Monotonicity methods for asymptotics of solutions to elliptic and parabolic equations near singularities of the potential

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Problem:

describe the behavior at singularity of solutions to equations associated to Schrödinger operators with singular homogeneous potentials

$$\mathcal{L}_a := -\Delta - \frac{a(x/|x|)}{|x|^2}, \quad x \in \mathbb{R}^N, \quad a : \mathbb{S}^{N-1} \to \mathbb{R}, \quad N \geqslant 3$$

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Examples: **Dipole-potential**

$$\boxed{-\frac{\hbar^2}{2m}\Delta + e\,\frac{x \cdot \mathbf{D}}{|x|^3}} \qquad a(\theta) = \lambda\theta \cdot \mathbf{D}$$

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Examples: **Quantum many-body**

$$\boxed{\sum_{j=1}^{M} \frac{-\Delta_j}{2m_j} + \sum_{\substack{j,m=1\\ j<m}}^{M} \frac{\lambda_j \lambda_m}{|x^j - x^m|^2}} \qquad a(\theta) = \sum \frac{\lambda_j \lambda_m}{|\theta^j - \theta^m|^2}$$

$$x^j \in \mathbb{R}^d, \quad N = Md, \quad \theta^j = x^j/|x|, \quad x = (x^1, \dots, x^M) \in \mathbb{R}^N$$

Problem:

describe the asymptotic behavior at the singularity of solutions to equations associated to Schrödinger operators with singular homogeneous **electromagnetic** potentials

$$\mathcal{L}_{\mathbf{A},a} := \left(-i\nabla + \frac{\mathbf{A}\left(\frac{x}{|x|}\right)}{|x|} \right)^2 - \frac{a\left(\frac{x}{|x|}\right)}{|x|^2}, \quad \mathbf{A} \in C^1(\mathbb{S}^{N-1}, \mathbb{R}^N)$$

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Example: **Aharonov-Bohm magnetic potentials** associated to thin solenoids; if the radius of the solenoid tends to zero while the flux through it remains constant, then the particle is subject to a δ-type magnetic field, which is called Aharonov-Bohm field.

$$\frac{\mathbf{A}\left(\frac{x}{|x|}\right)}{|x|} = \boldsymbol{\alpha}\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right), \quad x = (x_1, x_2) \in \mathbb{R}^2$$

$$\mathbf{A}(\theta_1, \theta_2) = \boldsymbol{\alpha}(-\theta_2, \theta_1), \quad (\theta_1, \theta_2) \in \mathbb{S}^1,$$

with $\boldsymbol{\alpha} =$ circulation around the solenoid.

Describe the behavior at singularities of solutions: **Motivation**

- **regularity theory** for elliptic operators with singularities of Fuchsian type

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- **construction of solutions** to equations with singular potentials

  [**Felli-Terracini**, Comm. PDE's(2006)]
  [**Felli**, J. Anal. Math.(2009)]
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- study of **spectral properties** (essential self-adjointness)

  [**Felli-Marchini-Terracini**, J. Funct. Anal.(2007)]
  [**Felli-Marchini-Terracini**, Indiana Univ. Math. J.(2009)]

Describe the behavior at singularities of solutions: **References**

[Felli-Schneider, Adv. Nonl. Studies (2003)]: Hölder continuity results for degenerate elliptic equations with singular weights; asymptotics of solutions for potentials $\lambda|x|^{-2}$ ($a(\theta)$ constant).

[Felli-Marchini-Terracini, DCDS-A (2008)]: asymptotics of solutions near the pole for $a(x/|x|)|x|^{-2}$ ($a(\theta)$ bounded), through separation of variables and comparison methods.

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Difficulties:

- in the presence of a singular magnetic potential, comparison methods are no more available!
- in the many-particle/cylindrical case separation of variables (radial and angular) does not “eliminate” the singularity; the angular part of the operator is also singular!

To overcome these difficulties

Almgren type monotonicity formula
blow-up methods

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[Felli-Ferrero-Terracini, J. Europ. Math. Soc. (2011)]: singular homogeneous electromagnetic potentials of Aharonov-Bohm type, by an Almgren type monotonicity formula and blow-up methods.

[Felli-Ferrero-Terracini, Preprint 2010]: behavior at collisions of solutions to Schrödinger equations with many-particle and cylindrical potentials by a nonlinear Almgren type formula and blow-up.

[Felli-Primo, DCDS-A, to appear]: local asymptotics for solutions to heat equations with inverse-square potentials.

