Stochastic dynamics of complex systems: mesoscopic description and beyond

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Probability and NonLocal PDEs: Interplay and Cross-Impact

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Stochastic dynamics of complex systems

Systems in continuum — sets of points distributed in a continuum X ($X = \mathbb{R}^d$, or $X = \Lambda \subset \mathbb{R}^d$, or $X = \mathbb{R}^d \times S$ with a space of marks S, *etc.*)

cf. **Discrete systems** — sets on lattices, graphs

Interpretation — particles in mathematical physics, individuals in population ecology, cells in biology, agents on the market in economics

Stochastic dynamics — particles randomly may:

- born (appear)
- die (disappear)
- move (continuously or with jumps)

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- Here infinity is a mathematical approximation, for a real ‘huge’ system.
- On the other hand,all real elements have some physical sizes, hence it is natural to assume that in any compact region of X (assuming that there is a topology on X) there exists finite number of elements only.

Let X be (for simplicity) a metric space.

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Definition

The space of (locally finite) configurations over X is

$$\Gamma = \Gamma(X) = \left\{ \gamma \subset X \,\middle|\, |\gamma \cap \Lambda| < \infty \text{ for all } \Lambda \in \mathcal{B}_c(X) \right\}.$$

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Structures:

- topology
- Borel σ-algebra $\mathcal{B}(\Gamma)$
- metrization of topology, Γ is a Polish space

- We are interested in random variables

$$|\gamma \cap \Lambda|,$$

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- Namely, let $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ be the space of probability measures on $\big(\Gamma, \mathcal{B}(\Gamma)\big)$ with finite local moments:

$$\int_\Gamma |\gamma \cap \Lambda|^n \, d\mu(\gamma) < \infty, \qquad \Lambda \in \mathcal{B}_{\mathrm{c}}(X), n \in \mathbb{N}.$$

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- The description of a system at moment $t \geq 0$ is a distribution (a measure) $\mu_t \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$.

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  $\Lambda \in \mathcal{B}_c(X)$, $n \in \mathbb{N}$.
- For example, $d\sigma(x) = z dx$, the Lebesgue measure on $X = \mathbb{R}^d$ with $z > 0$, or $d\sigma(x) = \rho(x) dx$.

- A random event: after a (small) time interval Δt, a finite subset ξ, $|\xi| = n$, $n \in \mathbb{N} \cup \{0\}$ of an existing configuration $\gamma \in \Gamma$ disappears, and a new finite group ω, $|\omega| = m$, $m \in \mathbb{N} \cup \{0\}$ of elements appears in a bounded region $\Lambda \subset X$.

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- The probability of such event is described by

$$\Delta t \int_{\Lambda^m} c(\xi, \omega, \gamma \setminus \xi) d\omega + o(\Delta t),$$

where $c \geq 0$ is a probability rate of the event, and

$$d\omega = \frac{1}{m!} d\sigma(x_1) \dots d\sigma(x_m),$$

for $\omega = (x_1, \dots, x_m)$ and a measure σ on X.

- Note that in

$$\Delta t \int_{\Lambda^m} c(\xi, \omega \mid \gamma \setminus \xi) d\omega + o(\Delta t),$$

the case $\omega = \emptyset$ corresponds to a death event and the case $\xi = \emptyset$ describes the birth.

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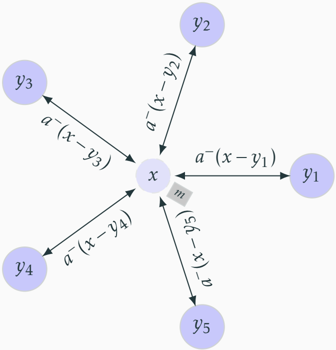
  the case $\omega = \emptyset$ corresponds to a death event and the case $\xi = \emptyset$ describes the birth.
- If $|\xi| = |\omega| \neq 0$, then one can speak about a 'jump'.

At a random moment of time, an existing element $x \in \gamma$ may disappear (die). The rate of this event depends on x itself, but also it is influenced by the rest of the population (say, because of the competition for resources).



