On stability of travelling wave solutions for integro-differential equations related to branching Markov processes.

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Motivation

Let $\mathbf{X} = (\mathbf{X}_t)_{t \geq 0}$ be a branching Brownian motion.
Informally the process with $\mathbf{X}_0 = \{0\}$ may be described as follows:
A particle starts from the origin and moves as a standard one-dimensional Brownian Motion on a real line. It dies at a random time with an exponential distribution of parameter 1. When the particle dies, it produces two new points at the place of its death. Each of the two particles repeats behaviour of the parent independently of each other. The process continues indefinitely.

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Denote $\mathbb{R}_+ := [0, \infty)$ and let $\mathbb{1}$ be an indicator function.

$$u(x,t) := \mathbf{E}_{\{x\}} \Big[\prod_{y \in \mathbf{X}_t} \mathbb{1}_{\mathbb{R}_+}(y) \Big] = \mathbf{P}_{\{0\}} \big[y \geq -x, \ \forall\, y \in \mathbf{X}_t \big]$$
$$= \mathbf{P}_{\{0\}} \big[\text{the left-most particle of } \mathbf{X}_t \text{ is} \geq -x \big].$$

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&= \mathbf{P}_{\{0\}} \big[\text{the left-most particle of } \mathbf{X}_t \text{ is } \geq -x \big].
\end{aligned}
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Then u solves the following equation

$$
\begin{cases}
\dfrac{\partial u}{\partial t}(x,t) = \dfrac{1}{2} \dfrac{\partial^2 u}{\partial x^2}(x,t) + u^2(x,t) - u(x,t), & x \in \mathbb{R}, \ t > 0, \\
u(x,0) = \mathbb{1}_{\mathbb{R}_+}(x), & x \in \mathbb{R}.
\end{cases} \tag{1}
$$

Hence, $1 - u$ solves the Fisher-KPP equation.

`MkKean, 1975`

Theorem 1.
Let $\mathbf{X} = (\mathbf{X}_t)_{t\geq 0}$ *be a branching Brownian motion, then for some constant* $C \in \mathbb{R}$ *its left-most particle* M_t *satisfies ,*

$$\lim_{t\to\infty} \mathbf{P}_{\{0\}}\left[M_t + c_* t - \frac{3}{2\lambda_*}\ln t + C \geq -x\right] = \phi(x), \quad x \in \mathbb{R},$$

where $c_* = \lambda_* = \sqrt{2}$, $\phi(x) = \mathbf{E}_{\{0\}}\left[e^{-e^{-\lambda_* x} D_\infty}\right]$, $\mathbf{E}_{\{0\}}\left[D_\infty > 0\right] = 1$,

$$\lim_{x\to+\infty} \phi(x) = 1, \qquad \lim_{x\to-\infty} \phi(x) = \mathbf{E}_{\{0\}}\left[D_\infty = 0\right] = 0.$$

In other words the solution to (1) *satisfies*

$$\lim_{t\to\infty} u(x + c_* t - \frac{3}{2\lambda_*}\ln t + C, t) = \phi(x), \quad x \in \mathbb{R}.$$

Moreover $(x,t) \to \phi(x - c_* t)$ *solves* (1)*, thus, it is a monotone travelling wave solution.*

Uchiama 1978; Bramson 1983; Lau 1985; Lalley, Selke 1987.

Goal: Generalize the previous theorem to more general branching Markov processes (Lévy instead of BM + more general branching mechanisms).

Relation between Branching Markov Processes and Evolution Equations

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If a process $\mathbf{X} = (\mathbf{X}_t)_{t \geq 0}$ takes values in $\mathbf{R}$, then $\mathbf{X}_t \in \mathbb{R}^n_{sym}$ means that at time t the process $\mathbf{X}$ consists of n points, which are located on the real line $\mathbb{R}$. $\mathbf{X}$ dies out if there exists $t > 0$ such that $\mathbf{X}_t \in \mathbb{R}^0_{sym} = \{\varnothing\}$.

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Let $\mathbf{X} = (\mathbf{X}_t, \mathbf{P}_{\mathbf{x}})$ *be a right-continuous temporally homogeneous Markov process on* $\mathbf{R}$*. Then* $\mathbf{X}$ *is called* *a branching Markov process* *if it satisfies*

$$\mathbf{E}_{\mathbf{x}}\Big[\prod_{y\in\mathbf{X}_t} f(y)\Big] = \prod_{x\in\mathbf{x}} \mathbf{E}_{\{x\}}\Big[\prod_{y\in\mathbf{X}_t} f(y)\Big],$$

for every $\mathbf{x} \in \mathbf{R}$, $t \geq 0$, $f \in B(\mathbf{R})$, $\|f\| < 1$.

