Optimizing the fractional power in a model with stochastic PDE constraints

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joint work with
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- supporting women in their careers
- giving prominence and visibility to women mathematicians
- providing a meeting place for like-minded people

Outline

Motivation for fractional operators

- foraging of wandering albatrosses use a Lévy flight strategy

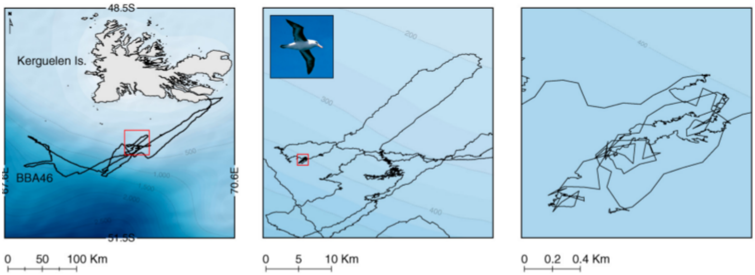

Figure: Humphries et al, Foraging success of biological Levy flights recorded in situ, PNAS 2012

Optimization problems with uncertain constraints

Optimization problems with uncertain constraints appear in...

1. biology: hunting strategies of predators: optimizing over the "average excursion" in the hunting procedure
2. finance: efficient portfolios: minimize the risk (uncertainty of the return) of a portfolio with finitely many assets
3. engineering:
   - want optimal response of a system and a quantification of the statistics of the system response, not only "mean response" (deterministic)
   - want to find optimal position of injection wells in oil fields

General Framework

Optimization problem under (S)PDE contraints

$$\min_{y \in Y, u \in U} \mathcal{J}(y,u) \quad \text{subject to } Constr(y,u) = 0$$

- $\mathcal{J}$ is a cost functional
- y state variable, solution to $Constr(y,u) = 0$
- u control variable
- *Constr* is a constraint (in our case: a (S)PDE)

Typical cost functional: $\mathcal{J}(y,u) = \|y - y_D\|^2 + c|u|^2$

Or, with a convex fct. Φ: $\mathcal{J}(y,u) = \|y - y_D\|^2 + \Phi(u)$

Minimization under SPDE constrains

For fixed ω minimize

$$\mathcal{J}(y, s, \omega) = \|y - y_{D_T}\|^2_{L^2(D\times[0,T])} + \Phi(s) \qquad \text{(costfct)}$$

subject to the state system

$$\begin{aligned} dy &= \mathcal{L}^s y dt + dW \qquad \text{in } D \times [0, T] \\ y(., 0) &= y_0 \qquad \text{in } D \end{aligned} \qquad \text{(SPDE)}$$

New features:

- optimization w.r.t. fractional exponent s of differential operator $\mathcal{L}$
- output of the optimization should be a random variable $\mathcal{J}(\omega)$, in applications one studies quantities such as $Var[J(\omega)]$

The penalty function

Let $L < \infty$ and $s \in (0, L)$. The penalty function $\Phi(s)$

- is given a priori
- modellises e.g. "natural search radius"
- should be strictly convex, growing to infinity at boundary

$$\lim_{s \to 0} \Phi(s) = \infty = \lim_{s \to L} \Phi(s).$$

- assume $\Phi \in C^2(0, L)$ non-negative (avoid degenerate or singular situations)
- has to be chosen such that the problem has sufficient compactness properties in s.

Possible choice: $\Phi(s) = \frac{1}{s(L-s)}$.