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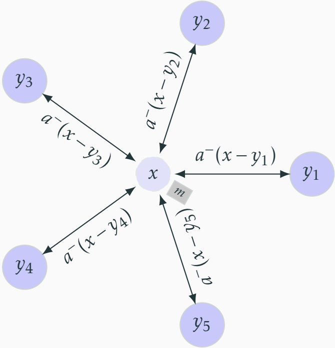

$$
\begin{aligned}
& c(\xi, \omega, \gamma \setminus \xi) \\
&= c(\{x\}, \varnothing, \gamma \setminus \{x\}) \\
&= m + \sum_{y \in \gamma \setminus x} a^-(x-y)
\end{aligned}
$$

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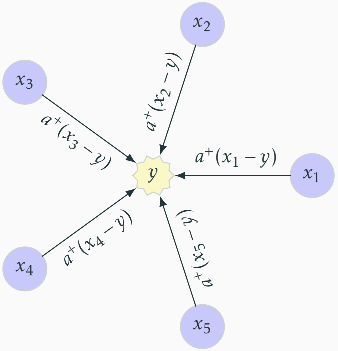
Also, at a random moment of time, an existing element x may send an off-spring to $y \in \mathbb{R}^d$. The rate of this event depends on both x and y only.



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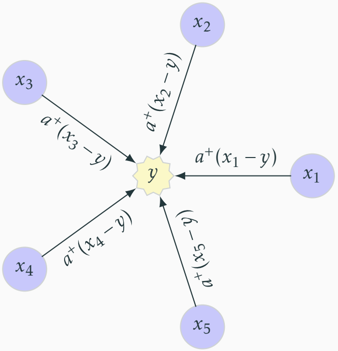

$$c(\xi, \omega, \gamma \setminus \xi)$$

$$= c(\varnothing, \{y\}, \gamma)$$

$$= \sum_{x \in \gamma} a^+(x - y)$$

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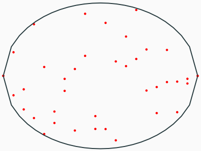
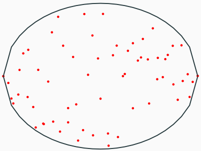
Observe in a (small) region Λ



time $= 0$
initial (random) number of points $= N_0^\Lambda$



time $= t$
(random) number of points $= N_t^\Lambda$

Averaged over (thousands of) simulations:

$$n_t^\Lambda = \mathbb{E}\left[N_t^\Lambda\right] = \int_\Gamma |\gamma \cap \Lambda| \, d\mu_t(\gamma),$$

$$\mathrm{cov}_t^{\Lambda_1,\Lambda_2} = \mathbb{E}\left[N_t^{\Lambda_1} N_t^{\Lambda_2}\right] - n_t^{\Lambda_1} n_t^{\Lambda_2} = \int_\Gamma |\gamma \cap \Lambda_1||\gamma \cap \Lambda_2| \, d\mu_t(\gamma) - n_t^{\Lambda_1} n_t^{\Lambda_2},$$

$$\ldots$$

Statistical description

The evolution of a system (the dynamics of measures) may be defined via the equality

$$\frac{\partial}{\partial t} \int_{\Gamma} F(\gamma)\, d\mu_t(\gamma) = \int_{\Gamma} (LF)(\gamma)\, d\mu_t(\gamma),$$

where $\mu_0 \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, $F : \Gamma \to \mathbb{R}$ is from some class of functions $\mathcal{F}(\Gamma)$,

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$$(LF)(\gamma) = \sum_{\xi \Subset \gamma} \int_{\Gamma_0} c(\xi, \omega \mid \gamma \setminus \xi)(F(\gamma \setminus \xi \cup \omega) - F(\gamma))d\omega.$$

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$$\Gamma_0 = \left\{\eta\subset X \,\middle|\, |\eta|<\infty\right\} \simeq \bigsqcup_{n=0}^{\infty} X^n.$$

- Birth-and-death generator:

$$(LF)(\gamma) := \sum_{x\in\gamma} d(x,\gamma\setminus x)\,[F(\gamma\setminus x) - F(\gamma)] + \int_X b(x,\gamma)\,[F(\gamma\cup x) - F(\gamma)]\,d\sigma(x).$$

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- Then the probability for an $x \in \gamma$ to die after the time δt is $d(x, \gamma \setminus x)\delta t + o(\delta t)$; probability that after the time δt in a region Λ will born a new x given γ is $\int_\Lambda b(x, \gamma)\,d\sigma(x) \cdot \delta t + o(\delta t)$.

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Mainly for finite systems in bounded or infinite domains:

- [Preston'75] (heuristic);
- [Holley/Stroock'78] (finite systems in finite volumes of $\mathbb{R}^d$);
- [Méléard/Fournier/Champagnat/...'04--17] (finite, multitype systems in $\mathbb{R}^d$);
- [Kolokoltsov'04--11] (finite systems in $\mathbb{R}^d$);
- [Bezborodov'14] (finite systems in $\mathbb{R}^d$, PhD Thesis);
- [Garsia/Kurtz'06] (infinite systems in $\mathbb{R}^d$ with $d(x, \gamma) \equiv 1$ and structural restrictions on b).

