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K-MOTIVES OF ALGEBRAIC VARIETIES

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Abstract

A kind of motivic algebra of spectral categories and modules over them is developed to introduce K-motives of algebraic varieties. As an application, bivariant algebraic K-theory $K(X,Y)$ as well as bivariant motivic cohomology groups $H^{p,q}(X,Y,\mathbb{Z})$ are defined and studied. We use Grayson's machinery [**12**] to produce the Grayson motivic spectral sequence connecting bivariant K-theory to bivariant motivic cohomology. It is shown that the spectral sequence is naturally realized in the triangulated category of K-motives constructed in the paper. It is also shown that ordinary algebraic K-theory is represented by the K-motive of the point.

Contents

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1. Introduction

The triangulated category of motives DM^{eff} in the sense of Voevodsky [**33**] does not provide a sufficient framework to study such a fundamental object as the motivic spectral sequence. In this paper we construct the triangulated category of K-motives over any field F. This construction provides a natural framework to study (bivariant) K-theory in the same fashion as the triangulated category of motives provides a framework for motivic cohomology.

The main idea is to use formalism of spectral categories over smooth schemes Sm/F and modules over them. In this language a transfer from one scheme X to another scheme Y is a symmetric spectrum $\mathcal{O}(X,Y)$ such that there is an associative composition law

$$\mathcal{O}(Y,Z) \wedge \mathcal{O}(X,Y) \to \mathcal{O}(X,Z).$$

The category of $\mathcal{O}$-modules $\operatorname{Mod}\mathcal{O}$ consists of presheaves of symmetric spectra having "$\mathcal{O}$-transfers". The main spectral categories we work with are $\mathcal{O}_{K^{Gr}}$, $\mathcal{O}_{K^{\oplus}}$, $\mathcal{O}_K$. They come from various symmetric K-theory spectra associated with the category of bimodules $\mathcal{P}(X,Y)$, $X,Y \in Sm/F$. By a bimodule we mean a coherent $\mathcal{O}_{X\times Y}$-module P such that $\operatorname{Supp} P$ is finite over X and the coherent $\mathcal{O}_X$-module $(p_X)_*(P)$ is locally free.

In order to develop a satisfactory homotopy theory of presheaves of symmetric spectra with $\mathcal{O}$-transfers we specify the condition of being a "motivically excisive spectral category". In this case one produces a compactly generated triangulated category $SH^{\mathrm{mot}}\mathcal{O}$ which plays the role of the triangulated category of motives associated with the spectral category of transfers $\mathcal{O}$. Voevodsky's category DM^{eff} can be recovered in this way from the Eilenberg–Mac Lane spectral category $\mathcal{O}_{cor}$ associated with the category of correspondences. The spectral categories $\mathcal{O}_{K^{Gr}}$, $\mathcal{O}_{K^{\oplus}}$, $\mathcal{O}_K$ produce equivalences of triangulated categories

$$SH^{\mathrm{mot}}\mathcal{O}_{K^{Gr}} \simeq SH^{\mathrm{mot}}\mathcal{O}_{K^{\oplus}} \simeq SH^{\mathrm{mot}}\mathcal{O}_K.$$

The K-motive $M_K(X)$ of a smooth scheme X over F is the image of the free $\mathcal{O}_K$-module $\mathcal{O}_K(-,X)$ in $SH^{\mathrm{mot}}\mathcal{O}_K$ (see Definition 8.3). Then there is an isomorphism (see Corollary 8.5)

$$K_i(X) \cong SH^{\mathrm{mot}}\mathcal{O}_K(M_K(X)[i], M_K(pt)), \quad i \in \mathbb{Z}.$$

Thus ordinary K-theory is represented by the K-motive of the point.

One of the main computational tools of the paper is the "Grayson motivic spectral sequence". It is a strongly convergent spectral sequence of the form

$$E_2^{pq} = H_{\mathcal{M}}^{p-q,-q}(X,\mathbb{Z}) \Longrightarrow K_{-p-q}(X)$$

where the groups on the left hand side are motivic cohomology groups of X. We show in Theorem 8.7 that it is recovered from the "Grayson tower" in $SH^{\mathrm{mot}}\mathcal{O}_{K^{Gr}}$

$$\cdots \xrightarrow{f_{q+1}} M_{K^{Gr}}(q)(pt) \xrightarrow{f_q} M_{K^{Gr}}(q-1)(pt) \xrightarrow{f_{q-1}} \cdots \xrightarrow{f_1} M_{K^{Gr}}(pt),$$

where $M_{K^{Gr}}(q)(pt)$-s are certain $\mathcal{O}_{K^{Gr}}$-modules. Thus the triangulated category of K-motives provides a sufficient framework to study the motivic spectral sequence. In fact, we construct its bivariant counterpart. It is used to show that $\mathcal{O}_{K^{Gr}}$, $\mathcal{O}_{K^{\oplus}}$, $\mathcal{O}_K$ are motivically excisive spectral categories.

One should stress that we do not construct a tensor product on $SH^{\mathrm{mot}}\mathcal{O}_K$.

