Heat kernel estimates for fractional evolution equations

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   - Tools from fractional calculus
   - Fractional evolution
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The equations

- Time-fractional diffusion for $\beta \in (0,1)$,

$$D_t^\beta f = \nabla \cdot (A(x)\nabla) f.$$

- Time-fractional pseudo-differential evolution (..constant coefficients) for $\beta \in (0,1)$ and $\alpha > 0$,

$$D_t^\beta f = \Psi_x(-i\nabla) f, \qquad \text{where } \Psi(p) = |p|^\alpha w(p/|p|).$$

- ...variable coefficients (time and space), asymptotic expansion, etc.

Some recent work

- Deng and Schilling, "Exact Asymptotic Formulas for the Heat Kernels of Space and Time-Fractional Equations"
- Chen et al., "Heat kernel estimates for time fractional equations"

Fractional calculus

The repeated Riemann integral at 0 of a function f is given by

$$I_0^n f(x) = \frac{1}{(n-1)!}\int_0^x (x-w)^{n-1} f(w)dw.$$

If $x > 0$, extend analytically to complex n with $Re(n) > 0$, which gives the Riemann-Liouville integral of order β with $Re(\beta) > 0$.

Riemann-Liouville integral of order $\beta \in (0,1)$

For some nice function f

$$I_0^\beta f(x) = \frac{1}{\Gamma(\beta)}\int_0^x (x-w)^{\beta-1} f(w)dw$$

Caputo-Djrbashian fractional derivative

The Caputo-Djrbashian derivative of order $\beta \in (0,1)$ is defined as the inverse operator of the fractional integral I_0^β,

$$I_0^\beta f(x) = \frac{1}{\Gamma(\beta)} \int_0^x f(w)(x-w)^{\beta-1} dw.$$

Caputo-Djrbashian derivative of order $\beta \in (0,1)$

$$D^\beta f(x) = I_0^{1-\beta}\left[\frac{d}{dx} f\right](x) = \frac{1}{\Gamma(-\beta)} \int_0^x \frac{f(x-y)-f(x)}{y^{1+\beta}} dy + \frac{f(x)-f(0)}{\Gamma(1-\beta)x^\beta}, \quad x > 0.$$

Stable densities

Stable densities are defined by:

$$w_\beta(x;1,1) = w_\beta(x) = \frac{1}{\pi}\Re\int_0^\infty \exp\{-ixy - |y|^\beta\}dy$$

(normalized, positively skewed)

Asymptotics of β-stable density near 0 and ∞

$$w_\beta(x) \sim c_\beta \begin{cases} x^{-1-\beta}, & x \to \infty \\ h(x), & x \to 0 \end{cases} \tag{1}$$

$$h(x) := x^{-\frac{2-\beta}{2(1-\beta)}} \exp\left\{-c_\beta x^{-\frac{\beta}{1-\beta}}\right\}$$

Mittag-Leffler functions

The Mittag-Leffler functions are defined by the series

$$E_\beta(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}, \quad \beta > 0, z \in \mathbb{C}.$$

Theorem (Pollard-Zolotarev Formula)

Let $\beta \in (0, 1)$. For any $s \in \mathbb{R}$,

$$E_\beta(s) = \frac{1}{\beta} \int_0^\infty e^{sx} x^{-1-\frac{1}{\beta}} w_\beta(x^{-\frac{1}{\beta}}) dx.$$

To prove: expand exponential in power series, then each term of sum is Mellin transform of stable densities, which recovers the infinite sum definition.

Operator-valued Mittag-Leffler function

Key importance of the formula for Mittag-Leffler functions is the ability to define the function $E_\beta(L)$ for any operator L that generates a strongly continuous semigroup $(T_t)_{t\geq 0}$

$$\begin{aligned} E_\beta(L) &= \frac{1}{\beta}\int_0^\infty e^{Lz} z^{-1-\frac{1}{\beta}} w_\beta(z^{-\frac{1}{\beta}})dz \\ &= \frac{1}{\beta}\int_0^\infty T_z z^{-1-\frac{1}{\beta}} w_\beta(z^{-\frac{1}{\beta}})dz \end{aligned}$$

Motivation

Take for example the (time-)fractional diffusion equation

$$D^{\beta} f = \Delta_x f$$

whose fundamental solution G is the law of time changed Brownian motion $B_x(\tau_0^{\beta}(t))$, where the time change process is an inverted stable subordinator:

$$G(t, x-y) = \frac{1}{\beta} \int_0^{\infty} \exp\left\{ -\frac{|x-y|^2}{t^{\beta} z} \right\} z^{-1-\frac{1}{\beta}} w_{\beta}(z^{-\frac{1}{\beta}}) dz$$

This (*non-Markovian*) time-changed process behaves like (a slowed down) Brownian motion with *trapping*.