The penalty function

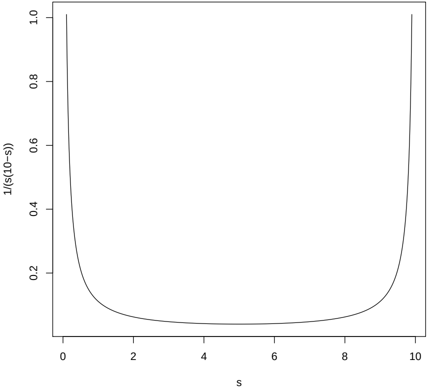

Figure: A possible choice of a penalty function with $L = 10$: $\Phi(s) = \frac{1}{s(L-s)}$

Solving a constrained control problem

Example in finite dimensions: $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$

$$\min_{y \in \mathbb{R}^n, u \in \mathbb{R}^m} \mathcal{J}(y, u) \quad \text{subject to } Ay = Bu$$

1. Define the solution matrix $S \in \mathbb{R}^{m \times n}$ by $y = A^{-1}Bu$
2. Get a *reduced cost functional* $\hat{\mathcal{J}}(u) = \mathcal{J}(Su, u)$
3. Derive necessary and sufficient optimality conditions
4. Prove the existence of optimal controls

N.B.: PDE case: instead of solution matrix S have operator $\mathcal{S}$, called the control-to-state operator

Outline

Fractional stochastic heat equation - definitions

$$\begin{aligned} dy(t) &= \mathcal{L}^s y(t)dt + dW(t) \qquad \text{in } D \times [0, T] \\ y(.,0) &= y_0 \qquad \text{in } D \end{aligned} \tag{SPDE}$$

(1) Q-Wiener Process: $L^2(D)$-valued stochastic process $W(t)$ s.t.

- $W(0) = 0$ a.s.
- For each $\omega \in \Omega$, the path $W(t) : [0, \infty) \to L^2(D)$ is continuous
- $W(t) - W(s) \sim \mathcal{N}(0, (t-s)Q)$, $TrQ < \infty$

Important property

$$W(t,x) = \sum_{j=1}^{\infty} \sqrt{\mu_j} e_j(x) B_j(t) \qquad \text{a.s.}$$

μ_j eigenvalues of Q, e_j eigenfunctions of Q, B_j i.i.d. BM

Fractional stochastic heat equation - definitions

$$
\begin{aligned}
dy(t) &= \mathcal{L}^s y(t)dt + dW(t) \qquad \text{in } D \times [0, T] \\
y(., 0) &= y_0 \qquad \text{in } D
\end{aligned}
\tag{SPDE}
$$

(2) The fractional operator(Prototype: minus Laplacian)

- $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset L^2(D) \to L^2(D)$ densely defined, linear, self-adjoint, positive operator with compact inverse.
- $\mathcal{L}e_j = \lambda_j e_j$ in D, e_j in suitable subspace of $L^2(D)$

Important property

$$
\mathcal{L}v = \sum_{j=1}^{\infty} \lambda_j \langle v, e_j \rangle e_j
$$

(makes sense if $v \in \mathcal{H}^1 := \{\phi \in L^2(D) : \{\lambda_j \langle \phi, e_j \rangle\}_{j \in \mathbb{N}} \in l^2\}$)

The domain of $\mathcal{L}^s$

Useful properties for us: Can characterize the domain of $\mathcal{L}$ by

$$D(\mathcal{L}) = \left\{ v \in L^2(D) : \sum_{j\in\mathbb{N}} \lambda_j^2 \langle v, e_j \rangle^2 < +\infty \right\}.$$

$\rightsquigarrow -\mathcal{L}$ generator of analytic semigroup $S(t) = \sum_{j=1}^{+\infty} e^{-\lambda_j t} v_j(x) v_j(y)$.

Analogously, define $\mathcal{H}^s := \left\{ v \in L^2(D) : \|v\|_{\mathcal{H}^s} < +\infty \right\}$ with the norm

$$\|v\|_{\mathcal{H}^s} := \left(\sum_{j\in\mathbb{N}} \lambda_j^{2s} |\langle v, e_j \rangle|^2 \right)^{1/2}.$$

Want: Solution to SPDE $y(s)(., t) \in L^2(\Omega, \mathcal{H}^s(D))$ for any $t \in (0, T]$

Solutions to the SPDE

1. For fixed s, expand $y(s)(x,t) = \sum_{j=1}^{\infty} y_j(s,t) e_j(x)$
2. Each $y_j(s,t) = \langle y(s)(.,t), e_j \rangle$ solves an SDE

$$y_j(t) = y_{j,0} - \lambda_j^s \int_0^t y_j(\tau) d\tau + \sqrt{\mu_j} \int_0^t dB_j(\tau)$$

with $y_{j,0} = \langle y_0, e_j \rangle$ deterministic.