Throughout the paper we denote by Sm/F the category of smooth separated schemes of finite type over the base field F.

2. Model structures for symmetric spectra

In this section we collect basic facts about symmetric spectra of simplicial sets Sp^Σ we shall need later. We refer the reader to [**16, 25**] for details.

Definition 2.1. (1) An object A of a model category $\mathcal{M}$ is *finitely presentable* if the set-valued Hom-functor $\mathrm{Hom}_{\mathcal{M}}(A,-)$ commutes with all filtered colimits.

(2) Following [**6**] a cofibrantly generated model category $\mathcal{M}$ is *weakly finitely generated* if I and J can be chosen such that the following conditions hold.

- ⋄ The domains and the codomains of the maps in I are finitely presentable.
- ⋄ The domains of the maps in J are small.
- ⋄ There exists a subset J' of J of maps with finitely presentable domains and codomains, such that a map $f\colon A\to B$ in $\mathcal{M}$ with fibrant codomain B is a fibration if and only if it is contained in $J'-inj$.

Recall from [**14**, Chapter V] that it is possible to define $X\otimes K$ for an object X in a model category and a simplicial set K, even if the model category is not simplicial.

Lemma 2.2. *Let $\mathcal{M}$ be a left proper, cellular, weakly finitely generated model category in which all objects are small and let S be a set of cofibrations in $\mathcal{M}$. Suppose that, for every domain or codomain X of S and every finite simplicial set K, $X\otimes K$ is finitely presentable. Then the Bousfield localization $\mathcal{M}/S$ is weakly finitely generated.*

Proof. The proof is like that of [**15**, 4.2]. □

We first define level projective and flat model structures for symmetric spectra. A morphism $f\colon A\to B$ of symmetric spectra is called a *flat cofibration* if and only if for every level cofibration $g\colon X\to Y$ the pushout product map

$$f\wedge g\colon B\wedge X\cup_{A\wedge X}A\wedge Y\to B\wedge Y$$

is a level cofibration. In particular, every flat cofibration is a level cofibration.

Theorem 2.3 ([**16, 25**])**.** *The category of symmetric spectra Sp^Σ admits the following two level model structures in which the weak equivalences are those morphisms $f\colon X\to Y$ such that for all $n\geqslant 0$ the map $f_n\colon X_n\to Y_n$ is a weak equivalence of simplicial sets.*

(1) *In the projective level model structure a morphism $f\colon X\to Y$ is a projective level fibration if and only if for every $n\geqslant 0$ the map $f_n\colon Y_n\to X_n$ is a Kan fibration of simplicial sets. A morphism $f\colon X\to Y$ is a projective cofibration if it has the left lifting property with respect to all acyclic projective level fibrations.*

(2) *In the flat level model structure the cofibrations are the flat cofibrations. A morphism $f\colon X\to Y$ is a flat level fibration if it has the right lifting property with respect to all acyclic flat cofibrations.*

*The two stable model structures are cellular, proper, simplicial, weakly finitely generated, symmetric monoidal with respect to the smash product of symmetric spectra and satisfy the monoid axiom in the sense of Schwede–Shipley [**26**].*

Proof. By [**16, 25**] (1) and (2) determine proper, simplicial, cofibrantly generated and symmetric monoidal model structures. By [**16**, 3.2.13] every symmetric spectrum is small. By [**16, 25**] the domains and codomains of generating (trivial) cofibrations are finitely presentable and projective (flat) cofibrations are levelwise injections of simplicial sets. It follows that both model structures are cellular and weakly finitely generated. The monoid axiom follows from [**25**, III.1.11] and the fact that in a weakly finitely generated model category the class of weak equivalences is closed under filtered colimits by [**6**, 3.5]. □

Recall that a symmetric spectrum X is *injective* if for every monomorphism which is also a level equivalence $i\colon A \to B$ and every morphism $f\colon A \to X$ there exists an extension $g\colon B \to X$ with $f = gi$. A morphism $f\colon A \to B$ of symmetric spectra is a *stable equivalence* if for every injective Ω-spectrum X the induced map $[f, X]\colon [B, X] \to [A, X]$ on homotopy classes of spectrum morphisms is a bijection.

Theorem 2.4 ([**15, 16, 25**])**.** *The category of symmetric spectra Sp^Σ admits the following two stable model structures in which the weak equivalences are the stable equivalences.*

(1) *In the projective stable model structure the cofibrations are the projective cofibrations. A morphism $f\colon X \to Y$ is a stable projective fibration if it has the right lifting property with respect to all acyclic projective cofibrations.*

(2) *In the flat stable model structure the cofibrations are the flat cofibrations. A morphism $f\colon X \to Y$ is a stable flat fibration if it has the right lifting property with respect to all acyclic flat cofibrations.*

*The two stable model structures are cellular, proper, simplicial, weakly finitely generated, symmetric monoidal with respect to the smash product of symmetric spectra and satisfy the monoid axiom in the sense of Schwede–Shipley [**26**].*

Proof. By [**16, 25**] (1) and (2) determine proper, simplicial, cofibrantly generated and symmetric monoidal model structures. The domains and codomains of generating cofibrations are finitely presentable and by [**16**, 3.2.13] every symmetric spectrum is small. By [**16, 25**] projective (flat) cofibrations are levelwise injections of simplicial sets. It follows that both model structures are cellular. By [**25**, III.2.2] and [**14**, 1.1.11] stable equivalences which are monomorphisms of symmetric spectra are closed under pushouts. By [**16**, 5.4.1] the monoid axiom is true for the stable projective model structure. The monoid axiom for the stable flat model structure is proved like that of [**16**, 5.4.1] (for this use as well [**25**, III.1.11] and the fact that in a weakly finitely generated model category the class of weak equivalences is closed under filtered colimits by [**6**, 3.5]).