Outline of the talk

1. Elliptic monotonicity formula
2. Asymptotics at singularities (elliptic case)
3. Parabolic monotonicity formula
4. Asymptotics at singularities (parabolic case)

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1. Elliptic monotonicity formula

Studying regularity of area-minimizing surfaces of codimension $\geqslant 1$, in 1979 F. Almgren introduced the *frequency function*

$$\mathcal{N}(r) = \frac{r^{2-N}\int_{B_r} |\nabla u|^2\,dx}{r^{1-N}\int_{\partial B_r} u^2}$$

and observed that, if u is harmonic, then $\boxed{\mathcal{N} \nearrow \text{ in } r}$.

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Proof:

$$\mathcal{N}'(r) = \frac{2r\left[\left(\int_{\partial B_r}\left|\frac{\partial u}{\partial \nu}\right|^2 dS\right)\left(\int_{\partial B_r}|u|^2 dS\right) - \left(\int_{\partial B_r} u \frac{\partial u}{\partial \nu} dS\right)^2\right]}{\left(\int_{\partial B_r}|u|^2 dS\right)^2} \underset{\text{inequality}}{\overset{\text{Schwarz's}}{\geqslant}} 0$$

$\square$

Why frequency?

If $\mathcal{N} \equiv \gamma$ is constant, then $\mathcal{N}'(r) = 0$, i.e.

$$\left(\int_{\partial B_r} \left|\frac{\partial u}{\partial \nu}\right|^2 dS\right) \cdot \left(\int_{\partial B_r} u^2 dS\right) - \left(\int_{\partial B_r} u \frac{\partial u}{\partial \nu} dS\right)^2 = 0$$

$\Longrightarrow$ u and $\frac{\partial u}{\partial \nu}$ are parallel as vectors in $L^2(\partial B_r)$, i.e. $\exists\ \eta(r)$ s.t.

$$\frac{\partial u}{\partial \nu}(r,\theta) = \eta(r) u(r,\theta), \quad \text{i.e.} \quad \frac{d}{dr} \log|u(r,\theta)| = \eta(r).$$

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$$\frac{\partial u}{\partial \nu}(r,\theta) = \eta(r) u(r,\theta), \quad \text{i.e.} \quad \frac{d}{dr} \log|u(r,\theta)| = \eta(r).$$

After integration we obtain

$$u(r,\theta) = e^{\int_1^r \eta(s)ds} u(1,\theta) = \varphi(r)\psi(\theta).$$

Why frequency?

$u(r,\theta) = \varphi(r)\psi(\theta)$ and $\Delta u = 0$

$$\Downarrow$$

$$\left(\varphi''(r) + \tfrac{N-1}{r}\varphi'(r)\right)\psi(\theta) + r^{-2}\varphi(r)\Delta_{\mathbb{S}^{N-1}}\psi(\theta) = 0.$$

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ψ is a spherical harmonic $\Rightarrow \exists\, k \in \mathbb{N} : -\Delta_{\mathbb{S}^{N-1}}\psi = k(N-2+k)\psi$

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$\varphi(r) = c_1 r^{\sigma^+} + c_2 r^{\sigma^-}$ with $\sigma^{\pm} = -\frac{N-2}{2} \pm \frac{1}{2}(2k+N-2)$

$\sigma^+ = k$, $\quad \sigma^- = -(N-2) - k$

$$|x|^{\sigma^-}\psi\left(\tfrac{x}{|x|}\right) \notin H^1(B_1) \rightsquigarrow c_2 = 0, \qquad \varphi(1) = 1 \rightsquigarrow c_1 = 1$$

$$\Downarrow$$

$$u(r,\theta) = r^k\psi(\theta)$$

From $\mathcal{N}(r) \equiv \gamma$, we deduce that $\gamma = k$.

Applications to elliptic PDE's

The Almgren monotonicity formula was used in

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- **[Garofalo-Lin, Indiana Univ. Math. J. (1986)]**: generalization to variable coefficient elliptic operators in divergence form (unique continuation)
- **[Athanasopoulos-Caffarelli-Salsa, Amer. J. Math. (2008)]**: regularity of the free boundary in obstacle problems.
- **[Caffarelli-Lin, J. AMS (2008)]** regularity of free boundary of the limit components of singularly perturbed elliptic systems.

