The question remains: what is the correct variational problem that leads to steady water waves

Conclusions

Two-dimensional Steady Rotational Surface Waves

- the dependence of vorticity on the stream function is not given data but is part of the solution that arises from a constraint on the vorticity
- Motivated by a variational principle a notion of weak solution is introduced
- Non-trivial weak solutions are shown to arise from solutions of a minimax principle.
- the existence of non-trivial waves with a prescribed distribution of vorticity on the surface of a fluid confined beneath an elastic sheet is proved.
- The existence of weak solutions of the corresponding irrotational-wave problem (zero vorticity) is a special case.

V.III Historical Perspective

This approach may be traced to a discussion by Kelvin of a principle that, among patches of fixed constant vorticity and equal area in otherwise irrotational fluid, those that provide extrema of kinetic energy represent steady flows.

The modern formulation for general vorticity, involving rearrangements of functions, is due to Arnol'd in the 1960s

In the 1970s Benjamin adapted Arnol'd's ideas to three-dimensional axisymmetric vortex rings in a uniform flow, and proposed a strategy for proving an existence theorem for vortex rings

The analysis background to underpin Benjamin's ideas and existence theorems in a similar vein to those he envisaged, are due to Burton in the 1980s

The present contribution may be the first application of these ideas to free boundary problems.

Probably The End

Thank You