It remains to verify that both model structures are weakly finitely generated. By [**15**, p. 109] the stable projective model structure on Sp^Σ is the Bousfield localization of the level projective model structure with respect to the set $\mathcal{S}$

$$F_{n+1}(C \wedge S^1) \to F_n(C)$$

as C runs through the domains and codomains of the generating cofibrations of pointed simplicial sets $SSets_*$ and each F_n is the left adjoint to the nth evaluation functor $Ev_n\colon Sp^\Sigma \to SSets_*$. It follows from Lemma 2.2 and Theorem 2.3 that the stable projective model structure on Sp^Σ is weakly finitely generated.

If we show that the stable flat model structure is the Bousfield localization of the level flat model structure with respect to the set $\mathcal{S}$, it will follow from Lemma 2.2 and Theorem 2.3 that it is weakly finitely generated. The cofibrations in both model structures are the same. Since level projective model structure is Quillen equivalent to level flat model structure, then the corresponding Bousfield localizations with respect to $\mathcal{S}$ are Quillen equivalent as well. Therefore weak equivalences in the stable flat model structure and in the Bousfield localization of the level flat model structure with respect to the set $\mathcal{S}$ coincide as was to be shown. □

We want to make several remarks about symmetric spectra. One of tricky points when working with symmetric spectra is that the stable equivalences of symmetric spectra can not be defined by means of stable homotopy groups. It is not enough to invert π_*-isomorphisms (=stable weak equivalence of ordinary spectra) to get a satisfactory homotopy category of symmetric spectra. Instead one inverts a bigger class – that of stable weak equivalences between symmetric spectra in the sense of [**16**].

Given a symmetric spectrum X and its stably fibrant model γX in Sp^{Σ}, stable homotopy groups of X can considerably be different from those of γX in general (see, e.g. [**16, 25**]). In particular, Hom-sets $\mathrm{Ho}(Sp)(S^i, X)$ are different from $\mathrm{Ho}(Sp^{\Sigma})(S^i, X)$. Nevertheless there is an important class of *semistable* symmetric spectra within which stable equivalences coincide with π_*-isomorphisms. Recall that a symmetric spectrum is semistable if some (hence any) stably fibrant replacement is a π_*-isomorphism. Here a stably fibrant replacement is a stable equivalence $X \to \gamma X$ with target an Ω-spectrum.

Suspension spectra, Eilenberg–Mac Lane spectra, Ω-spectra or Ω-spectra from some point X_n on are examples of semistable symmetric spectra (see [**25**, Example I.4.48]). So Waldhausen's algebraic K-theory symmetric spectrum we shall discuss later is semistable. Semistability is preserved under suspension, loop, wedges and shift [**25**, Example I.4.51].

In what follows we shall use these facts without further comments.

3. Spectral categories and modules over them

To define K-motives, we work in the framework of spectral categories and modules over them in the sense of Schwede–Shipley [**27**]. We start with preparations.

A biexact functor of Walddhausen categories is a functor $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$, $(A, B) \mapsto A \otimes B$, having the property that for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$ the partial functors $A \otimes -$ and $- \otimes B$ are exact, and for every pair of cofibrations $A \rightarrowtail A'$ and $B \rightarrowtail B'$ in $\mathcal{A}$ and $\mathcal{B}$ respectively, the map $A' \otimes B \coprod_{A \otimes B} A \otimes B' \to B \otimes B'$ is a cofibration in $\mathcal{C}$.

For example let $\mathcal{P}(X, Y)$, $X, Y \in Sm/F$, be the category of coherent $\mathcal{O}_{X \times Y}$-modules P such that $\operatorname{Supp} P$ is finite over X and the coherent $\mathcal{O}_X$-module $(p_X)_*(P)$ is locally free. For example, let $f\colon X \to Y$ be a morphism of smooth schemes, and let Γ_f be its graph; then $\Gamma_f \in \mathcal{P}(X, Y)$. Let $X, Y, U \in Sm/F$ be three smooth schemes.

In this case we have a natural functor

$$\begin{aligned}\mathcal{P}(X,Y)\times\mathcal{P}(Y,U)\to\mathcal{P}(X,U)\\ P\times Q\mapsto P\otimes Q:=(p_{X,U})_*(p^*_{X,Y}(P)\otimes_{\mathcal{O}_{X\times Y\times U}}p^*_{Y,U}(Q))\end{aligned}\quad (1)$$

By [**28**, section 1] the sheaf on the right really belongs to $\mathcal{P}(X,U)$ and the above functor is biexact.