Example (Garofalo-Lin). Let $u \in H^1_{\mathrm{loc}}(B_1)$ be a weak solution to

$$-\Delta u(x) + V(x)u = 0 \quad \text{in } B_1,$$

with $V \in L^\infty_{\mathrm{loc}}(B_1)$. Define

$$D(r) = \frac{1}{r^{N-2}} \int_{B_r} \left[\left| \nabla u \right|^2 + V|u|^2 \right] dx$$

$$H(r) = \frac{1}{r^{N-1}} \int_{\partial B_r} |u|^2 \, dS$$

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$$\mathcal{N}(r) = \frac{D(r)}{H(r)}$$

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Perturbed monotonicity

$$\mathcal{R}(r) = -\frac{2\Big[\int_{B_r} Vu\,(x\cdot\nabla u)\,dx + \frac{N-2}{2}\int_{B_r} Vu^2\,dx - \frac{r}{2}\int_{\partial B_r} Vu^2\,dS\Big]}{\int_{\partial B_r}|u|^2 dS}$$

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integrate between r and $\bar{r}$ $\quad \mathcal{N}(r) + C_2 \leqslant (\mathcal{N}(\bar{r}) + C_2)e^{\frac{C_1}{2}\bar{r}^2} \leqslant \text{const}$

In particular, $\mathcal{N}$ is bounded.

Perturbed monotonicity

$$H(r) = \frac{1}{r^{N-1}} \int_{\partial B_r} |u|^2 \, dS \quad \Longrightarrow \quad H'(r) = \frac{2}{r^{N-1}} \int_{\partial B_r} u \frac{\partial u}{\partial \nu} \, dS$$

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Test $-\Delta u(x) + V(x)u = 0$ with $u \quad \Longrightarrow$

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Test $-\Delta u(x) + V(x) u = 0$ with u $\quad \Longrightarrow$

$$r^{N-2} D(r) = \int_{B_r} \left[\left| \nabla u \right|^2 + V |u|^2 \right] dx = \int_{\partial B_r} u \frac{\partial u}{\partial \nu} \, dS = \frac{r^{N-1}}{2} H'(r)$$

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Hence $\frac{H'(r)}{H(r)} = \frac{2}{r} \mathcal{N}(r) \leqslant \frac{\mathrm{const}}{r}$, i.e. $\frac{d}{dr} \log H(r) \leqslant \frac{C}{r}$.

Integrating between R and $2R$

$$\log \frac{H(2R)}{H(R)} \leqslant C \log 2 \quad \text{i.e.} \quad \int_{\partial B_{2R}} u^2 \, dS \leqslant \mathrm{const} \int_{\partial B_R} u^2 \, dS$$

Doubling condition

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$$\Downarrow$$

Strong unique continuation property

If u vanishes of infinite order at 0, i.e.

$$\int_{B_R} u^2\,dx = O(R^m) \quad \text{as } R \to 0 \quad \forall\, m \in \mathbb{N},$$

then $u \equiv 0$ in B_1.

2. Asymptotics at singularities (elliptic case)

describe the behavior at the singularity of solutions to equations associated to Schrödinger operators with singular homogeneous potentials (with the same order of homogeneity of the operator)

$$\mathcal{L}_a := -\Delta - \frac{a(x/|x|)}{|x|^2}, \quad x \in \mathbb{R}^N, \quad a : \mathbb{S}^{N-1} \to \mathbb{R}, \quad N \geqslant 3$$

Hardy type inequalities

$$\Lambda(a) := \sup_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |x|^{-2} a(x/|x|)\, u^2(x)\, dx}{\displaystyle\int_{\mathbb{R}^N} |\nabla u(x)|^2\, dx}$$

- Classical Hardy's inequality: $\Lambda(1) = \left(\frac{2}{N-2}\right)^2$.

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- If $a \in L^\infty(\mathbb{S}^{N-1})$, classical Hardy's inequality $\Rightarrow \Lambda(a) < +\infty$.
- For many body potentials $\frac{a(x/|x|)}{|x|^2} = \sum_{j<m}^{M} \frac{\lambda_j \lambda_m}{|x^j - x^m|^2}$ $\xRightarrow[\text{inequalities}]{\text{Maz'ja and Badiale-Tarantello}}$ $\Lambda(a) < +\infty$.

