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Glauber dynamics on the cone of discrete Radon measures

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- Continuous state space
- “locally finite” system
- agents have an additional property (“mark”) $\in (0, \infty)$ to model mortality, fecundity, etc.

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This leads to

Definition

The cone of discrete positive measures is defined by

$$\mathbb{K} = \mathbb{K}(\mathbb{R}^d) := \left\{ \omega = \sum_i s_i \delta_{x_i} \in \mathbb{M}(\mathbb{R}^d) \middle| s_i \in (0, \infty), x_i \in \mathbb{R}^d \right\}$$

where $\mathbb{M}(\mathbb{R}^d)$ is the set of all Radon measures on $\mathbb{R}^d$. Denote by $\tau(\omega)$ the support of $\omega \in \mathbb{K}$.

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Important technical connection between $\mathbb{K}$ and $\mathbb{K}_0$:

Definition

The K-transform $K \colon B_{\mathrm{bs}}(\mathbb{K}_0) \to L^1(\mathbb{K}, d\pi)$ is defined by

$$(KG)(\omega) = \sum_{\eta \Subset \omega} G(\eta),$$

where

$$B_{\mathrm{bs}}(\mathbb{K}_0) := \{G \colon \mathbb{K}_0 \to \mathbb{R} \mid \exists N, \Lambda \colon G(\eta) = 0 \text{ if } |\tau(\eta)| > N \text{ or } \eta(\Lambda^C) \neq 0\}$$

Note that K can be extended to $K \colon L^1(\mathbb{K}_0, d\lambda_\pi) \to L^1(\mathbb{K}, d\pi)$, $\lambda_\pi = K^*\pi$.

Kolmogorov equation:

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We want to consider the quasi-Kolmogorov equation

$$\frac{\partial}{\partial t} G_t(\eta) = \hat{L} G_t(\eta)$$

where $\hat{L} := K^{-1} L K$ and $G \in L^1(\mathbb{K}_0)$ with an appropriate measure defined below.

Definition of generator

Definition

Let $\lambda = \frac{1}{s} e^{-s} ds$ on $(0, \infty)$ and σ a nonatomic Radon measure on $\mathbb{R}^d$. The generator L on functions $F \colon \mathbb{K} \to \mathbb{R}$ is defined by

$$(LF)(\omega) = \sum_{x \in \tau(\omega)} s_x \left[F(\omega - s_x \delta_x) - F(\omega) \right] + \int_{\mathbb{R}_+^* \times \mathbb{R}^d} \left[F(\omega + s_x \delta_x) - F(\omega) \right] e^{-\Phi((s,x);\omega))} s \lambda(ds) \sigma(dx)$$

where for $\omega := (s_y, y)_{y \in \tau(\omega)} \in \mathbb{K}$

$$\Phi\left((s,x);\omega\right) := 2s \sum_{y \in \tau(\omega)} s_y \phi(x,y)$$

with a pair potential $\phi \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$.

Derivation of generator

Consider the "gradient"

$$D^-_{(s_x,x)}F(\omega) := F(\omega - s_x\delta_x) - F(\omega)$$

and define the Dirichlet form

$$\begin{aligned}\mathcal{E}(F,G) &:= \frac{1}{2}\int_{\mathbb{K}}\int_{\mathbb{R}^d} D^-_{(s_x,x)}F(\omega)D^-_{(s_x,x)}G(\omega)\omega(dx)\mu(d\omega)\\ &= \frac{1}{2}\int_{\mathbb{K}}\sum_{x\in\tau(\omega)} s_x D^-_{(s_x,x)}F(\omega)D^-_{(s_x,x)}G(\omega)\mu(d\omega)\end{aligned}$$

for a Gibbs measure μ and $F, G \in K(B_{\text{bs}}(\mathbb{K}))$.

Proposition

For $F, G \in K(B_{bs}(\mathbb{K}))$*, we have*

$$\mathcal{E}(F,G) = (LF, G)_{L^2(\mathbb{K},\mu)}$$

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Main ingredient:

Lemma (Georgii-Nguyen-Zessin identity)

Let $F : \mathbb{R}^d \times \mathbb{K} \to \mathbb{R}_+$ measurable and μ a Gibbs measure. Then

$$\int_{\mathbb{K}} \int_{\mathbb{R}^d} F(x,\omega)\omega(dx)\mu(d\omega) = \int_{\mathbb{K}} \int_{\mathbb{R}_+^* \times \mathbb{R}^d} F(x, \omega + s\delta_x) e^{-\Phi((s,x);\omega)} s\lambda(ds)\sigma(dx)\mu(d\omega) \tag{GNZ}$$

where Φ is defined as above.