Let Cor_{virt} be a category whose objects are those of Sm/F and satisfying the following conditions:

(a) For any pair of smooth schemes $X,Y\in Sm/F$ there is a Waldhausen category $(\mathcal{C}_{virt}(X,Y),w)$ with w a family of weak equivalences in $\mathcal{C}_{virt}(X,Y)$ such that $\operatorname{Ob}\mathcal{C}_{virt}(X,Y)=\operatorname{Mor}Cor_{virt}(X,Y)$.

(b) For any triple $X,Y,Z\in Sm/F$ there is an associative biexact functor of Waldhausen categories

$$\mathcal{C}_{virt}(X,Y)\times\mathcal{C}_{virt}(Y,Z)\xrightarrow{\varphi}\mathcal{C}_{virt}(X,Z).\quad (2)$$

By associativity we mean that the diagram

$$\begin{array}{ccc}\mathcal{C}_{virt}(X,Y)\times(\mathcal{C}_{virt}(Y,Z)\times\mathcal{C}_{virt}(Z,W)) & \xrightarrow{1\times\varphi} & \mathcal{C}_{virt}(X,Y)\times\mathcal{C}_{virt}(Y,W)\\ \downarrow{\cong} & & \\ (\mathcal{C}_{virt}(X,Y)\times\mathcal{C}_{virt}(Y,Z))\times\mathcal{C}_{virt}(Z,W) & & \downarrow{\varphi}\\ \downarrow{\varphi\times 1} & & \\ \mathcal{C}_{virt}(X,Z)\times\mathcal{C}_{virt}(Z,W) & \xrightarrow{\varphi} & \mathcal{C}_{virt}(X,W)\end{array}$$

is commutative.

(c) For any $X\in Sm/F$ there is a distinguished object $1_X\in\mathcal{C}_{virt}(X,X)$ (the "tensor product unit object") such that

$$\varphi(1_X,P)=\varphi(P,1_Y)$$

for all $P\in\mathcal{C}_{virt}(X,Y)$.

In other words, the category Cor_{virt} is "enriched" over Waldhausen categories. Sometimes we refer to Cor_{virt} as the category of "virtual" correspondences.

Example 3.1. Given $X,Y\in Sm/F$, consider the category $\mathcal{P}(X,Y)$. It is an exact category, and therefore it can be regarded as a Waldhausen category with the family of weak equivalences being isomorphisms. Clearly, biproduct (1) is associative up to isomorphism but not strictly associative. We replace the exact categories $\mathcal{P}(X,Y)$-s by equivalent exact categories $\mathcal{P}'(X,Y)$-s and define a strictly associative biproduct

$$\varphi=\varphi_{XYU}\colon\mathcal{P}'(X,Y)\times\mathcal{P}'(Y,U)\to\mathcal{P}'(X,U).$$

One approach is to define $\mathcal{P}'(X,Y)$ as follows. Consider sequences $(P_1,\dots,P_k)$, $k\geqslant 1$, of objects in other categories $\mathcal{P}(U_i,V_i)$, so that $U_1=X$, $V_k=Y$ and $V_i=U_{i+1}$ for any $i<k$. If $X=Y$ the object $1_X=\Gamma_1$ of $\mathcal{P}(X,X)$ is a monoidal unit. There is a relation for sequences $(P)=(1_X,P)=(P,1_Y)$ with $P\in\mathcal{P}(X,Y)$. We also require $(\Gamma_f,\Gamma_g)=(\Gamma_{gf})$ to hold for any morphisms of smooth schemes $f\colon X\to U,g\colon U\to Y$.

We say that a sequence $(P_1, \dots, P_k)$ is of minimal length if it can not be reduced to a sequence of smaller length by means of the relations above. We set

$$P_1 \otimes \cdots \otimes P_k := ((\cdots(P_1 \otimes P_2) \otimes \cdots) \otimes P_{k-1}) \otimes P_k.$$

By definition, objects of $\mathcal{P}'(X,Y)$ are the sequences $(P_1, \dots, P_k)$ of minimal length. Define the arrows between two sequences of minimal length $(P_1, \dots, P_k)$ and $(Q_1, \dots, Q_l)$ by

$$\mathrm{Hom}_{\mathcal{P}'(X,Y)}((P_1, \dots, P_k),(Q_1, \dots, Q_l)) := \mathrm{Hom}_{\mathcal{P}(X,Y)}(P_1 \otimes \cdots \otimes P_k, Q_1 \otimes \cdots \otimes Q_l).$$

One easily sees that $\mathcal{P}'(X,Y)$ is an exact category and the natural exact functor $\mathcal{P}(X,Y) \to \mathcal{P}'(X,Y)$ sending an object P to the sequence of length one is an equivalence.

The tensor product

$$\varphi = \varphi_{XYU} \colon \mathcal{P}'(X,Y) \times \mathcal{P}'(Y,U) \to \mathcal{P}'(X,U)$$

on the new objects is simply concatenation of sequences, which is strictly associative. By construction,

$$\varphi(1_X,(P_1, \dots, P_k)) = \varphi((P_1, \dots, P_k), 1_Y) = (P_1, \dots, P_k)$$

for any $(P_1, \dots, P_k) \in \mathcal{P}'(X,Y)$.