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The quadratic form associated to

$$\mathcal{L}_a = -\Delta - \frac{a(x/|x|)}{|x|^2} \qquad \Longleftrightarrow \qquad \boxed{\Lambda(a) < 1}$$

is positive definite in $\mathcal{D}^{1,2}(\mathbb{R}^N)$

Perturbations of $\mathcal{L}_a$

linear/semilinear perturbations of $\mathcal{L}_a$: $\boxed{\mathcal{L}_a u = \boldsymbol{h}(\boldsymbol{x})\, u(x) + \boldsymbol{f}(\boldsymbol{x}, u(x))}$

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(H) h **"negligible" with respect to** $\frac{a(x/|x|)}{|x|^2}$ **at singularity:**

$$\lim_{r\to 0^+} \eta_0(r) = 0, \quad \frac{\eta_0(r)}{r} \in L^1, \quad \frac{1}{r}\int_0^r \frac{\eta_0(s)}{s}\, ds \in L^1, \quad \frac{\eta_1(r)}{r} \in L^1, \quad \frac{1}{r}\int_0^r \frac{\eta_1(s)}{s}\, ds \in L^1,$$

where

$$\eta_0(r) = \sup_{\substack{u\in H^1(B_r)\\ u\not\equiv 0}} \frac{\int_{B_r} |h(x)| u^2 dx}{\int_{B_r} \left(|\nabla u|^2 - \frac{a(x/|x|)}{|x|^2} u^2\right) dx + \frac{N-2}{2r} \int_{\partial B_r} u^2 dS},$$

$$\eta_1(r) = \sup_{\substack{u\in H^1(B_r)\\ u\not\equiv 0}} \frac{\int_{B_r} |\nabla h \cdot x| u^2 dx}{\int_{B_r} \left(|\nabla u|^2 - \frac{a(x/|x|)}{|x|^2} u^2\right) dx + \frac{N-2}{2r} \int_{\partial B_r} u^2 dS}$$

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(H) ***h*** **"negligible" with respect to $\frac{a(x/|x|)}{|x|^2}$ at singularity:**

$$\lim_{r\to 0+} \eta_0(r) = 0, \quad \frac{\eta_0(r)}{r} \in L^1, \quad \frac{1}{r}\int_0^r \frac{\eta_0(s)}{s}\, ds \in L^1, \quad \frac{\eta_1(r)}{r} \in L^1, \quad \frac{1}{r}\int_0^r \frac{\eta_1(s)}{s}\, ds \in L^1,$$

Examples:

- $h, (x \cdot \nabla h) \in L^s$ for some $s > N/2$
- $a \in L^\infty(\mathbb{S}^{N-1}) \rightsquigarrow h(x) = O(|x|^{-2+\varepsilon})$
- $\displaystyle \frac{a(x/|x|)}{|x|^2} = \sum_{j,m} \frac{\lambda_j \lambda_m}{|x^j - x^m|^2} \rightsquigarrow |h(x)| + |\nabla h \cdot x| = O\Big(\sum |x^j - x^m|^{-2+\varepsilon}\Big)$

Perturbations of $\mathcal{L}_a$

linear/semilinear perturbations of $\mathcal{L}_a$: $\boxed{\mathcal{L}_a u = h(x)\, u(x) + f(x, u(x))}$

(H) h **"negligible" with respect to $\frac{a(x/|x|)}{|x|^2}$ at singularity:**

$$\lim_{r\to 0^+} \eta_0(r) = 0, \quad \frac{\eta_0(r)}{r} \in L^1, \quad \frac{1}{r}\int_0^r \frac{\eta_0(s)}{s}\,ds \in L^1, \quad \frac{\eta_1(r)}{r} \in L^1, \quad \frac{1}{r}\int_0^r \frac{\eta_1(s)}{s}\,ds \in L^1,$$

Examples:

- $h, (x\cdot\nabla h) \in L^s$ for some $s > N/2$
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(F) f **at most critical:** $|f(x,s)s| + |f'_s(x,s)s^2| + |\nabla_x F(x,s)\cdot x| \leqslant C_f(|s|^2 + |s|^{2^*})$

where $F(x,s) = \int_0^s f(x,t)\,dt, \quad 2^* = \frac{2N}{N-2}.$

The angular operator

We aim to describe the rate and the shape of the singularity of solutions, by relating them to the eigenvalues and the eigenfunctions of a Schrödinger operator on $\mathbb{S}^{N-1}$ corresponding to the angular part of $\mathcal{L}_a$:

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Positivity of the quadratic form associated to $\mathcal{L}_a$ $\Lambda(a) < 1$	$\Longleftrightarrow$	$\mu_1(a) > -\left(\frac{N-2}{2}\right)^2$ **(PD)**