3. By Ito formula, get the explicit representation

$$y_j(t) = y_{j,0} e^{-\lambda_j^s t} + \sqrt{\mu_j} \int_0^t e^{-\lambda_j^s (t-\tau)} dB_j(\tau)$$

(Semi-)explicit form of solutions to (SPDE)

$$y(s)(x,t) = \sum_{j=1}^{+\infty} e_j(x) y_{j,0} e^{-\lambda_j^s t} + \sum_{j=1}^{+\infty} e_j(x) \sqrt{\mu_j} \int_0^t e^{-\lambda_j^s (t-\tau)} dB_j(\tau).$$

A priori estimates on the solution

By standard estimates, $y(s)(x,t) = \sum_{j=1}^{\infty} y_j(s,t)e_j(x)$ is

- a $L^2(D)$-valued stochastic process with continuous sample paths
- the r.v. $\omega \mapsto \|y(s,\omega)\|_{L^2(\Omega, L^2(D\times T))}$ is a.s. finite

$\rightsquigarrow$ This is sufficient to prove optimality conditions!

To show the existence of pathwise optimal controls, need more:

- pathwise interpretation of the stochastic integral
- $y(s,t,.) \in L^2(\Omega, \mathcal{H}^s(D))$
- the sample paths of $y(s)(x,t)$ are $C^\delta([0,T], L^2(D))$ for $\delta \in (0, \frac{1}{2})$

$\rightsquigarrow$ Need to restrict the set of admissible controls s

Example: set of admissible controls

Additional assumptions

- on the fractional Diffusion operator: $\sum_{j=1}^{\infty} \lambda_j^{-s} \; < \; \infty$
- on the Covariance operator: $\mu_j \sim \lambda_j^{-2s-\epsilon}$

Example: $\mathcal{L} = -\Delta$ on $(0, \pi)$ with Dirichlet boundary conditions

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^{s+\varepsilon}} = \sum_{j=1}^{\infty} \frac{1}{j^{2s+\varepsilon}} \qquad < \infty \quad \text{for } s \geq \frac{1}{2}$$

$\rightsquigarrow$ Set of admissible controls is $s \in \left(\frac{1}{2}, L\right)$

Remark: No such extra condition needed in deterministic case.

Outline

Optimality conditions

Naive idea:

- necessary condition for optimality: $\mathcal{J}'(s) = 0$
- sufficient condition for optimality: $\mathcal{J}''(s) > 0$

Make it rigorous, step 1: show that the map

$$s \mapsto \mathcal{J}(s) := \mathcal{J}(y(s), s)$$

is twice differentiable on $(0, +\infty)$. Then apply the chain rule

$$\mathcal{J}'(\bar{s}) = \frac{d}{ds}\mathcal{J}(y(\bar{s}), \bar{s}) = \partial_y \mathcal{J}(y(\bar{s}), \bar{s}) \circ \partial_s y(\bar{s}) \ + \ \partial_s \mathcal{J}(y(\bar{s}), \bar{s}).$$

Step 2: Define the *s*-derivative of a Wiener Integral

Deriving explicit optimality conditions

A property of Wiener Integrals

Let $B_j(t)$ be standard Brownian motion. Then

$$\frac{d}{ds}\int_0^t g(s,\tau)dB_j(\tau) = \int_0^t \partial_s g(s,\tau)dB_j(\tau) \qquad \text{(WInt)}$$

As

$$\left|\frac{d^k}{ds^k}\exp(-\lambda_j^s\tau)\right| \leq \frac{C_k}{s^k}(1+|\ln(t)|)^k \in L^2([0,T])$$

we get, using (WInt) with $g(s,\tau) = exp(-\lambda_j^s(t-\tau))$,

$$\partial_s y(\bar{s}),\ \partial_{ss}^2 y(\bar{s}) \in L^2(\Omega, L^2(D\times[0,T])).$$