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Such definition is too general. We need more restrictions on the process to be able to work with it.

`Ikeda, Nagasawa and Watanabe, 1968,1969`

Definition 2.

Let $\mathbf{X} = (\mathbf{X}_t, \mathbf{P}_{\mathbf{x}})$ *be* a branching strong Markov process *and there exist a random time* τ*, which satisfies the following conditions*

1. *There exists a stochastic kernel* $\pi(x, E)$ *on* $\mathbb{R} \times \mathbf{R}$ *such that for each* $x \in \mathbb{R}$*, and* E *– Borel in* $\mathbf{R}$*, on* $\{\tau < \infty\}$*:*

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Also, up to time τ *we identify* $\mathbf{X}$ *started from* $\mathbf{X}_0 = \{x\}$*,* $x \in \mathbb{R}$*, with a Markov process on* $\mathbb{R}$ *which we denote by* $X^0 = (X_t^0, \mathbb{P}_x^0)$*. We terminate* X^0 *at* τ*.*

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Then, we call $\mathbf{X}$ *a* (X^0, π)*-branching Markov process.*

Moreover, X^0 *is called a non-branching part of* $\mathbf{X}$*,* τ *– the first branching time of* $\mathbf{X}$*, and* π *– a branching law of* $\mathbf{X}$*.*

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Starting from X^0, τ and π one can construct a (X^0, π)-branching Markov process. Such process is unique up to finite dimensional distributions.

`Ikeda, Nagasawa and Watanabe, 1968,1969`

Let $\mathbf{X}_t$ be a (X^0, π)-branching Markov process and $f \in \mathrm{B}(\mathbb{R})$, $\|f\| < 1$.

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$$\Big(\mathbf{E}_{\mathbf{X}_s=\mathbf{z}}\big[\prod_{y\in\mathbf{X}_t} f(y)\big] = \mathbf{E}_{\mathbf{z}}\big[\prod_{y\in\mathbf{X}_{t-s}} f(y)\big] = \prod_{z\in\mathbf{z}} \mathbf{E}_{\{z\}}\big[\prod_{y\in\mathbf{X}_{t-s}} f(y)\big] = \prod_{z\in\mathbf{z}} u(z, t{-}s) \Big)$$

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We derive the so-called S-equation:

$$\begin{cases} u(x,t) = \mathbf{E}_{\{x\}}\big[f(X_t^0), \tau > t\big] \\ \qquad\qquad + \displaystyle\int_0^t\int_{\mathbb{R}}\int_{\mathbf{R}} \mathbf{P}_{\{x\}}\big[\tau \in ds, X_{s-}^0 \in dy\big]\pi(X_{s-}^0, d\mathbf{z})\prod_{z\in\mathbf{z}} u(z,t-s), \\ u(x,0) = f(x). \end{cases}$$

Ikeda, Nagasawa and Watanabe, 1968,1969

Under additional regularity assumptions on the (X^0, π)-branching Markov process $\mathbf{X}$ we could derive the following PDE:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = (A^0 u)(x,t) + k \displaystyle\int_{\mathbf{R}} \pi(x, d\mathbf{z}) \prod_{z \in \mathbf{z}} u(z,t), & x \in \mathbb{R},\ t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}. \end{cases}$$

where A^0 is the generator of the non-branching part X^0, $k := \frac{\mathrm{d}\mathbf{P}_{\{x\}}[\tau \in dt]}{\mathrm{d}t}(0)$ - value of the probability density of the branching time τ at 0.