The Almgren type frequency function

In an open bounded $\Omega \ni 0$, let u be a $H^1(\Omega)$-weak solution to

$$\mathcal{L}_a u = h(x)\, u(x) + f(x, u(x))$$

For small $r > 0$ define

$$D(r) = \frac{1}{r^{N-2}} \int_{B_r} \left(|\nabla u(x)|^2 - \frac{a(\frac{x}{|x|})}{|x|^2} u^2(x) - h(x)\, u^2(x) - f(x, u(x))u(x) \right) dx,$$

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If **(PD)** holds and $u \not\equiv 0$,
$\Rightarrow H(r) > 0$ for small $r > 0$ $\rightsquigarrow$

Almgren type frequency function

$$\mathcal{N}(r) = \mathcal{N}_{u,h,f}(r) = \frac{D(r)}{H(r)}$$

is well defined in a suitably small interval $(0, r_0)$.

“Perturbed monotonicity”

$\mathcal{N}'$ is an integrable perturbation of a nonnegative function: enough to prove the existence of $\lim_{r\to 0^+} \mathcal{N}(r)$

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$\mathcal{N} \in W^{1,1}_{\mathrm{loc}}(0, r_0)$ and, in a distributional sense and for a.e. $r \in (0, r_0)$,

$$\mathcal{N}'(r) = \frac{2r\left[\left(\int_{\partial B_r}\left|\frac{\partial u}{\partial \nu}\right|^2 dS\right)\left(\int_{\partial B_r}|u|^2 dS\right) - \left(\int_{\partial B_r} u\frac{\partial u}{\partial \nu}\, dS\right)^2\right]}{\left(\int_{\partial B_r}|u|^2 dS\right)^2} + (\mathcal{R}_h + \mathcal{R}_f^1 + \mathcal{R}_f^2)(r)$$

$$\mathcal{R}_h(r) = -\frac{\int_{B_r}(2h(x) + \nabla h(x)\cdot x)|u|^2\, dx}{\int_{\partial B_r}|u|^2\, dS}, \quad \mathcal{R}_f^1(r) = \frac{r\int_{\partial B_r}\big(2F(x,u) - f(x,u)u\big)\, dS}{\int_{\partial B_r}|u|^2\, dS}$$

$$\mathcal{R}_f^2(r) = \frac{\int_{B_r}\big((N-2)f(x,u)u - 2NF(x,u) - 2\nabla_x F(x,u)\cdot x\big)\, dx}{\int_{\partial B_r}|u|^2\, dS}.$$

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$\mathcal{N}'$ = nonnegative function + L^1-function on $(0, \tilde{r}) \Longrightarrow$

$$\mathcal{N}(r) = \mathcal{N}(\tilde{r}) - \int_r^{\tilde{r}} \mathcal{N}'(s)\, ds$$

admits a limit γ as $r \to 0^+$ which is necessarily finite

Blow-up:

Set

$$w^\lambda(x) = \frac{u(\lambda x)}{\sqrt{H(\lambda)}}, \quad \text{so that } \int_{\partial B_1} |w^\lambda|^2 dS = 1.$$

$\{w^\lambda\}_{\lambda\in(0,\bar\lambda)}$ is bounded in $H^1(B_1) \Longrightarrow$ for any $\lambda_n \to 0^+$, $w^{\lambda_{n_k}} \rightharpoonup w$ in $H^1(B_1)$ along a subsequence $\lambda_{n_k} \to 0^+$, and $\int_{\partial B_1} |w|^2 dS = 1$.

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$$(E_k) \quad -\mathcal{L}_a w^{\lambda_{n_k}} = \lambda_{n_k}^2 h(\lambda_{n_k} x)\, w^{\lambda_{n_k}} + \frac{\lambda_{n_k}^2}{\sqrt{H(\lambda_{n_k})}} f\left(\lambda_{n_k} x, \sqrt{H(\lambda_{n_k})}\, w^{\lambda_{n_k}}\right)$$

$$\text{weak} \; \big\downarrow \; \text{limit}$$

$$(E) \quad \mathcal{L}_a w(x) = 0$$

$(E_k) - (E)$ tested with $w^{\lambda_{n_k}} - w \Rightarrow w^{\lambda_{n_k}} \to w$ in $H^1(B_1)$

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If $\mathcal{N}_k(r)$ the Almgren frequency function associated to (E_k) and $\mathcal{N}_w(r)$ is the Almgren frequency function associated to (E), then

$$\lim_{k\to\infty} \mathcal{N}_k(r) = \mathcal{N}_w(r) \quad \text{for all } r \in (0,1).$$