Kondratiev, Lytvynov ’05: Existence of Hunt process for homogeneous particles

Kondratiev, Kutoviy, Zhizhina ’06: Existence of dynamics for homogeneous particles

Hagedorn, Kondratiev, Pasurek, Röckner ’13: Properties of $\mathbb{K}$, Gibbsian formalism

Kondratiev, Kuna, Lytvynov ’15: Characterization of probability measures on $\mathbb{K}$

Generator on $\mathbb{K}_0$

Proposition

The operator $\hat{L} = K^{-1}LK$ on $G\colon \mathbb{K}_0 \to \mathbb{R}$ corresponding to L is given by

$$(\hat{L}G)(\eta) = -\left(\sum_{x\in\tau(\eta)} s_x\right) G(\eta)$$

$$+ \int_{\mathbb{R}_+^*\times\mathbb{R}^d} s \sum_{\xi\subset\eta} G(\xi + s\delta_x) e_\lambda(e^{-2ss_y\phi(x,y)}, \xi)\times$$

$$\times\, e_\lambda(e^{-2ss_y\phi(x,y)} - 1, \eta - \xi)\lambda(ds)\sigma(dx)$$

where

$$e_\lambda(f,\eta) := \prod_{x\in\tau(\eta)} f(s_x, x)$$

Theorem (Finkelshtein, Kondratiev, K '18)

For $C > 0$ and $\alpha \in \mathbb{R}$, define the family of weighted L^1-type spaces

$$\mathbf{L}_{\alpha,C} := L^1\left(\mathbb{K}_0, C^{|\tau(\eta)|} e^{\alpha \sum_{x\in\tau(\eta)} s_x} d\lambda_\mu\right)$$

with the usual L^1-norm, $\lambda_\mu = K^*\mu$ for μ Gibbs measure on $\mathbb{K}(\mathbb{R}^d)$. Assume

$$C \leq \frac{(1-\alpha)\alpha}{2\int_{\mathbb{R}^d} \phi(x,y)\sigma(dy)} \tag{1}$$

and $C \geq 2$. Then $(\hat{L}, \mathcal{D}(\hat{L}))$ generates an analytic semigroup on the space $\mathbf{L}_{\alpha,C}$, where

$$\mathcal{D}(\hat{L}) := \{G \in \mathbf{L}_{\alpha,C} | \sum_{x\in\tau(\eta)} s_x \cdot G(\eta) \in \mathbf{L}_{\alpha,C}\}$$

Sketch of Proof

First step: Split the operator $\hat{L} = \hat{L}_0 + \hat{L}_1$, where

$$\hat{L}_0 G(\eta) = -\left(\sum_{x \in \tau(\eta)} s_x\right) G(\eta) \text{ and } \hat{L}_1 = \hat{L} - \hat{L}_0.$$

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Proposition

For any α, C, the operator $\hat{L}_0$ defined above with the domain

$$\mathcal{D}(\hat{L}_0) := \left\{ G \in \mathbf{L}_{\alpha,C} \middle| \sum_{x\in\tau(\eta)} s_x \cdot G(\eta) \in \mathbf{L}_{\alpha,C} \right\}$$

generates a contraction semigroup on $\mathbf{L}_{\alpha,C}$. Moreover, this semigroup is analytic.
Proof: Hille-Yosida.

Second step: View $\hat{L}_1$ as perturbation of $\hat{L}_0$.

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Lemma (relative bound for $\hat{L}_1$)

Let $\alpha < 1$. For any $C > 0$ such that (1) *holds, the following estimate holds:*

$$\|\hat{L}_1 G\|_{\alpha,C} \leq \frac{1}{C}\|\hat{L}_0 G\|_{\alpha,C}, G \in \mathcal{D}(\hat{L}_0) = \mathcal{D}(\hat{L}_1)$$

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Note that (1) is crucial: Calculation yields

$$\|\hat{L}_1 G\| \leq C^{-1} \int_{\mathbb{K}_0} |G(\xi_1)| \sum_{x \in \tau(\xi_1)} s_x \exp\left[\Psi(s_x, x)\right] \mathbf{C}(\xi_1) \lambda_\mu(d\xi_1) \overset{!}{\leq} \frac{1}{C}\|\hat{L}_0 G\|$$

where $\mathbf{C}(\eta) = C^{|\tau(\eta)|} e^{\alpha \sum_{x \in \tau(\eta)} s_x}$ and

$$\Psi(s_x, x) = \left(2C \int_{\mathbb{R}_+^* \times \mathbb{R}^d} \phi(x,y) e^{(\alpha-1)s} ds\sigma(dy) - \alpha\right) s_x$$

In the homogeneous case, the following holds:

Theorem (Kondratiev, Kutoviy, Zhizhina ’06)

For any $C, \beta > 0$ *which satisfy*

$$2\exp(C(\beta)C) < C$$

the operator $\hat{L}$ *generates an analytic semigroup on*

$$\mathcal{L}_{C,\beta} = L^1(\Gamma_0, C^{|\eta|} e^{-\beta E(\eta)} \lambda_\mu(d\eta))$$

and $C(\beta)$ *is defined as*

$$C(\beta) := \int_{\mathbb{R}^d} |1 - \exp[-\beta\phi(x)]| dx.$$

Note that there is no a-priori restriction on C.

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Thank You for your attention!