Let Cor_K (respectively, $Cor_{K^\oplus}$) be the category whose objects are those of Sm/F, $\mathrm{Mor}\, Cor_K(X,Y) = \mathrm{Ob}\,\mathcal{P}'(X,Y)$ (respectively, $\mathrm{Mor}\, Cor_{K^\oplus}(X,Y) = \mathrm{Ob}\,\mathcal{P}'(X,Y)$) and for any $X, Y \in Sm/F$ let $\mathcal{C}_K(X,Y)$ (respectively, $\mathcal{C}_{K^\oplus}(X,Y)$) be the exact category $\mathcal{P}'(X,Y)$ (respectively, the same category $\mathcal{P}'(X,Y)$ but considered as an additive category). Note that the map taking a morphism of smooth schemes $f \colon X \to Y$ to Γ_f determines a functor $Sm/F \to Cor_K$ (respectively, a functor $Sm/F \to Cor_{K^\oplus}$).

Using coherence properties for tensor product (see [**21**, Ch. VII]), we have that for any triple $X, Y, Z \in Sm/F$ there are biexact functors of Waldhausen categories

$$\mathcal{C}_K(X,Y) \times \mathcal{C}_K(Y,Z) \xrightarrow{\varphi} \mathcal{C}_K(X,Z)$$

and

$$\mathcal{C}_{K^\oplus}(X,Y) \times \mathcal{C}_{K^\oplus}(Y,Z) \xrightarrow{\varphi} \mathcal{C}_{K^\oplus}(X,Z)$$

satisfying conditions (b)-(c) above.

Thus $\mathcal{P}'$ yields two examples for $\mathcal{C}_{virt}$: $\mathcal{C}_K$ and $\mathcal{C}_{K^\oplus}$, respectively. We also get two categories of correspondences Cor_K and $Cor_{K^\oplus}$ which are the same as categories on smooth schemes but with different Waldhausen categories on objects.

Given a (multisimplicial) additive category $\mathcal{M}$, we shall sometimes write $K^\oplus\mathcal{M}$ to denote the K-theory symmetric spectrum spectrum of $\mathcal{M}$.

Definition 3.2. (1) Following [**27**] a *spectral category* is a category $\mathcal{O}$ which is enriched over the category Sp^Σ of symmetric spectra (with respect to smash product, i.e., the monoidal closed structure of [**16**, 2.2.10]). In other words, for every pair of objects $o, o' \in \mathcal{O}$ there is a morphism symmetric spectrum $\mathcal{O}(o, o')$, for every object o of O there is a map from the sphere spectrum S to $\mathcal{O}(o, o)$ (the "identity element" of o), and for each triple of objects there is an associative and unital composition map of symmetric spectra $\mathcal{O}(o', o'') \wedge \mathcal{O}(o, o') \to \mathcal{O}(o, o'')$. An $\mathcal{O}$-module M

is a contravariant spectral functor to the category Sp^Σ of symmetric spectra, i.e., a symmetric spectrum $M(o)$ for each object of $\mathcal{O}$ together with coherently associative and unital maps of symmetric spectra $M(o) \wedge \mathcal{O}(o', o) \to M(o')$ for pairs of objects $o, o' \in \mathcal{O}$. A morphism of $\mathcal{O}$-modules $M \to N$ consists of maps of symmetric spectra $M(o) \to N(o)$ strictly compatible with the action of $\mathcal{O}$. The category of $\mathcal{O}$-modules will be denoted by $\operatorname{Mod} \mathcal{O}$.

(2) A *spectral functor* or a *spectral homomorphism* F from a spectral category $\mathcal{O}$ to a spectral category $\mathcal{O}'$ is an assignment from $\operatorname{Ob} \mathcal{O}$ to $\operatorname{Ob} \mathcal{O}'$ together with morphisms $\mathcal{O}(a,b) \to \mathcal{O}'(F(a), F(b))$ in Sp^Σ which preserve composition and identities.

(3) The *monoidal product* $\mathcal{O} \wedge \mathcal{O}'$ of two spectral categories $\mathcal{O}$ and $\mathcal{O}'$ is the spectral category where $\operatorname{Ob}(\mathcal{O} \wedge \mathcal{O}') := \operatorname{Ob} \mathcal{O} \times \operatorname{Ob} \mathcal{O}'$ and $\mathcal{O} \wedge \mathcal{O}'((a,x),(b,y)) := \mathcal{O}(a,b) \wedge \mathcal{O}'(x,y)$.

(4) A *monoidal spectral category* consists of a spectral category $\mathcal{O}$ equipped with a spectral functor $\diamond \colon \mathcal{O} \wedge \mathcal{O} \to \mathcal{O}$, a unit $u \in \operatorname{Ob} \mathcal{O}$, a Sp^Σ-natural associativity isomorphism and two Sp^Σ-natural unit isomorphisms. Symmetric monoidal spectral categories are defined similarly.