- Recall that the evolution of measures is given by

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- Let, for $t \geq 0$, $\mu_t \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$. Suppose that there exists a family of measurable symmetric functions $k_t^{(n)} = k_{\mu_t}^{(n)} : X^n \to [0;+\infty)$, $n \in \mathbb{N}$, such that

$$\int_\Gamma \sum_{\{x_1,\ldots,x_n\}\subset\gamma} G^{(n)}(x_1,\ldots,x_n)d\mu_t(\gamma)$$

$$= \frac{1}{n!}\int_{X^n} G^{(n)}(x_1,\ldots,x_n)k_t^{(n)}(x_1,\ldots,x_n)d\sigma(x_1)\ldots d\sigma(x_n)$$

for all symmetric functions $G^{(n)}$ with bounded supports.

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- Let $N^\Lambda(\gamma) := |\gamma \cap \Lambda|$, then

$$\mathbb{E}_{\mu_t}\left[N^\Lambda\right] = \int_\Lambda k_t^{(1)}(x)\,d\sigma(x),$$

$$\mathbb{E}_{\mu_t}\left[N^{\Lambda_1}N^{\Lambda_2}\right] = \int_{\Lambda_1}\int_{\Lambda_2} k_t^{(2)}(x_1,x_2)\,d\sigma(x_1)\,d\sigma(x_2),$$

$$\mathbb{E}_{\mu_t}\left[N^{\Lambda_1}\ldots N^{\Lambda_n}\right] = \int_{\Lambda_1}\ldots\int_{\Lambda_n} k_t^{(n)}(x_1,\ldots,x_n)\,d\sigma(x_1)\ldots d\sigma(x_n)$$

- Equation for μ_t is reduced to a (linear) equation for k_t:

$$\frac{\partial}{\partial t} k_t = L^{\triangle} k_t$$

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- Based on the so-called harmonic analysis on the configuration spaces: [Kondratiev/Kuna'02], [Lenard'75].
- The structure is the following

$$k_t = \begin{pmatrix} k_t^{(0)} \\ k_t^{(1)} \\ \vdots \\ k_t^{(n)} \\ \vdots \end{pmatrix}, \qquad L^{\triangle} = \begin{pmatrix} * & * & \cdots & * & \cdots & * & \cdots \\ * & * & \ddots & * & \cdots & * & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots \\ * & * & \cdots & * & \cdots & L_{n,m} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

where $L_{n,m} : \mathcal{F}(X^n) \to \mathcal{F}(X^m)$

$$d(x,\gamma\setminus x)=m+\sum_{y\in\gamma\setminus x}a^{-}(x-y),\qquad b(x,\gamma)=\sum_{y\in\gamma}a^{+}(x-y),$$

then

$$L^{\triangle}=\begin{pmatrix} L_{0,0} & L_{1,0} & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ L_{0,1} & L_{1,1} & L_{2,0} & \cdots & 0 & \cdots & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots \\ 0 & 0 & \cdots & L_{n-1,n} & L_{n,n} & L_{n+1,n} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Here, for example,

$$(L_{n,n}k^{(n)})(x_1,\ldots,x_n)=-\Big(m+\sum_{i=1}^{n}\sum_{j\neq i}a^{-}(x_i-x_j)\Big)k^{(n)}(x_1,\ldots,x_n)+\text{jumps}$$

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Hierarchy!

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$$\mathscr{K}_C = \left\{ k : \Gamma_0 \to \mathbb{R} \,\middle|\, |k(\eta)| \leq \mathrm{const} \cdot C^{|\eta|} \;\; d\eta\text{-a.e.} \right\}.$$

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- In other words,

$$|k(x_1, \ldots, x_n)| \le \mathrm{const} \cdot C^n.$$

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- Technique: analytic semigroups, $\odot$-dual semigroups, evolution in:

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- Evolution exists for all $t > 0$.

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- Price: the evolution can be constructed on $[0, T)$ only for some (small) $T > 0$.

Mesoscopic description

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- We know more than after an existence theorem about a Markov process on Γ, namely, we know that

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- Idea: to consider a properly scaled system by a small parameter $\varepsilon > 0$. The original system will correspond to $\varepsilon = 1$ and will remain unsolvable, however, for small values of ε, one gets an info.