`Ikeda, Nagasawa and Watanabe, 1968,1969`

1) Let the non-branching part X^0 be a standard Brownian motion up to the branching time τ, which is exponentially distributed with rate 1:

$$(A^0 u)(x) = \frac{1}{2}\partial^2_{xx} u(x) - u(x).$$

Suppose that a particle at the moment of its death gives birth to two children which are positioned at the same point, where the parent dies:

$$\pi(x, d\mathbf{z}) = \mathbb{1}_{\mathbb{R}^2_{sym}}(\mathbf{z})\delta_x(dz_1)\delta_x(dz_2).$$

As a result, we have

$$\partial_t u(x,t) = \frac{1}{2}\partial^2_{xx} u(x,t) - u(x,t) + u^2(x,t).$$

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$$\partial_t u(x,t) = \frac{1}{2}\partial^2_{xx} u(x,t) - u(x,t) + u^2(x,t).$$

2) Let the non-branching part X^0 be trivial: the point does not move and dies with a random exponentially distributed time with rate 1.

$$(A^0 u)(x) = -u(x).$$

Next, we assume that a particle gives birth to n children with a probability p_n, and children are placed at the same point where the parent dies.

$$\pi(x, d\mathbf{z}) = \sum_{n\in\mathbb{N}_0} p_n \mathbb{1}_{\mathbb{R}^n_{sym}}(\mathbf{z}) \prod_{j=1..n} \delta_x(dz_j).$$

Hence, we have

$$\partial_t u(t) = -u(t) + \sum_{n\in\mathbb{N}_0} p_n u^n(t).$$

3) Let the non-branching part X^0 of a branching Markov process $\mathbf{X}$ be the pure-jump Markov process with a bounded jump-kernel $a \in L^1(\mathbb{R}\to\mathbb{R}_+)$ and the jump rate 1. Namely, starting from a point $x \in \mathbb{R}$, the process X^0 waits a random exponentially distributed time with rate 1, and, then, it jumps from x to a point $y \in \mathbb{R}$ with probability $a(y-x)dy$. Next, we suppose, that the branching time τ is exponentially distributed with rate 1. At time τ the particle $X^0_{\tau-}$ dies.

$$(A^0 u)(x) = \int_{\mathbb{R}} a(x-y)(u(y)-u(x))dy - u(x) = (a*u)(x) - 2u(x).$$

Suppose, a particle at the moment of its death gives birth to two children which are positioned at the same point, where the parent dies. Then,

$$\partial_t u(x,t) = (a*u)(x,t) - 2u(x,t) + u^2(x,t).$$

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$$(A^0 u)(x) = \int_{\mathbb{R}} a(x-y)(u(y)-u(x))dy - u(x) = (a*u)(x) - 2u(x).$$

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$$\partial_t u(x,t) = (a*u)(x,t) - 2u(x,t) + u^2(x,t).$$

4) In the previous example assume that a particle at the moment of its death gives birth to two children, one of which is positioned at the same point where the parent dies, and the second one is placed randomly at $z \in \mathbb{R}$ with a probability $b(z - X^0_{\tau-})dz$, where b is a bounded probability density:

$$\pi(y, d\mathbf{z}) = 1\!\!1_{\mathbb{R}^2_{sym}}(\mathbf{z})\delta_y(dz_1)b(z_2-y)dz_2.$$

Then,

$$\partial_t u(x,t) = (a*u)(x,t) - 2u(x,t) + u(x,t)(\bar{b}*u)(x,t), \quad \bar{b}(x) := b(-x).$$

Assumption 1.

The (X_0, π)*-branching Markov process does not explode in finite time. This is equivalent to the fact that* $u \equiv 1$ *is the unique solution to the corresponding S-equation with the initial condition* $f \equiv 1$*.*

Theorem 2.

Let $\mathbf{X}$ *satisfy Assumption 1. Then, for* $f \in B(\mathbb{R})$, $0 \le f \le 1$,

$$u(x,t) = \mathbf{E}_{\{x\}}\Big[\prod_{y \in \mathbf{X}_t} f(y)\Big], \qquad x \in \mathbb{R},\ t \ge 0,$$

is the minimal (in the class of non-negative functions) solution to the S-equation with the initial condition f*.*

Ikeda, Nagasawa and Watanabe, 1968,1969

On Limiting Behaviour of a Branching Random Walk

Let $\mathbf{Y} = (\mathbf{Y}_n, \mathbf{P}_\mathbf{x})_{n\in\mathbb{N}}$ be a branching random walk on the real line. Informally the process started from $\mathbf{Y}_0 = \{x\}$, $x \in \mathbb{R}$, may be described as follows: Its children, who form the first generation, are scattered in $\mathbb{R}$ according to the distribution of a point process Ξ:

$$\mathbf{Y}_1 = \{x + y \,\big|\, y \in \Xi\}.$$

Each of the particles in the first generation produces its own children:

$$\mathbf{Y}_2 = \cup\{z + y \,\big|\, z \in \mathbf{Y}_1,\ y \in \Xi\},$$

who are thus in the second generations and are positioned (with respect to their parent) according to the same distribution of Ξ. Each individual in the n-th generation $\mathbf{Y}_n$ produces independently of each other and member of earlier generations. The system goes on indefinitely, but can possibly die if there is no particle at a generation.