Example 3.3. (1) A naive spectral category $\mathcal{O}_{naive}$ on Sm/F is defined as follows. $\mathcal{O}_{naive}$ has the same set of objects as Sm/F and the morphism spectra are defined by

$$\mathcal{O}_{naive}(X,Y)_p = \operatorname{Hom}_{Sm/F}(X,Y)_+ \wedge S^p.$$

Here S^p denotes the pointed simplicial set $S^p = S^1 \wedge \cdots \wedge S^1$ (p factors) and the symmetric group permutes the factors. Composition is given by the composite

$$\begin{aligned}
\mathcal{O}_{naive}(Y,Z)_p \wedge \mathcal{O}_{naive}(X,Y)_q \\
&= (\operatorname{Hom}_{Sm/F}(Y,Z)_+ \wedge S^p) \wedge (\operatorname{Hom}_{Sm/F}(X,Y)_+ \wedge S^q) \\
&\xrightarrow{shuffle} \operatorname{Hom}_{Sm/F}(Y,Z)_+ \wedge \operatorname{Hom}_{Sm/F}(X,Y)_+ \wedge S^p \wedge S^q \\
&\to \operatorname{Hom}_{Sm/F}(X,Z)_+ \wedge S^{p+q} = \mathcal{O}_{naive}(X,Z)_{p+q}.
\end{aligned}$$

$\mathcal{O}_{naive}$ is a symmetric monoidal spectral category equipped with a spectral functor $\mathcal{O}_{naive} \wedge \mathcal{O}_{naive} \to \mathcal{O}_{naive}, \quad \mathcal{O}_{naive}(X,Y) \wedge \mathcal{O}_{naive}(U,V) \to \mathcal{O}_{naive}(X \times U, Y \times V)$, and a unit $\operatorname{Spec} F \in \operatorname{Ob} \mathcal{O}_{naive}$. It is straightforward to verify that the category of $\mathcal{O}_{naive}$-modules can be regarded as the category of presheaves $Pre^\Sigma(Sm/F)$ of symmetric spectra on Sm/F. This is used in the sequel without further comment.

(2) Any *ringoid* $\mathcal{A}$, that is a category whose Hom-sets are abelian groups with bilinear composition, gives rise to a spectral category $H\mathcal{A}$ also called the Eilenberg–Mac Lane spectral category of $\mathcal{A}$. In more detail, $H\mathcal{A}$ has the same set of objects as $\mathcal{A}$ and the morphism spectra are defined by $H\mathcal{A}(a,b)_p = \mathcal{A}(a,b) \otimes \widetilde{\mathbb{Z}}[S^p]$. Here $\widetilde{\mathbb{Z}}[S^p]$ denotes the reduced simplicial free abelian group generated by the pointed simplicial set S^p and the symmetric group permutes the factors. Composition is defined as above.

In Appendix we shall present another sort of Eilenberg–Mac Lane spectral categories associated with ringoids. It will always be clear from the context which of these sorts is used.

An important example of a ringoid is the category correspondences. For any $X, Y \in Sm/F$ define $Cor(X,Y)$ to be the free abelian group generated by closed integral subschemes $Z \subset X \times_F Y$ which are finite and surjective over a component of X. Let $X, Y, W \in Sm/F$ be smooth schemes and let $Z \in Cor(X, Y), T \in Cor(Y, W)$ be cycles on $X \times Y$ and $Y \times W$ each component of which is finite and surjective over a component of X (respectively, over a component of Y). One checks easily that the cycles $Z \times W$ and $X \times T$ intersect properly on $X \times Y \times W$ and each component of the intersection cycle $(Z \times W)_\bullet(X \times T)$ is finite and surjective over a component of X. Thus setting $T \circ Z = (pr_{1,3})_*((Z \times Y)_\bullet(X \times T))$ we get a bilinear composition map

$$Cor(Y, W) \times Cor(X, Y) \to Cor(X, W).$$

In this way we get a ringoid (denoted $SmCor/F$) whose objects are those of Sm/F and $\mathrm{Hom}_{SmCor/F}(X, Y) = Cor(X, Y)$ – see [**32**] for details. The Eilenberg–Mac Lane spectral category corresponding to the ringoid will be denoted by $\mathcal{O}_{cor}$.

For any smooth schemes X, Y, U, V the external product of cycles defines a homomorphism

$$Cor(X, Y) \otimes Cor(U, V) \to Cor(X \times U, Y \times V)$$

which gives the structure of symmetric monoidal spectral category for $\mathcal{O}_{cor}$.

(3) There are two other important ringoids $K_0^\oplus$ and K_0 on Sm/F. Namely, for any $X, Y \in Sm/F$ define $K_0^\oplus(X, Y)$ (respectively, $K_0(X, Y)$) to be the abelian group for the split exact (respectively, exact) category $\mathcal{P}'(X, Y)$. Composition is given by tensor product. Denote by $\mathcal{O}_{K_0^\oplus}$ and $\mathcal{O}_{K_0}$ their Eilenberg–Mac Lane spectral categories. There are canonical homomorphisms

$$K_0^\oplus(X, Y) \to K_0(X, Y) \to Cor(X, Y), \quad X, Y \in Sm/F.$$

Here, the first map is the obvious surjective homomorphism. The second one takes the class $[P]$ of the coherent sheaf $P \in \mathcal{P}(X, Y)$ to $\sum_Z \ell_{\mathcal{O}_{X \times Y, z}} P_z \cdot [Z]$, where the sum is taken over all closed integral subschemes $Z \subset X \times Y$ that are finite and surjective over a component of X and z denotes the generic point of the corresponding scheme Z.