- We will study a ‘condensed’ system. Namely, we suppose that, at $t = 0$, the system is described by correlation functions $k_{0,\varepsilon}$ such that, for all $n \in \mathbb{N}$, one has that, point-wise,

$$\exists \lim_{\varepsilon \to 0} \varepsilon^n k_{0,\varepsilon}^{(n)}(x_1, \ldots, x_n) =: r_0(x_1, \ldots, x_n).$$

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- We have to introduce a scaling L_ε of the operator L which will give the corresponding operator $L_\varepsilon^\triangle$ such that the solution to the equation

$$\frac{\partial}{\partial t} k_{t,\varepsilon} = L_\varepsilon^\triangle k_{t,\varepsilon}$$

will keep the same property:

$$\exists \lim_{\varepsilon \to 0} \varepsilon^n k_{t,\varepsilon}^{(n)}(x_1, \ldots, x_n) =: r_t(x_1, \ldots, x_n).$$

- Moreover, the limiting evolution

$$r_0(\eta) \mapsto r_t(\eta)$$

should preserve the correlation functions of Poisson measures, namely,

$$r_0(\eta) = \prod_{x\in\eta} u_0(x)$$

should lead to

$$r_t(\eta) = \prod_{x\in\eta} u(x,t).$$

- Beside the evolution $r_0 \mapsto r_t$ is linear one, the dependence $u_0(\cdot) \mapsto u(\cdot, t)$ is, in general, non-linear; and it is given by

$$\frac{\partial}{\partial t} u(x,t) = v\Big(u(x,t)\Big),$$

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- Hence, if we are able to solve (by studying properties or numerically) the mesoscopic equation, then we will have that

$$k_{0,\varepsilon}^{(n)}(x_1, \ldots, x_n) = \varepsilon^{-n} \prod_{j=1}^{n} u(x_j, 0) + o(\varepsilon^{-n})$$

leads to

$$k_{t,\varepsilon}^{(n)}(x_1, \ldots, x_n) = \varepsilon^{-n} \prod_{j=1}^{n} u(x_j, t) + o(\varepsilon^{-n}).$$

[F/Kondratiev/Kutoviy'10, J.Stat.Phys. & '12, J.Funct.Anal.]

- We consider the operator L_ε with a^- replaced by εa^- only:

$$(L_\varepsilon F)(\gamma) = \sum_{x\in\gamma}\Bigg(m + \varepsilon \sum_{y\in\gamma\setminus x} a^-(x-y)\Bigg)\Big(F(\gamma\setminus x) - F(\gamma)\Big) + \sum_{y\in\gamma}\int_{\mathbb{R}^d} a^+(x-y)\Big(F(\gamma\cup x) - F(\gamma)\Big)dx.$$

- One gets then the corresponding equation

$$\frac{\partial}{\partial t}k_{t,\varepsilon}(\eta) = (L_\varepsilon^\triangle k_{t,\varepsilon})(\eta) = (Ak_{t,\varepsilon})(\eta) + \varepsilon(Bk_{t,\varepsilon})(\eta).$$

Theorem

Let $u_0 : \mathbb{R}^d \to \mathbb{R}$ be a bounded function, and $k_{0,\varepsilon}$ be such that

$$\lim_{\varepsilon \to 0} \varepsilon^n k_{0,\varepsilon}(x_1, \dots, x_n) = \prod_{j=1}^{n} u_0(x_j)$$

for each $\{x_1, \dots, x_n\} \subset \mathbb{R}^d$, $n \in \mathbb{N}$. Then

$$\lim_{\varepsilon \to 0} \varepsilon^n k_{t,\varepsilon}(x_1, \dots, x_n) = \prod_{j=1}^{n} u(x_j, t)$$

(point-wise and in $\mathcal{K}_C$), where

$$\frac{\partial}{\partial t} u(x,t) = \int_{\mathbb{R}^d} a^+(x-y) u(y,t) dy - m u(x,t) \\ - u(x,t) \int_{\mathbb{R}^d} a^-(x-y) u(y,t) dy.$$

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- In particular, $k^{(1)}_{t,\varepsilon}(x)$ is such that

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- Then

$$\varepsilon k_{0,\varepsilon}^{(1)}(x) \to u_0(x), \quad x \in \mathbb{R}^d, \varepsilon \to 0, \quad \text{e.g. } k_{0,\varepsilon}^{(1)}(x) = \varepsilon^{-1}u_0(x)$$

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- Note that here $o(1) = o_{\varepsilon,x}(1)$.

[F/Kondratiev/Kozitsky/Kutoviy'15, M.Mod.&Meth.Appl.Sci.]