Blow-up:

By scaling

$$\mathcal{N}_k(r) = \mathcal{N}(\lambda_{n_k} r)$$

$$\Downarrow$$

$$\mathcal{N}_w(r) = \lim_{k\to\infty} \mathcal{N}(\lambda_{n_k} r) = \gamma \quad \forall r \in (0,1)$$

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Then $\mathcal{N}_w$ is constant in $(0,1)$ and hence $\mathcal{N}_w'(r) = 0$ for any $r \in (0,1)$

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$$\left(\int_{\partial B_r} \left|\frac{\partial w}{\partial \nu}\right|^2 dS\right) \cdot \left(\int_{\partial B_r} |w|^2 dS\right) - \left(\int_{\partial B_r} w\frac{\partial w}{\partial \nu} dS\right)^2 = 0$$

Therefore w and $\frac{\partial w}{\partial \nu}$ are parallel as vectors in $L^2(\partial B_r)$, i.e. $\exists$ a real valued function $\eta = \eta(r)$ such that $\frac{\partial w}{\partial \nu}(r,\theta) = \eta(r)w(r,\theta)$ for $r \in (0,1)$.

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After integration we obtain

$$w(r,\theta) = e^{\int_1^r \eta(s)ds} w(1,\theta) = \varphi(r)\psi(\theta).$$

Blow-up:

$$w(r,\theta) = \varphi(r)\psi(\theta)$$

Rewriting equation (E) $\mathcal{L}_a w(x) = 0$ in polar coordinates we obtain

$$\left(-\varphi''(r) - \frac{N-1}{r}\varphi'(r)\right)\psi(\theta) + r^{-2}\varphi(r)L_a\psi(\theta) = 0.$$

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Then ψ is an eigenfunction of the operator L_a.
Let $\mu_{k_0}(a)$ be the corresponding eigenvalue $\implies$ $\varphi(r)$ solves

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Then $\varphi(r) = r^{\sigma^+}$ with $\sigma^+ = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(a)}$.

$$\Downarrow$$

$$w(r,\theta) = r^{\sigma^+}\psi(\theta)$$

From $\mathcal{N}_w(r) \equiv \gamma$, we deduce that $\gamma = \sigma^+$.

Step 1: any $\lambda_n \to 0^+$ admits a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ s.t.

$$\frac{u(\lambda_{n_k}x)}{\sqrt{H(\lambda_{n_k})}} \to |x|^{\gamma}\psi\left(\frac{x}{|x|}\right) \quad \text{strongly in } H^1(B_1)$$

$\gamma = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(a)}$, ψ eigenfunction associated to μ_{k_0}.

Asymptotics at the singularity

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Step 2: $\lim_{r\to 0^+} \frac{H(r)}{r^{2\gamma}}$ is finite and > 0 (Step 1 + separation of variables)

Step 3: So $\quad \lambda_{n_k}^{-\gamma} u(\lambda_{n_k}\theta) \to \sum_{i=j_0}^{j_0+m-1} \beta_i \psi_i(\theta)$ in $L^2(\mathbb{S}^{N-1})$

where $\{\psi_i\}_{i=j_0}^{j_0+m-1}$ is an $L^2(\mathbb{S}^{N-1})$-orthonormal basis for the eigenspace associated to μ_{k_0}.

Expanding $u(\lambda\,\theta) = \sum_{k=1}^{\infty} \varphi_k(\lambda)\psi_k(\theta)$, we compute the β_i's.

Asymptotics at the singularity

$$\beta_i = \lim_{k\to\infty} \lambda_{n_k}^{-\gamma} \varphi_i(\lambda_{n_k})$$

$$= \int_{\mathbb{S}^{N-1}} \left[R^{-\gamma} u(R\theta) + \int_0^R \frac{h(s\theta)u(s\theta) + f\big(s\theta, u(s\theta)\big)}{2\gamma + N - 2} \left(s^{1-\gamma} - \frac{s^{\gamma+N-1}}{R^{2\gamma+N-2}} \right) ds \right] \psi_i(\theta)\, dS(\theta)$$

depends neither on the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ nor on its subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$

$\Longrightarrow$ **the convergences actually hold as $\lambda \to 0^+$**.