Subdiffusive behaviour of time-changed Brownian motion

The scaling of an inverted β-stable subordinator and for Brownian motion is

$$\tau_0^\beta(t) = t^\beta \tau_0^\beta(1), \quad B_x(t) = t^{1/2} B_x(1)$$

$$\mathbb{E}(B_x(\tau_0^\beta(t)))^2 = \mathbb{E}\left(B_x(t^\beta \tau_0^\beta(1))\right)^2 = t^\beta < t = \mathbb{E}(B_x(t))^2$$

Fractional divergence equation

The equation

$$D^{\beta} f(t,x) = Lf(t,x) := \mathrm{div}(A(x)\nabla f(t,x)), \quad f(0,x) = Y(x)$$

where L is uniformly elliptic second order differential operator in divergence form, has solution

$$f(t,x) = E_{\beta}(Lt^{\beta})Y(x)$$

where

$$E_{\beta}(s) = \frac{1}{\beta}\int_0^{\infty} e^{sz} z^{-1-1/\beta} w_{\beta}(z^{-1/\beta})dz.$$

Let $G(t,x,y)$ be the heat kernel associated with L, i.e, the fundamental solution of

$$\partial_t f = Lf$$

Heat kernel for fractional diffusion

Then the solution to $D^\beta f = Lf$ is

$$f(x,t) = \frac{1}{\beta}\int_0^\infty \int_{\mathbb{R}^d} G(t^\beta z, x, y) Y(y) z^{-1-\frac{1}{\beta}} w_\beta(z^{-\frac{1}{\beta}}) dy dz$$

Heat kernel for fractional diffusion

Then the solution to $D^\beta f = Lf$ is

$$f(x,t) = \frac{1}{\beta}\int_0^\infty \int_{\mathbb{R}^d} G(t^\beta z, x, y) Y(y) z^{-1-\frac{1}{\beta}} w_\beta(z^{-\frac{1}{\beta}}) dy\, dz = \int_{\mathbb{R}^d} G^{(\beta)}(t,x,y) Y(y) dy$$

where

$$G^{(\beta)}(t,x,y) := \frac{1}{\beta}\int_0^\infty G(t^\beta z, x, y) z^{-1-\frac{1}{\beta}} w_\beta(z^{-\frac{1}{\beta}}) dz$$

Main result for second order elliptic operator

Theorem (IJ, V. Kolokoltsov, (forthcoming))

Let $\Omega := |x-y|^2 t^{-\beta}$. *The heat kernel* $G^{(\beta)}(t,x,y)$, *satisfies the following two-sided estimates for all* $t > 0$ *and* $x, y \in \mathbb{R}^d$,

- *For* $\Omega \leq 1$,

$$G^{(\beta)}(t,x,y) \asymp C \begin{cases} t^{-\frac{\beta}{2}}, & d = 1, \\ t^{-\beta}(|\log \Omega| + 1), & d = 2, \\ t^{-\frac{d\beta}{2}} \Omega^{1-\frac{d}{2}}, & d \geq 3. \end{cases}$$

- *For* $\Omega \geq 1$,

$$G^{(\beta)}(t,x,y) \asymp C t^{-\frac{d\beta}{2}} \Omega^{-\frac{d}{2}\left(\frac{1-\beta}{2-\beta}\right)} \exp\left\{-C_\beta \Omega^{\frac{1}{2-\beta}}\right\}.$$

Some estimates used for proof

Global Aronson estimate

For all $t \geq 0$, $x, y \in \mathbb{R}^d$,

$$\frac{1}{C} t^{-\frac{d}{2}} \exp\left\{-c\frac{|x-y|^2}{t}\right\} \leq G(t,x,y) \leq C t^{-\frac{d}{2}} \exp\left\{-c\frac{|x-y|^2}{t}\right\}.$$

Asymptotic formula (Laplace method)

For $a \in (0,\infty)$ such that $\phi(a) \geq 0$, $\phi''(a) > 0$, the following holds

$$\int_0^\infty \exp\{-\Omega\phi(z)\}dz \sim \sqrt{\frac{2\pi}{\Omega\phi''(a)}} \exp\{-\Omega\phi(a)\}, \quad \Omega \to \infty$$

Also the estimates/asymptotics of the stable densities w_β.