BDLP model: analysis of the mesoscopic equation

- Recall the equation:

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- Stationary solutions $u \equiv 0$ and

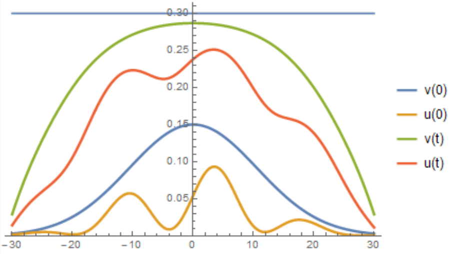
$$u \equiv \frac{\int_{\mathbb{R}^d} a^+(x)\,dx - m}{\int_{\mathbb{R}^d} a^-(x)\,dx} =: \theta > 0.$$

Comparison principle holds **iff**

$$a^{+}(x) \geq \theta a^{-}(x), \quad x \in \mathbb{R}^{d}.$$

Namely, if $0 \leq u_0 \leq v_0 \leq \theta$ and u, v are the corresponding solutions, then

$$0 \leq u(x,t) \leq v(x,t) \leq \theta, \qquad x \in \mathbb{R}^{d}, t \geq 0.$$



[F/Tkachov'2018, Nonlinearity]

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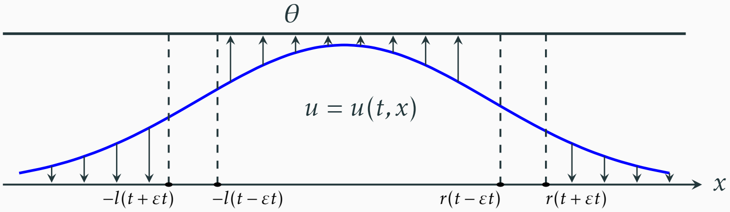
- Let $r, l : \mathbb{R}_+ \to \mathbb{R}$ and let $|r(t)|, |l(t)|$ be increasing to ∞ functions, such that the following holds.

Case (C1) For each $\varepsilon \in (0, 1)$,

$$\lim_{t\to\infty} \operatorname*{essinf}_{[-l(t-\varepsilon t), r(t-\varepsilon t)]} u(x,t) = \theta.$$

$$\lim_{t\to\infty} \operatorname*{esssup}_{(-\infty,-l(t+\varepsilon t)]\cup[r(t+\varepsilon t),\infty)} u(x,t) = 0.$$



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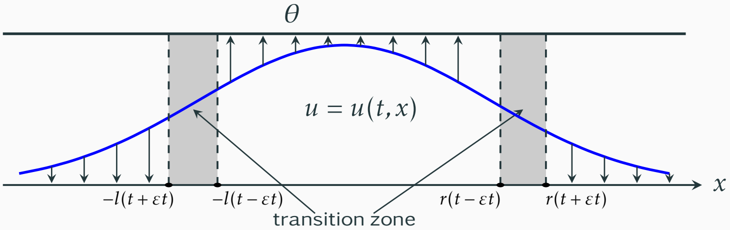

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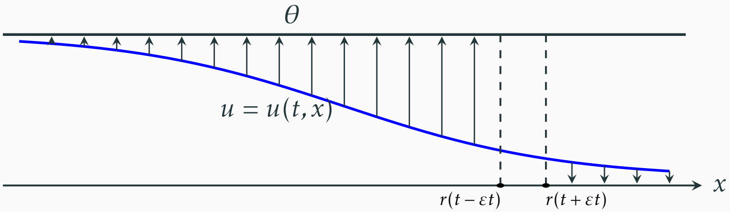

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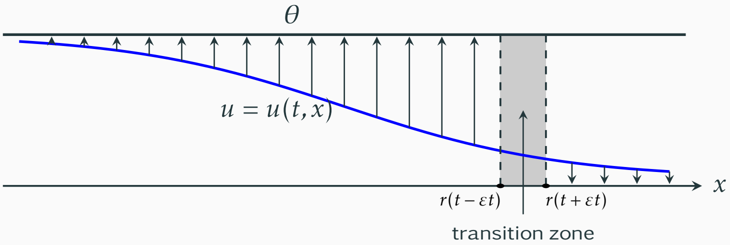

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- Acceleration (to the right):

$$\lim_{t \to \infty} \frac{r(t)}{t} = \infty.$$

Informal theorem

- Acceleration takes place if **either** a **or** u_0 (or both) has heavy tails.