Optimality conditions

Let $y_0 \in L^2(D)$ be deterministic, and let $y = y(s)$ be a solution to the state equation (SPDE) Then the following holds true for a fixed realisation $\omega \in \Omega$:

(i) necessary condition: If $\bar{s}$ is an optimal parameter for (IP) and $y(\bar{s})$ the associated unique solution to the state system (SPDE), then

$$\int_0^T \int_D (y(\bar{s}) - y_D)\partial_s y(\bar{s})\, dxdt \;+\; \Phi^{'}(\bar{s}) \;=\; 0 \tag{1}$$

(ii) sufficient condition: If $\bar{s} \in (\frac{1}{2}, L)$ satisfies the necessary condition (1) and if in addition

$$\int_0^T \int_D (\partial_s y(\bar{s}))^2 + (y(\bar{s}) - y_D)\partial^2_{ss} y(\bar{s})\, dxdt \;+\; \Phi^{''}(\bar{s}) \;>\; 0 \tag{2}$$

then $\bar{s}$ is optimal for (IP).

Outline

Natural and optimal exponents

- $\bar{s}$ - optimal exponent found by our optimization problem
- s_0 - minimum of $\Phi(s)$

The optimal $\bar{s}$ can be different from s_0!

Case 1 - equality: $\bar{s} = s_0$ iff $\Phi^{'}(\bar{s}) = 0 = \Phi^{'}(s_0)$
Reason: optimality conditions

$$-\Phi^{'}(\bar{s}) = \int_0^T \int_D (y(\bar{s}) - y_D)\partial_s y(\bar{s})\, dxdt$$

Case 2 - the optimal exponent is different from the "natural" choice: Example for $\bar{s} < s_0$ on next page

Natural and optimal exponents

Choose zero noise and $\mathcal{L} = -\Delta$ on $(0, \pi)$ with Dirichlet B.C.

For fixed $j_0 \in \mathbb{N}$

- Initial data: $y_0 = \epsilon\, e_{j_0}(x)$ for all $[0, \pi]$
- target function $y_D(x, t) := \epsilon e_{j_0}(x)$.

1. Calculate solution of PDE: $y(s)(x, t) = \epsilon\, e_{j_0}(x) e^{-j_0^{2s} t}$
2. Optimality condition gives

$$0 = \int_0^T \int_D (y(\bar{s}) - y_D)\partial_s y(\bar{s})\, dxdt \; + \; \Phi'(\bar{s})$$

3. some calculation $\Rightarrow \Phi'(\bar{s}) < 0$.
4. convexity of $\Phi \Rightarrow \bar{s} < s_0$

$\rightsquigarrow$ The optimal exponent given by the full cost functional is smaller than the natural one

The calculation of the last slide

The solution can be written as the sum

$$y(s)(x,t) \;=\; \sum_{j=1}^{+\infty} e_j(x) y_{j,0} e^{-\lambda_j^s t} + \sum_{j=1}^{+\infty} e_j(x)\sqrt{\mu_j}\int_0^t e^{-\lambda_j^s(t-\tau)} dB_j(\tau)$$

Plug in eigenfunctions $e_j(x) := c_j \sin(jx)$, eigenvalues $\lambda_j = j^2$.
Then, take the necessary optimality condition, plug in

$$\partial_s y(s) = -2\epsilon\, j_0^{2s} \ln(j_0) \cdot t \cdot e_{j_0}(x) e^{-j_0^{2s} t}$$

$$\int_0^T \int_D (y(\bar{s}) - y_D)\partial_s y(\bar{s})\, dxdt \;=\; -2\epsilon^2\, j_0^{-2\bar{s}} \ln(j_0) \int_0^{j_0^{2s} T} \vartheta(e^{-\vartheta} - 1) e^{-\vartheta} d\vartheta \qquad (\star)$$

(we substituted $\vartheta := j_0^{2s} t$)
For $\epsilon \neq 0$ and $j_0 \geq 1$, obtain from ($\star$) that $\Phi'(\bar{s}) < 0$.

