LMS Undergraduate Summer School

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Mini-course

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Can one see the forest without losing sight of the trees?

Abstract
As in the above cited proverb, it can be very challenging to see the essential among too much information or detail available. This situation appears in a lot of areas: molecules in a glass of water, stars in a galaxy, blade of grass in a field, snakes in pit, any kind of data classification. In order to get a handle on the problem, a typical approach is to compute a few numbers, the so-called characteristics, and the hope is that these characteristics capture important information about the system. Think for example of a liquid: we know its density, we know its colour, we may X-ray it and so on. What do these characteristics tell us about the molecules building up the liquid? In other contexts though, it may not be so clear what are useful characteristics to be considered (e.g. the mean, the median, correlations, clusters etc.) It is an art to develop good characteristics for the problem under study. A myriad of characteristics have and are developed in this way.

In these lectures we will take a step back and take a conceptual view on the problem. What do a particular collection of characteristics will tell us about our system? Assume the only information we have about our system are the values of a few characteristics, what can we conclude about our system, or in other words, which information do the characteristics contain?

In order to be a bit more concrete, we have to formulate what do we mean by a general system. In the lectures, we will concentrate on systems which are given as a set on which we have a probability. For examples, liquids are described by probability distributions on the space of all possible positions of the molecules. We will consider easier sets in the lectures, like for example the natural numbers.

Let me give a concrete and simple example. Consider a probability on the real numbers (that is a "random" number). Assume we know the mean and the variance of the probability but nothing else about the probability distribution. We have the suspicion that our system is discrete or in other words quantised. Can that be or do our characteristics rule that out? What can you say about other restrictions? What happens if you choose a pair of numbers?

These may seem to be simple questions, but they are very challenging, have a long history and go under the name of "Moment Problem". Moment problems were among the motivating examples for the development of modern probability, functional analysis and convex analysis. There are interesting connections with algebra geometry and number theory. Quickly one reaches unsolved problems. The lectures will not require any pre-knowledge nor of probability or functional analysis, but they require an enthusiasm for real analysis and a love for constructing mathematical objects and a passion for solving puzzles.