Asymptotics at the singularity

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Theorem [F.-Ferrero-Terracini (2010)] Let $\Omega \ni 0$ be a bounded open set in $\mathbb{R}^N$, $N \geqslant 3$, **(PD)**, **(H)**, **(F)** hold. If $u \not\equiv 0$ weakly solves $\mathcal{L}_a u = hu + f(x,u)$ in Ω, then $\exists\, k_0 \in \mathbb{N}$, $k_0 \geqslant 1$, s. t.

$$\gamma = \lim_{r\to 0^+} \mathcal{N}_{u,h,f}(r) = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(a)}.$$

Furthermore, as $\lambda \to 0^+$,

$$\lambda^{-\gamma}u(\lambda x) \to |x|^{\gamma} \sum_{i=j_0}^{j_0+m-1} \beta_i \psi_i\left(\frac{x}{|x|}\right) \qquad \text{in } H^1(B_1).$$

3. Parabolic monotonicity formula

[Poon, Comm. PDE's(1996)]

For u solving $u_t + \Delta u = 0$ in $\mathbb{R}^N \times (0, T)$, define

$$D(t) = t\int_{\mathbb{R}^N} |\nabla u(x,t)|^2 G(x,t)\,dx$$

$$H(t) = \int_{\mathbb{R}^N} u^2(x,t) G(x,t)\,dx$$

Parabolic Almgren frequency

$$\mathcal{N}(t) = \frac{D(t)}{H(t)}$$

where

$$G(x,t) = t^{-N/2} \exp\left(-\frac{|x|^2}{4t}\right)$$

is the heat kernel satisfying

$$G_t - \Delta G = 0 \quad \text{and} \quad \nabla G(x,t) = -\frac{x}{2t} G(x,t).$$

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$$\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$$

$$0$$

$\square$

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$\square$

Doubling properties and unique continuation:

[Escauriaza-Fernández, Ark. Mat. (2003)]
[Fernández, Comm. PDEs (2003)]
[Escauriaza-Fernández-Vessella, Appl. Analysis (2006)]
[Escauriaza-Kenig-Ponce-Vega, Math. Res. Lett. (2006)]

See also **[Caffarelli-Karakhanyan-Lin, J. Fixed Point Theory Appl. (2009)]**

4. Asymptotics at singularities (parabolic case)

$$u_t + \Delta u + \frac{a(x/|x|)}{|x|^2}\, u + h(x,t)u = 0, \quad \text{in } \mathbb{R}^N \times (0,T)$$

where $T > 0$, $N \geqslant 3$, $a \in L^\infty(\mathbb{S}^{N-1})$, and

$$(\mathbf{H}) \begin{cases} |h(x,t)| \leqslant C_h(1+|x|^{-2+\varepsilon}) & \text{for all } t \in (0,T), \text{ a.e. } x \in \mathbb{R}^N \\ h, h_t \in L^r\big((0,T), L^{N/2}\big), r > 1, & h_t \in L^\infty_{\mathrm{loc}}\big((0,T), L^{N/2}\big). \end{cases}$$

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$$u_t + \Delta u + \frac{a(x/|x|)}{|x|^2}\, u + h(x,t)u = 0, \quad \text{in } \mathbb{R}^N \times (0,T)$$

where $T > 0$, $N \geqslant 3$, $a \in L^\infty(\mathbb{S}^{N-1})$, and

$$(\mathbf{H}) \begin{cases} |h(x,t)| \leqslant C_h(1+|x|^{-2+\varepsilon}) & \text{for all } t \in (0,T), \text{ a.e. } x \in \mathbb{R}^N \\ h, h_t \in L^r\big((0,T), L^{N/2}\big), r > 1, \quad h_t \in L^\infty_{\mathrm{loc}}\big((0,T), L^{N/2}\big). \end{cases}$$

Remark: also subcritical semilinear perturbations $f(x,t,s)$ can be treated

4. Asymptotics at singularities (parabolic case)

$$u_t + \Delta u + \frac{a(x/|x|)}{|x|^2}\, u + h(x,t)u = 0, \quad \text{in } \mathbb{R}^N \times (0,T)$$

where $T > 0$, $N \geqslant 3$, $a \in L^\infty(\mathbb{S}^{N-1})$, and

$$(\mathbf{H}) \begin{cases} |h(x,t)| \leqslant C_h(1+|x|^{-2+\varepsilon}) & \text{for all } t \in (0,T), \text{ a.e. } x \in \mathbb{R}^N \\ h, h_t \in L^r\big((0,T), L^{N/2}\big), r > 1, & h_t \in L^\infty_{\mathrm{loc}}\big((0,T), L^{N/2}\big). \end{cases}$$