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[F/Tkachov'17, Applicable Analysis]

[F/Tkachov'18, Advances in Applied Probability]

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then

$$b(x) = x^{-q}, \qquad r(t) = \exp\left(\frac{\beta}{q} t\right);$$

$$b(x) = \exp\left(-p(\log x)^q\right), \qquad r(t) = \exp\left(\left(\frac{\beta}{p} t\right)^{\frac{1}{q}}\right);$$

$$b(x) = \exp\left(-x^\alpha\right), \qquad r(t) = (\beta t)^{\frac{1}{\alpha}};$$

$$b(x) = \exp\left(-\frac{x}{(\log x)^q}\right), \qquad r(t) \sim \beta t (\log t)^q, t \to \infty.$$

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[F/Kondratiev/Tkachov'15&16, arXiv]

Beyond the mesoscopic limit

What did we lose?

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then one can rewrite

$$\varepsilon^{|\eta|} k_{t,\varepsilon}(\eta) = \prod_{x\in\eta} u(x,t) + o(1),$$

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hence we have 'lost' exactly the $o(1)$, that is a function of η and t.
- For example, the first-order correlation function:

$$\varepsilon k^{(1)}_{t,\varepsilon}(x) = u(x,t) + \varepsilon w(x,t) + O(\varepsilon^2)(t;x)$$

- Consider the covariance

$$\begin{aligned}\mathrm{Cov}_{t,\varepsilon}^{\Lambda_1,\Lambda_2} &:= \mathbb{E}\Big[\Big(N_{t,\varepsilon}^{\Lambda_1} - \mathbb{E}[N_{t,\varepsilon}^{\Lambda_1}]\Big)\Big(N_{t,\varepsilon}^{\Lambda_2} - \mathbb{E}[N_{t,\varepsilon}^{\Lambda_2}]\Big)\Big] \\ &= \mathbb{E}[N_{t,\varepsilon}^{\Lambda_1} N_{t,\varepsilon}^{\Lambda_2}] - \mathbb{E}[N_{t,\varepsilon}^{\Lambda_1}]\mathbb{E}[N_{t,\varepsilon}^{\Lambda_2}] \\ &= \int_{\Lambda_1}\int_{\Lambda_2}\Big(\tilde{k}_{t,\varepsilon}^{(2)}(x,y) - \tilde{k}_{t,\varepsilon}^{(1)}(x)\tilde{k}_{t,\varepsilon}^{(1)}(y)\Big)\,dx\,dy.\end{aligned}$$

- We have proved that

$$\lim_{\varepsilon\to 0} \mathrm{Cov}_{t,\varepsilon}^{\Lambda_1,\Lambda_2} = 0.$$

- We are interested now to find the next order of approximation:

$$\mathrm{Cov}_{t,\varepsilon}^{\Lambda_1,\Lambda_2} = \varepsilon\int_{\Lambda_1}\int_{\Lambda_2} g(x,y,t)\,dx\,dy + o(\varepsilon),$$

- One can of course guess that

$$\varepsilon^{|\eta|} k_{t,\varepsilon}(\eta) = \prod_{x \in \eta} u(x,t) + \varepsilon^m s(\eta,t) + o(\varepsilon^m),$$

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- However, it will be a hierarchical equation and it is absolutely nontrivial to guess which class of functions the evolution $s(0) \mapsto s(t)$ preserves (like $r_0 \mapsto r_t$ preserved products).
- We proceed to work with cumulants instead.

If $k : \Gamma_0 \to \mathbb{R}$ with $k(\varnothing) = 1$ there exists a unique $v : \Gamma_0 \to \mathbb{R}$ with $u(\varnothing) = 0$, such that, $k^{(1)}(x_1) = v^{(1)}(x_1)$,

$$\begin{aligned} k^{(2)}(x_1, x_2) &= v^{(2)}(x_1, x_2) + v^{(1)}(x_1)v^{(1)}(x_2), \\ k^{(3)}(x_1, x_2, x_3) &= v^{(3)}(x_1, x_2, x_2) + v^{(1)}(x_1)v^{(2)}(x_2, x_3) \\ &\quad + v^{(1)}(x_2)v^{(2)}(x_1, x_3) + v^{(1)}(x_3)v^{(2)}(x_1, x_2), \end{aligned}$$

and so on. In particular, if $k = k_\mu$ is the correlation function of a measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, then

$$\begin{aligned} &\mathbb{E}_\mu\Big[\Big(N^{\Lambda_1} - \mathbb{E}_\mu[N^{\Lambda_1}]\Big)\dots\Big(N^{\Lambda_n} - \mathbb{E}_\mu[N^{\Lambda_n}]\Big)\Big] \\ &\qquad = \int_{\Lambda_1} \dots \int_{\Lambda_n} v_\mu^{(n)}(x_1, \dots, x_n)\, d\sigma(x_1) \dots d\sigma(x_n), \end{aligned}$$

for compact $\Lambda_i \cap \Lambda_j = \varnothing$, $i \neq j$.