Outline

Existence of pathwise optimal controls - Idea

1. Fix ω. Pick a minimizing sequence s_k of controls and consider the solution $y_k = y(s_k)$ to (SPDE).
2. By properties of Φ, s_k is bounded and wlog $s_k \to \bar{s}$ (and $\bar{s}$ is in the admissible set $\mathscr{S}$)
3. A priori estimates + compactness $\Rightarrow$ for fixed ω, a subsequence $\{y_k(\omega)\}_{k\in\mathbb{N}}$ converges strongly in $L^2(D \times [0, T])$ to $\bar{y}(\omega)$
   Challenge: $y_k(\omega) \in L^2([0, T], \mathcal{H}^{s_k}(D))$

   $\rightsquigarrow$ with every s_k also the Banach space $\mathcal{H}^{s_k}(D)$ changes

   $\rightsquigarrow$ Need a compactness result which can deal with varying Banach spaces
4. Identify $\bar{y} = y(\bar{s})$ - open problem

Existence of optimal controls - prerequisites

Need a compactness result which can deal with varying Banach spaces.

Summary of properties of solutions to the state equation

1. For almost every $\omega \in \Omega$,
$$\sup_k \left(\|y_k(\omega)\|_{L^2(0,T,\mathcal{H}^{s_k}(D))} \right) < \infty$$
2. For almost every $\omega \in \Omega$,
$$\sup_k \left(\|y_k(\omega)\|_{L^2([0,T]\times D)} \right) < \infty$$
3. The trajectories of the family of stochastic processes $y_k(t)$ are in $C^{\delta_k}([0,T], L^2(D))$ for every k and $\delta_k \geq \delta_* \geq 1/4$

A compactness result

Compactness lemma

Given a sequence $\{y_{s_k}\}_{k\in\mathbb{N}}$ of $L^2(D)$-valued stochastic processes with δ-hölder continuous sample paths and $y_{s_k}(\omega) \in L^2(0, T, \mathcal{H}^{s_k}(D))$.
Then $\{y_k\}_{k\in\mathbb{N}}$ contains a subsequence that converges strongly in $L^2(D \times [0, T])$ for fixed ω.

Idea of the proof:

Solution properties $\Rightarrow$ the infinite string $\left(\{y_{k,1}\}_{k\in\mathbb{N}}, \{y_{k,2}\}_{k\in\mathbb{N}}, \dots\right)$ lies in

$$\mathfrak{C} := C^{1/4}([0, T]) \times C^{1/4}([0, T]) \times \dots$$

$\Rightarrow$ existence of a subsequence y_{k_m} which converges in $\mathfrak{C}$ to an infinite string $(y_1^*, y_2^*, \dots$, and every $y_j^* \in C^{1/4}([0, T])$.

Existence of optimal controls

Theorem: existence of pathwise optimal controls

Assume the eigenvalues of $\mathcal{L}^s$ and Q are such that $\sum_{j=1}^{\infty} \mu_j \lambda_j^s < \infty$.
Let the initial data y_0 be deterministic and satisfy
$\sup_{s \in \mathscr{S}} \|y_0\|_{\mathcal{H}^s} < +\infty$.
Then for almost every fixed $\omega \in \Omega$, the functional $\mathcal{J}(\omega)$ attains a minimum in the interior of $\mathscr{S}$ (the set of admissible controls).
Moreover

$$\inf_{s \in \mathscr{S}} \mathcal{J}(\omega) < +\infty.$$

Idea of the proof:

Show that for a fixed realisation $\omega \in \Omega$, the sequence $\{y_{s_k}(\omega)\}_{k \in \mathbb{N}}$ of solutions to the state equation (SPDE) with initial datum y_0 contains a subsequence that converges strongly in $L^2(D \times [0, T])$.

Thanks....

Дякую за увагу!

Thank you all for your attention

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