`Z. Shi, Lecture Notes, 2015`

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As a result, $\mathbf{Y}$ is a spatially and temporally homogenous Markov chain, which satisfies

$$\mathbf{E}_\mathbf{x}\Big[\prod_{y\in\mathbf{Y}_n} f(y)\Big] = \prod_{x\in\mathbf{x}} \mathbf{E}_{\{x\}}\Big[\prod_{y\in\mathbf{Y}_n} f(y)\Big],$$

for every $\mathbf{x} \in \mathbf{R}$, $n \in \mathbb{N}_0$, $f \in \mathrm{B}(\mathbf{R})$, $\|f\| < 1$.

Denote the log-Laplace transform of Ξ (i.e. $\mathbf{Y}_1$ started from $\mathbf{Y}_0 = \{0\}$) by

$$\psi(\lambda) := \ln \mathbf{E}_{\{0\}}\Big[\sum_{y \in \mathbf{Y}_1} e^{-\lambda y}\Big], \qquad \lambda \in \mathbb{R}.$$

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With the following assumption we ensure that $\mathbf{Y}$ survives with a positive probability and its left-most particle propagates proportionally to n:

$$\psi(0) \in (0,\infty) \quad \text{and} \quad \exists\, \lambda > 0 : \ \psi(\lambda) < \infty. \tag{A2}$$

Then, speed of the propagation equals $c_* = \inf_{\lambda>0} \frac{\psi(\lambda)}{\lambda}$.

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$$\exists \lambda_* \in (0, \infty) : \quad \inf_{\lambda > 0} \frac{\psi(\lambda)}{\lambda} = \frac{\psi(\lambda_*)}{\lambda_*}, \qquad \frac{\psi(\lambda)}{\lambda} \in C^1(\{\lambda_*\}), \tag{A3}$$

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Moreover, we suppose that

$$\mathbf{E}_{\{0\}}\Big[\Big(\sum_{x\in\mathbf{Y}_1} g_1(x)\Big)\Big(\ln_+ \sum_{x\in\mathbf{Y}_1} g_1(x)\Big)^2\Big] < \infty \tag{H1}$$

$$\mathbf{E}_{\{0\}}\Big[\Big(\sum_{x\in\mathbf{Y}_1} g_2(x)\Big)\Big(\ln_+ \sum_{x\in\mathbf{Y}_1} g_2(x)\Big)\Big] < \infty \tag{H2}$$

where, $\ln_+ x := \ln \max\{x, 1\}$, and for $y \in \mathbb{R}$,

$$h(y) = \lambda_* y + \psi(\lambda_*), \quad g_1(y) = e^{-h(y)}, \quad g_2(y) = \max\{0, h(y)\} e^{-h(y)}.$$

Let (M_n) denote a position of the left-most particle of $\mathbf{Y}$,

$$M_n := \min\{x \in \mathbb{R} \,|\, x \in \mathbf{Y}_n\}, \quad n \in \mathbb{N}_0. \tag{2}$$

Theorem 3 (E. Aïdékon, 2013).
Under (A2), (A3), (A4), (H1), (H2), *if* $\mathbf{Y}_1 \not\subset \{a + b\mathbb{Z}\}$ *for any* $a, b \in \mathbb{R}$, *then there exists a constant* $C_* > 0$, *such that for any* $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbf{E}_{\{0\}}\left[M_n + c_* n - \frac{3}{2\lambda_*}\ln n + C_* \geq x\right] = \mathbf{E}_{\{0\}}\left[e^{-e^{\lambda_* x} D_\infty}\right],$$

where D_∞ *is the almost sure limit of the derivative martingale*

$$D_n = \prod_{y\in\mathbf{Y}_n} (\lambda_* y + n\psi(\lambda_*))e^{-\lambda_* y - n\psi(\lambda_*)}.$$

Moreover,

$$\mathbf{P}_{\{0\}}\left[D_\infty > 0 \,\middle|\, \mathbf{Y} \text{ does not extinct}\right] = 1.$$

Aïdékon included: $\lambda_* \in [0, \infty]$, $\lambda_* = \sup\{\mu > 0 : \psi(\mu) < \infty\}$.