Pseudo-differential operator

Consider the fractional evolution equation

$$D^{\beta} f(t,x) = \Psi_{\alpha}(-i\nabla) f(t,x), \quad f(0,x) = Y(x)$$

with symbol

$$\Psi_{\alpha}(p) = |p|^{\alpha} w(p/|p|), \quad p \in \mathbb{R}^d, \alpha > 0, \quad w \in C^k(\mathbb{S}^{d-1}).$$

The solution is given by

$$\begin{aligned} f(t,x) &= E_{\beta}(\Psi_{\alpha}(-i\nabla) t^{\beta}) Y(x) \\ &= \int_{\mathbb{R}^d} \frac{1}{\beta} \int_0^{\infty} G_{\psi_{\alpha}}(t^{\beta} z, x-y) z^{-1-1/\beta} w_{\beta}(z^{-1/\beta}) dz Y(y) dy \end{aligned}$$

The fundamental solution G_{ψ_α} for the equation

$$\partial_t f = \Psi_\alpha(-i\nabla) f$$

satisfies the following estimates (for all $t > 0$)

$$\begin{aligned} G_{\psi_\alpha}(t,x) &\asymp C \min\left(t^{-d/\alpha}, \frac{t}{|x|^{d+\alpha}}\right) \\ &= C \min(t^{-d/\alpha}, t^{-d/\alpha}\Omega^{-1-d/\alpha}) \end{aligned}$$

where $\Omega = |x|^\alpha t^{-\beta}$, like before.

Assumption

Let $\alpha > 0$ and $\beta \in (0,1)$, assume that $w \in C^k(\mathbb{S}^{d-1})$, and that $w \geq w_0 > 0$ for some strictly positive function w_0.

Main result for ΨDO

Theorem (IJ, V. Kolokoltsov (forthcoming))

The fundamental solution $G^{(\beta)}_{\psi_\alpha}$ satisfies the following bounds:

- *For* $\Omega \le 1$

$$G^{(\beta)}_{\psi_\alpha}(t,x) \le \begin{cases} t^{-\frac{d\beta}{\alpha}} & d<\alpha, \\ t^{-\beta}(|\log(\Omega)|+1) & d=\alpha, \\ t^{-\frac{d\beta}{\alpha}}\Omega^{1-\frac{d}{\alpha}} & d>\alpha. \end{cases}$$

- *For* $\Omega > 1$

$$|G^{(\beta)}_{\psi_\alpha}(t,x)| \le Ct^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}}$$

Same kind of estimates hold (but only for finite time) if we allow ψ to have coefficients depending on space.

Proof

Proof technique is quite similar as one used in previous result.
Solution to $D^\beta f = \Psi_\alpha(-i\nabla)f$ is

$$f_t = E_\beta(\Psi_\alpha(-i\nabla)t^\beta)f_0.$$

Then

$$G^{(\beta)}_{\psi_\alpha}(t,x) = \frac{1}{\beta}\int_0^\infty G_{\psi_\alpha}(t,x)z^{-1-1/\beta}w_\beta(z^{-1/\beta})dz.$$

Use known estimates for G_{ψ_α} and w_β.

Future work

Using similar approaches we will tackle some of the following extensions:

- Allow coefficients to depend on time, and add some potential
- Asymptotic expansions
- Replace D^{β} with a generalised Caputo operator $D^{(\nu)}$, for some Lévy kernel ν...

Caputo-type operator

Consider the generator $D_+^{(\nu)}$ of an $\mathbb{R}$-valued jump-type Feller process (with negative jumps) of the form

$$D_+^{(\nu)} f(t) := \int_0^\infty (f(t-s) - f(t))\nu(t, ds), \quad t \in \mathbb{R}, \qquad (2)$$

for a (Lévy) kernel $\nu(t, \cdot)$ on $(0, \infty)$ such that $\sup_t \int_0^\infty \min(1, |s|)\nu(t, ds) < \infty$.
Now let us heuristically force all jumps that fall below 0 to fall exactly on 0 by modifying $D^{(\nu)}$ as follows

$$D^{(\nu)} f(t) := \int_0^t (f(t-s) - f(t))\nu(t, ds) + (f(0) - f(t)) \int_t^\infty \nu(t, ds), \quad t > 0.$$

(Last) Mittag-Leffler generalisation

The operator $D^{(\nu)}$ generates a Feller process with transition kernel $G_{(\nu)}(t, dy)$. Looking at evolutions of the type

$$D^{(\nu)} u = Au$$

for some operator A, reveals a further generalised family of Mittag-Leffler functions:

$$E_{(\nu),x}(A) = \int_0^\infty e^{tA} \frac{\partial}{\partial t} \left(\int_x^\infty G_{(\nu)}(t, dy) \right) dt.$$

For $\nu(t, dy) = \frac{\beta}{\Gamma(1-\beta)} \frac{dy}{y^{1+\beta}}$ and A being generator of strongly continuous semigroup, reduces to previous Mittag-Leffler generalisation.

Chen, Zhen-Qing et al. "Heat kernel estimates for time fractional equations". In: *Forum Mathematicum*. Vol. 30. 5. De Gruyter. 2018, pp. 1163–1192.

Deng, Chang-Song and René L Schilling. "Exact Asymptotic Formulas for the Heat Kernels of Space and Time-Fractional Equations". In: *arXiv preprint arXiv:1803.11435* (2018).

Thanks for listening!