- Rewrite all above on the language of cumulants. Note that

$$k^{(n)}(x_1,\ldots,x_n) = \prod_{j=1}^{n} f(x_j)$$

has the cumulant

$$v^{(1)}(x_1) = k^{(1)}(x_1) = f(x_1), \qquad v^{(n)} \equiv 0, \quad n \neq 1.$$

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- Let $v_{t,\varepsilon}$ be the cumulant of $k_{t,\varepsilon}$. One can then show that

$$\varepsilon^{|\eta|} v_{0,\varepsilon}(\eta) \to \mathbb{1}_{\eta=\{x\}} u_0(x), \quad \varepsilon \to 0$$

implies

$$\varepsilon^{|\eta|} v_{t,\varepsilon}(\eta) \to \mathbb{1}_{\eta=\{x\}} u(x,t), \quad \varepsilon \to 0.$$

(This is a non-trivial statement, the evolution $v_{0,\varepsilon} \mapsto v_{t,\varepsilon}$ is non-linear.)

- However, now one can 'guess' the next term, namely, we prove that

$$\varepsilon^{|\eta|} v_{t,\varepsilon}(\eta) = \mathbb{1}_{\eta=\{x\}} u(x,t) + \varepsilon\left(\mathbb{1}_{\eta=\{x\}} p(x,t) + \mathbb{1}_{\eta=\{x,y\}} g(x,y,t)\right) + o(\varepsilon).$$

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- Equations for p and g are linear with coefficients dependent on u.
- Then, in particular,

$$\varepsilon k^{(1)}_{t,\varepsilon}(x) = u(x,t) + \varepsilon p(x,t) + o(\varepsilon).$$

$$\frac{\partial}{\partial t}p(x,t) = (a^+ * p)(x,t) - mp(x,t) - u(x,t)(a^- * p)(x,t) \\ - p(x,t)(a^- * u)(x,t) - \int_{\mathbb{R}^d} g(x,y,t)a^-(x-y)dy,$$

and

$$\frac{\partial}{\partial t}g(x,y,t) = \int_{\mathbb{R}^d} [g(x,z,t)a^+(y-z) + g(z,y,t)a^+(x-z)]\,dz \\ + a^+(x-y)[u(x,t)+u(y,t)] \\ - 2mg(x,y,t) - 2a^-(y-x)u(x,t)u(y,t) \\ - \int_{\mathbb{R}^d} [a^-(x-z)u(x,t)g(z,y,t) + a^-(y-z)u(y,t)g(x,z,t)]\,dz \\ - g(x,y,t)\int_{\mathbb{R}^d} [a^-(x-z) + a^-(y-z)]u(z,t)\,dz.$$

Let $u_0(x) = u_0 \in \mathbb{R}$, $a^\pm(x) = a^\pm(-x)$, $x \in \mathbb{R}^d$, $\varkappa^\pm := \int_{\mathbb{R}^d} a^\pm(x)\,dx$.
Then

$$\frac{d}{dt}u(t) = (\varkappa^+ - m)u(t) - \varkappa^- u(t)^2,$$

$$\frac{d}{dt}p(t) = (\varkappa^+ - m)p(t) - 2\varkappa^- u(t)p(t) - \varkappa^- \int_{\mathbb{R}^d} g(x,t)a^-(x)\,dx,$$

$$\frac{\partial}{\partial t}g(x,t) = 2\varkappa^+(a^+ * g)(x,t) + 2\varkappa^+ u(t)a^+(x) - 2mg(x,t) - 2\varkappa^- u(t)^2 a^-(x) - 2\varkappa^- u(t)(a^- * g)(x,t) - 2\varkappa^- u(t)g(x,t)$$

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$$\frac{d}{dt}p(t) = (\varkappa^+ - m)p(t) - 2\varkappa^- u(t)p(t) - \varkappa^- \int_{\mathbb{R}^d} \hat{g}_t(\xi)\hat{a}^-(\xi)\,d\xi,$$

$$\frac{\partial}{\partial t}g(x,t) = 2\varkappa^+(a^+ * g)(x,t) + 2\varkappa^+ u(t)a^+(x) - 2mg(x,t) - 2\varkappa^- u(t)^2 a^-(x)$$
$$- 2\varkappa^- u(t)(a^- * g)(x,t) - 2\varkappa^- u(t)g(x,t)$$

$$\Downarrow$$

$$\frac{\partial}{\partial t}\hat{g}(\xi,t) = 2\Big(\varkappa^+\hat{a}^+(\xi) - \varkappa^- u(t)\hat{a}^-(\xi) - \varkappa^- u(t) - m\Big)\hat{g}(\xi,t)$$
$$+2u(t)\Big(\varkappa^+\hat{a}^+(\xi) - \varkappa^- u(t)\hat{a}^-(\xi)\Big)$$