Main Result

Proposition 1.

Let $\mathbf{X}$ *be a spatially homogeneous* (X^0, π)*-branching Markov process. Then for any* $T > 0$ *its sampling* $\{X_{nT}\}_{n\in\mathbb{N}_0}$ *is a branching random walk.*

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Idea: Apply the result of Aïdékon to $\{\mathbf{X}_{\frac{n}{2^k}}\}_{n \in \mathbb{N}}$, $k \in \mathbb{N}$. Take $k \to \infty$.

Problem: All parameters and assumptions a priori depend on k. Such dependence must be clarified.

Proposition 2.

Let $\mathbf{X}$ *be a spatially homogeneous* (X^0, π)*-branching Markov process satisfying Assumption 1. Denote, for* $x \in \mathbb{R}$*,* $t > 0$*,* $\lambda \in \mathbb{R}$*,*

$$v_\lambda(x,t) := \mathbf{E}_{\{x\}}\Big[\sum_{y \in \mathbf{X}_t} e^{-\lambda y}\Big] \in [0, \infty].$$

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Then $v_\lambda(x,0) = e^{-\lambda x}$ *and* v *is the minimal non-negative solution to*

$$v_\lambda(x,t) = \mathbf{E}_{\{x\}}\big[f(X^0_t), \tau > t\big] + \int_0^t \int_{\mathbb{R}} \int_{\mathbf{R}} \mathbf{P}_{\{x\}}\big[\tau \in ds, X^0_{s-} \in dy\big] \pi(X^0_{s-}, d\mathbf{z}) \sum_{z \in \mathbf{z}} v_\lambda(z, t-s),$$

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Under additional regularity assumptions on $\mathbf{X}$,

$$\frac{\partial v_\lambda}{\partial t}(x,t) = (A^0 v_\lambda)(x,t) + k \int_{\mathbf{R}} \pi(x, d\mathbf{z}) \sum_{z \in \mathbf{z}} v_\lambda(z,t),$$

where A^0 *is the generator of* X^0, $k := \frac{d\mathbf{P}_{\{x\}}[\tau \in dt]}{dt}(0)$.

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where A^0 *is the generator of* X^0, $k := \frac{d\mathbf{P}_{\{x\}}[\tau \in dt]}{dt}(0)$.

Moreover, if $v_\lambda(0,t) < \infty$ *and* $v_0(0,t) < \infty$, *then*

$$v_\lambda(0,t) = v_\lambda(0,t-s) v_\lambda(0,s) \quad \text{and} \quad v_\lambda(0,s) < \infty \quad \text{for} \quad s \in [0,t].$$

Corollary 1.

Denote $\psi_k(\lambda) := \ln v_\lambda(0, 2^{-k})$. *Then,* $2^k \psi_k(\lambda) = \psi_0(\lambda)$.

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Therefore, (A2), (A3) *and* (A4) *for* ψ_0 *imply analogous assumptions for* ψ_k *with the same* λ *and* λ_*, *namely*

$$\psi_k(0) \in (0, \infty), \quad \psi_k(\lambda) < \infty, \quad \frac{\psi_k(\lambda_*)}{\lambda_*} = \inf_{\lambda > 0} \frac{\psi_k(\lambda)}{\lambda} = \frac{c_*}{2^k}.$$

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As a result, in order to check (A2), (A3) *and* (A4) *uniformly in* k *it is sufficient to compute* $v_\lambda(0, 1)$*, e.g. by solving the corresponding evolution equation, and then check that there exist* λ*,* $\lambda_* \in (0, \infty)$*:*

$$\ln v_\lambda(0, 1) \in (0, \infty), \quad \ln v_\lambda(0, 1) < \infty, \quad \frac{\ln v_{\lambda_*}(0, 1)}{\lambda_*} = \inf_{\lambda > 0} \frac{\ln v_\lambda(0, 1)}{\lambda},$$

$$\lambda_* < \sup\{\mu > 0 : \ln v_\mu(0, 1) < \infty\}.$$

$$\mathbf{E}_{\{0\}}\Big[\Big(\sum_{x\in\mathbf{X}_{2-k}} g_1(x)\Big)\Big(\ln_+ \sum_{x\in\mathbf{X}_{2-k}} g_1(x)\Big)^2\Big] < \infty \qquad (H1_k)$$