By [**28**, section 1] and [**36**, section 6] the canonical homomorphisms yield maps between ringoids

$$K_0^\oplus \to K_0 \to Cor.$$

These induce spectral functors between spectral categories

$$\mathcal{O}_{K_0^\oplus} \to \mathcal{O}_{K_0} \to \mathcal{O}_{cor}.$$

We shall prove below that the spectral categories $\mathcal{O}_{K_0^\oplus}$ and $\mathcal{O}_{K_0}$ are symmetric monoidal (see Corollary 5.10).

We want to construct a spectral category out of the category of virtual correspondences Cor_{virt}. Let $X, Y \in Sm/F$ and let $\mathcal{O}_{virt}(X, Y) := K(\mathcal{C}_{virt}(X, Y))$ be the K-theory spectrum of the Waldhausen category $\mathcal{C}_{virt}(X, Y)$. By [**8**, 6.1.1] and [**25**, Example I.2.11] $\mathcal{O}_{virt}(X, Y)$ naturally has the structure of a symmetric spectrum. Note that $\mathcal{O}_{virt}(X, Y)_0 = |N.w\mathcal{C}_{virt}(X, Y)|$.

It follows from [**8**, section 6.1] and [**25**, Example 2.11] that associative biexact functors (2) induce an associative law

$$\mathcal{O}_{virt}(Y,Z) \wedge \mathcal{O}_{virt}(X,Y) \to \mathcal{O}_{virt}(X,Z).$$

Moreover, for any $X \in Sm/F$ there is a map $\mathbf{1}\colon S \to \mathcal{O}_{virt}(X,X)$ which is subject to the unit coherence law (see [**8**, section 6.1]). Note that $\mathbf{1}_0\colon S^0 \to \mathcal{O}_{virt}(X,X)_0$ is the map which sends the basepoint to the null object and the non-basepoint to the unit object 1_X for the tensor product. Thus the triple $(\mathcal{O}_{virt}, \wedge, \mathbf{1})$ determines a spectral category on Sm/F.

Observe that $\mathcal{O}_{virt}$ is a symmetric monoidal spectral category provided that there is a biexact functor

$$\mathcal{C}_{virt}(X,Y) \times \mathcal{C}_{virt}(U,V) \to \mathcal{C}_{virt}(X\times U, Y\times V)$$

for all $X, Y, U, V \in Sm/F$ satisfying natural associativity, symmetry, unit isomorphisms (where $\operatorname{Spec} F$ is a unit).

In what follows the category of $\mathcal{O}_{virt}$-modules will be denoted by $\mathcal{M}_{virt}$.

Proposition 3.4. *The map*

$$Cor_{virt} \to \mathcal{M}_{virt}, \quad X \mapsto \mathcal{O}_{virt}(-,X),$$

determines a fully faithful functor.

Proof. It follows from [**6**, sections 2.1-2.2] that

$$\operatorname{Hom}_{\mathcal{M}_{virt}}(M,N) = \operatorname{Hom}_{Sp^\Sigma}(S, Sp^\Sigma(M,N)).$$

Using "Enriched Yoneda Lemma" one has,

$$\operatorname{Hom}_{Sp^\Sigma}(S, Sp^\Sigma(\mathcal{O}_{virt}(-,X), \mathcal{O}_{virt}(-,Y))) \cong \operatorname{Hom}_{Sp^\Sigma}(S, \mathcal{O}_{virt}(X,Y)) =$$
$$\operatorname{Hom}_{SSets}(\Delta^0, |N.w\mathcal{C}_{virt}(X,Y)|) = |N.w\mathcal{C}_{virt}(X,Y)|_0 = Cor_{virt}(X,Y).$$

Our statement now follows. □

If Cor_{virt} is either Cor_K or $Cor_{K^\oplus}$ then we shall denote the corresponding spectral categories by $\mathcal{O}_K$ and $\mathcal{O}_{K^\oplus}$, respectively. There is another spectral category $\mathcal{O}_{K^{Gr}}$ we shall use later associated with $Cor_{K^\oplus}$. It is equivalent to $\mathcal{O}_{K^\oplus}$ and is based on $S^\oplus$-construction of Grayson [**12**]. We let Ord denote the category of finite nonempty ordered sets. For $A \in Ord$ we define a category $Sub(A)$ whose objects are the pairs (i,j) with $i \leqslant j \in A$, and where there is an (unique) arrow $(i',j') \to (i,j)$ exactly when $i' \leqslant i \leqslant j \leqslant j'$. Given an additive category $\mathcal{M}$, we say that a functor $M\colon Sub(A) \to \mathcal{M}$ is *additive* if $M(i,i) = 0$ for all $i \in A$, and for all $i \leqslant j \leqslant k \in A$ the map $M(i,k) \to M(i,j) \oplus M(j,k)$ is an isomorphism. Here 0 denotes a previously chosen zero object of $\mathcal{M}$. The set of such additive functors is denoted by $Add(Sub(A), \mathcal{M})$.