Let (informally) $t \to \infty$. Then

$$u_\infty = \theta = \frac{\varkappa^+ - m}{\varkappa^-},$$

$$\hat{g}_\infty(\xi) = \frac{(\varkappa^+ - m)\Big(\varkappa^+ \hat{a}^+(\xi) - (\varkappa^+ - m)\hat{a}^-(\xi)\Big)}{\varkappa^+ - \Big(\varkappa^+ \hat{a}^+(\xi) - (\varkappa^+ - m)\hat{a}^-(\xi)\Big)},$$

$$p_\infty = -\frac{\int_{\mathbb{R}^d} \hat{g}_t(\xi)\hat{a}^-(\xi)\, d\xi}{\varkappa^+ - m}$$

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$$p_\infty = -\frac{\int_{\mathbb{R}^d} \hat{g}_t(\xi)\hat{a}^-(\xi)\,d\xi}{\varkappa^+ - m}$$

The extinction ('up to ε') will be if

$$k_\infty^{(1)} \approx u_\infty + \varepsilon p_\infty = 0$$

The question is to find the asymptotic of

$$m = m(\varepsilon).$$

When $d = 2$ and $a^+(x) = a^-(x)$ are Gaussian,

$$m(\varepsilon) = \varkappa^+ - \varkappa^- \exp\left(-1 - W\left(\frac{1-4\pi\varepsilon}{4e\pi\varepsilon}\right)\right)$$

$$\approx \varkappa^+ - 4\pi\varepsilon\varkappa^-\left(\log\left(\frac{1}{4\pi e\varepsilon} - 1\right) - 1\right)^{1+\frac{1}{1-\log\left(\frac{1}{4\pi e\varepsilon}-1\right)}}$$

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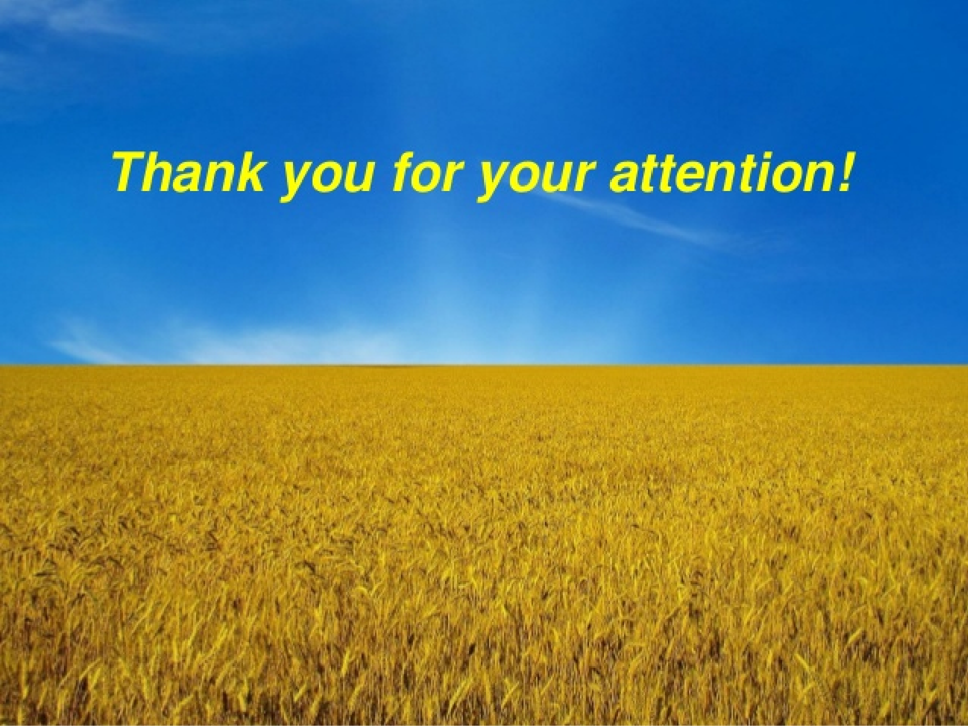
Thank you for your attention!