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where $h(y) = \lambda_* y + \psi(\lambda_*), \quad g_1(y) = e^{-h(y)}, \quad g_2(y) = \max\{0, h(y)\}e^{-h(y)}.$

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$$w_{\lambda,\mu}(x,t) := \mathbf{E}_{\{x\}}\Big[\sum_{y\in\mathbf{X}_t} e^{-\lambda y} \sum_{y\in\mathbf{X}_t} e^{-\mu y}\Big] \in [0,\infty].$$

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Then $w_{\lambda,\mu}(x,0) = e^{-(\lambda+\mu)x}$ and under additional regularity assumptions on $\mathbf{X}$, $w_{\lambda,\mu}$ is the minimal non-negative solution to

$$\frac{\partial w_{\lambda,\mu}}{\partial t}(x,t) = (A^0 w_{\lambda,\mu})(x,t) + k\int_{\mathbf{R}} \pi(x,d\mathbf{z})\Big[\sum_{z\in\mathbf{z}} w_{\lambda,\mu}(z,t) + \sum_{\substack{z,\tilde{z}\in\mathbf{z},\\ z\neq\tilde{z}}} v_\lambda(z,t)v_\mu(\tilde{z},t)\Big].$$

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Proposition 3.

If there exists $\delta > 0$ such that $w_{0,0}(0,1) + w_{0,\lambda_*}(0,1) + w_{\delta,\lambda_*}(0,1) < \infty$, then $(H1_k)$ and $(H2_k)$ hold for all $k \in \mathbb{N}$.

The corresponding to $\{\mathbf{X}_{\frac{n}{2^k}}\}_{n\in\mathbb{N}_0}$ derivative martingale $\{D_n(k)\}_{n\in\mathbb{N}_0}$ satisfies

$$D_n(k) \to D_\infty(k),\ n \to \infty,\ a.s.$$

Since $D_{2^k n}(k) = D_n(0)$, then a.s. $D_\infty(k) = D_\infty(0) = D_\infty$, $k \in \mathbb{N}$.

Similarly $C_*(k) = C_*(0) + \frac{3}{2} k \ln 2$.

As a result $M_t = \min\{x \in \mathbf{X}_t\}$ satisfies,

$$\lim_{n\to\infty} \mathbf{E}_{\{0\}}\left[M_{\frac{n}{2^k}} + \frac{n}{2^k} c_* - \frac{3}{2\lambda_*} \ln \frac{n}{2^k} + C_* \geq x\right] = \mathbf{E}_{\{0\}}\left[e^{-e^{\lambda_* x} D_\infty}\right].$$

Taking k large one can show that the limit holds for all $t \in \mathbb{R}$.

Theorem 4 (Main result).

Let $\mathbf{X}$ *be a spatially homogeneous* (X^0, π)*-branching Markov process satisfying Assumption 1 (i.e.* $\mathbf{X}$ *does not blow up in finite time). Suppose that the log-Laplace transform of* $\mathbf{X}_1$ *satisfies* (A2), (A3) *and* (A4) *(which may be checked in termes of* v_λ*). Assume also that there exists* $\delta > 0$ *such that*

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M_t *and the minimal solution* u *to the S-equation are connected by*

$$u(x,t) = \mathbf{E}_{\{x\}}\Big[\prod_{y\in\mathbf{X}_t} \mathbb{1}_{\mathbb{R}_+}(y)\Big] = \mathbf{P}_{\{0\}}\left[M_t > -x\right].$$

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Therefore,

$$\lim_{t\to\infty} u(x + c_* t - \frac{3}{2\lambda_*}\ln t + C_*, t) = \mathbf{E}_{\{0\}}\left[e^{-e^{-\lambda_* x} D_\infty}\right] =: \phi(x).$$

Theorem 4 (Main result).