Parabolic Almgren type frequency function

$$\mathcal{N}(t) = \frac{t \int_{\mathbb{R}^N} \Big(|\nabla u(x,t)|^2 - \frac{a(x/|x|)}{|x|^2} u^2(x,t) - h(x,t)u^2(x,t)\Big) G(x,t)\, dx}{\int_{\mathbb{R}^N} u^2(x,t)\, G(x,t)\, dx}$$

where $G(x,t) = t^{-N/2} \exp\Big(-\frac{|x|^2}{4t}\Big)$

"Perturbed monotonicity"

$\mathcal{N}'$ is an L^1-perturbation of a nonnegative function as $t \to 0^+$

$$\Downarrow$$

$$\exists \lim_{t \to 0^+} \mathcal{N}(t) = \gamma.$$

"Perturbed monotonicity"

$\mathcal{N}'$ is an L^1-perturbation of a nonnegative function as $t \to 0^+$

$$\Downarrow$$

$$\exists \lim_{t \to 0^+} \mathcal{N}(t) = \gamma.$$

Blow-up for scaling $u_\lambda(x,t) = u(\lambda x, \lambda^2 t)$

$$\rightsquigarrow$$

γ is an eigenvalue of

Ornstein-Uhlenbeck type operator

$$\mathfrak{L}_a = -\Delta + \frac{x}{2} \cdot \nabla - \frac{a(x/|x|)}{|x|^2}$$

i.e.

$$\gamma = \gamma_{m,k} = m - \frac{\alpha_k}{2}, \quad \alpha_k = \frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_k(a)},$$

for some $k, m \in \mathbb{N}, k \geqslant 1$.

Asymptotics at the singularity

Theorem [F.-Primo, to appear in DCDS-A]

Let $u \not\equiv 0$ be a weak solution to $u_t + \Delta u + \frac{a(x/|x|)}{|x|^2} u + h(x,t)u = 0$, with h satisfying **(H)** and $a \in L^\infty(\mathbb{S}^{N-1})$ satisfying **(PD)**. Then $\exists\, m_0, k_0 \in \mathbb{N}$, $k_0 \geqslant 1$, such that $\lim_{t\to 0^+} \mathcal{N}(t) = \gamma_{m_0,k_0}$. Furthermore $\forall\, \tau \in (0,1)$

$$\lim_{\lambda\to 0^+} \int_\tau^1 \left\| \lambda^{-2\gamma_{m_0,k_0}} u(\lambda x, \lambda^2 t) - t^{\gamma_{m_0,k_0}} V(x/\sqrt{t}) \right\|_{\mathcal{H}_t}^2 dt = 0,$$

$$\lim_{\lambda\to 0^+} \sup_{t\in[\tau,1]} \left\| \lambda^{-2\gamma_{m_0,k_0}} u(\lambda x, \lambda^2 t) - t^{\gamma_{m_0,k_0}} V(x/\sqrt{t}) \right\|_{\mathcal{L}_t} = 0,$$

where V is an eigenfunction of $\mathfrak{L}_a$ associated to the eigenvalue γ_{m_0,k_0}.

$$\|u\|_{\mathcal{H}_t} = \left(\int_{\mathbb{R}^N} \left(t|\nabla u(x)|^2 + |u(x)|^2\right) G(x,t)\, dx \right)^{1/2}, \quad \|u\|_{\mathcal{L}_t} = \left(\int_{\mathbb{R}^N} |u(x)|^2 G(x,t)\, dx \right)^{1/2}$$

Eigenfunctions of $\mathfrak{L}_a$

A basis of the eigenspace corresponding to $\gamma_{m,k} = m - \frac{\alpha_k}{2}$ is

$$\left\{ V_{n,j} : j, n \in \mathbb{N}, j \geqslant 1, \gamma_{m,k} = n - \frac{\alpha_j}{2} \right\},$$

where

$$V_{n,j}(x) = |x|^{-\alpha_j} P_{j,n}\left(\frac{|x|^2}{4}\right) \psi_j\left(\frac{x}{|x|}\right),$$

ψ_j is an eigenfunction of the operator $L_a = -\Delta_{\mathbb{S}^{N-1}} - a(\theta)$ associated to the j-th eigenvalue $\mu_j(a)$, and

$$P_{j,n}(t) = \sum_{i=0}^{n} \frac{(-n)_i}{\left(\frac{N}{2} - \alpha_j\right)_i} \frac{t^i}{i!},$$

$(s)_i = \prod_{j=0}^{i-1}(s+j)$, $(s)_0 = 1$.