Let $\mathbf{X}$ *be a spatially homogeneous* (X^0, π)*-branching Markov process satisfying Assumption 1 (i.e.* $\mathbf{X}$ *does not blow up in finite time). Suppose that the log-Laplace transform of* $\mathbf{X}_1$ *satisfies* (A2), (A3) *and* (A4) *(which may be checked in termes of* v_λ*). Assume also that there exists* $\delta > 0$ *such that*

$$w_{0,0}(0,1) + w_{0,\lambda_*}(0,1) + w_{\delta,\lambda_*}(0,1) < \infty.$$

Then the left-most particle of $\mathbf{X}_t$*:* $M_t = \min\{x \in \mathbf{X}_t\}$ *satisfies,*

$$\lim_{n\to\infty} \mathbf{E}_{\{0\}}\left[M_t + c_* t - \frac{3}{2\lambda_*}\ln t + C_* \geq x\right] = \mathbf{E}_{\{0\}}\left[e^{-e^{\lambda_* x} D_\infty}\right].$$

M_t *and the minimal solution* u *to the S-equation are connected by*

$$u(x,t) = \mathbf{E}_{\{x\}}\Big[\prod_{y\in\mathbf{X}_t} \mathbb{1}_{\mathbb{R}_+}(y)\Big] = \mathbf{P}_{\{0\}}\left[M_t > -x\right].$$

Therefore,

$$\lim_{t\to\infty} u(x + c_* t - \frac{3}{2\lambda_*}\ln t + C_*, t) = \mathbf{E}_{\{0\}}\left[e^{-e^{-\lambda_* x} D_\infty}\right] =: \phi(x).$$

Moreover, $(x,t) \to \phi(x - c_* t)$ *is a monotone solution to the S-equation, and*

$$\lim_{x\to+\infty} \phi(x) = 1, \qquad \lim_{x\to-\infty} \phi(x) = \mathbf{E}_{\{0\}}\left[D_\infty = 0\right] \in [0,1).$$

1) Branching Brownian Motion.

$$\partial_t u(x,t) = \frac{1}{2}\partial^2_{xx} u(x,t) - u(x,t) + u^2(x,t);$$

$$\partial_t v_\lambda(x,t) = \frac{1}{2}\partial^2_{xx} v_\lambda(x,t) + v_\lambda(x,t), \quad \ln v_\lambda(x,t) = \frac{\lambda^2 t}{2} + t - \lambda x;$$

$$\partial_t w_{\lambda,\mu}(x,t) = \frac{1}{2}\partial^2_{xx} w_{\lambda,\mu}(x,t) + w_{\lambda,\mu}(x,t) + 2v_\lambda(x,t)v_\mu(x,t);$$

$$v_\lambda(x,t) < \infty, \quad w_{\lambda,\mu}(x,t) < \infty, \quad x \in \mathbb{R},\ t \geq 0,\ \lambda > 0,\ \mu > 0;$$

$$c_* = \inf_{\lambda > 0} \frac{\ln v_\lambda(0,1)}{\lambda} = \inf_{\lambda > 0} \frac{\frac{\lambda^2}{2} + 1}{\lambda} = \sqrt{2}, \quad \lambda_* = \sqrt{2}.$$

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2) Galton-Watson Process.

$$\partial_t u(t) = -u(t) + \sum_{j\in\mathbb{N}_0} p_j u^j(t);$$

$$\partial_t v_\lambda(x,t) = -v_\lambda(x,t) + \sum_{j\in\mathbb{N}_0} j p_j v_\lambda(x,t), \quad \ln v_\lambda(x,t) = (-1 + \sum_{j\in\mathbb{N}_0} j p_j)t - \lambda x.$$

$$\ln v_\lambda(0,1) = -1 + \sum_{j\in\mathbb{N}_0} j p_j, \quad \lambda_* = \infty.$$

3) Branching Pure-jump Process.

$$\partial_t u(x,t) = (a * u)(x,t) - 2u(x,t) + u^2(x,t);$$

$$\partial_t v_\lambda(x,t) = (a * v_\lambda)(x,t), \quad \ln v_\lambda(x,t) = t \int\limits_{\mathbb{R}} e^{\lambda y} a(y) dy - \lambda x;$$

$$c_* = \inf_{\lambda > 0} \frac{\ln v_\lambda(0,1)}{\lambda} = \inf_{\lambda > 0} \frac{\int\limits_{\mathbb{R}} e^{\lambda y} a(y) dy}{\lambda} = \frac{\int\limits_{\mathbb{R}} e^{\lambda_* y} a(y) dy}{\lambda_*}.$$

Let there exist $l, \delta, \lambda > 0$ such that,

$$I := \inf_{y \in (-l-\delta, -l)} a(y) > 0, \qquad \int\limits_{\mathbb{R}} e^{\lambda y} a(y) dy < \infty, \tag{3}$$

and λ_* be less than the abscissa of the Laplace transform of a. Then conditions of the main theorem are satisfied.

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