We define the simplicial set $S^\oplus\mathcal{M}$ by setting

$$(S^\oplus\mathcal{M})(A) = Add(Sub(A), \mathcal{M}).$$

An n-simplex $M \in S^\oplus_n\mathcal{M}$ may be thought of as a compatible collection of direct sum diagrams $M(i,j) \cong M(i,i+1) \oplus \cdots \oplus M(j-1,j)$. There is a natural map $S^\oplus\mathcal{M} \to S\mathcal{M}$ which converts each direct sum diagram $M(i,k) \cong M(i,j) \oplus M(j,k)$ into the short exact sequence $0 \to M(i,j) \to M(i,k) \to M(j,k) \to 0$.

Given a finite set Q, one can define the $|Q|$-fold iterated $S^{\oplus}$-construction $S^{\oplus,Q}\mathcal{M}$ similar to [**8**, section 6]. Then the nth space of Grayson's K-theory spectrum is given by

$$K^{Gr}(\mathcal{M})_n = |\operatorname{Ob} S^{\oplus,Q}\mathcal{M}|,$$

where $Q = \{1,\dots,n\}$. It is verified similar to [**8**, section 6] that $K^{Gr}(\mathcal{M})$ is a symmetric spectrum.

If we consider additive categories $\mathcal{P}'(X,Y)$, $X,Y \in Sm/F$, together with associative biexact functors

$$\mathcal{P}'(X,Y) \times \mathcal{P}'(Y,Z) \to \mathcal{P}'(X,Z),$$

then we shall obtain a spectral category $\mathcal{O}_{K^{Gr}}$ on Sm/F such that $\mathcal{O}_{K^{Gr}}(X,Y) = K^{Gr}(\mathcal{P}'(X,Y))$. The natural map described above $S^{\oplus}\mathcal{P}'(X,Y) \to S\mathcal{P}'(X,Y)$, $X,Y \in Sm/F$, determines a stable equivalence of symmetric spectra

$$\mathcal{O}_{K^{Gr}}(X,Y) \to \mathcal{O}_{K^{\oplus}}(X,Y).$$

Altogether these maps give an equivalence of spectral categories

$$\mathcal{O}_{K^{Gr}} \to \mathcal{O}_{K^{\oplus}}.$$

Let $\mathcal{O}$ be a spectral category. The category $\operatorname{Mod}\mathcal{O}$ of $\mathcal{O}$-modules is enriched over symmetric spectra. Namely, to any $M,N \in \operatorname{Mod}\mathcal{O}$ we associate the symmetric spectrum

$$Sp^{\Sigma}(M,N) := \int_{o\in \operatorname{Ob}\mathcal{O}} \underline{Sp}^{\Sigma}(M(o),N(o)),$$

where the integral stands for the coend and $\underline{Sp}^{\Sigma}(-,-)$ stands for the internal symmetric spectrum (see [**6**, section 2.2]). By the "Enriched Yoneda Lemma" there is a natural isomorphism of symmetric spectra

$$M(o) \cong Sp^{\Sigma}(\mathcal{O}(-,o),M)$$

for all $o \in \operatorname{Ob}\mathcal{C}$ and $M \in \operatorname{Mod}\mathcal{O}$.

Let $\mathcal{O}$ be symmetric monoidal and let $\diamond\colon \mathcal{O}\wedge\mathcal{O} \to \mathcal{O}$ be the structure spectral functor (see Definition 3.2(4)). By a theorem of Day [**4**] $\operatorname{Mod}\mathcal{O}$ is a closed symmetric monoidal category with smash product $\wedge$ and $\mathcal{O}(-,u)$ being the monoidal unit. The smash product is defined as

$$M \wedge_{\mathcal{O}} N = \int^{\operatorname{Ob}\mathcal{O}\otimes\mathcal{O}} M(o) \wedge N(p) \wedge \mathcal{O}(-, o\diamond p). \tag{3}$$

The internal Hom functor, right adjoint to $-\wedge_{\mathcal{O}} M$, is given by

$$\underline{\operatorname{Mod}}\mathcal{O}(M,N)(o) := Sp^{\Sigma}(M, N(o\diamond -)) = \int_{p\in\operatorname{Ob}\mathcal{O}} \underline{Sp}^{\Sigma}(M(p), N(o\diamond p)).$$

It follows from [**6**, 2.7] that there is a natural isomorphism

$$\mathcal{O}(-,o) \wedge_{\mathcal{O}} \mathcal{O}(-,p) \cong \mathcal{O}(-,o\diamond p).$$

A *morphism* $\Psi\colon \mathcal{O} \to \mathcal{R}$ of spectral categories is simply a spectral functor. The *restriction of scalars*

$$\Psi^*\colon \operatorname{Mod}\mathcal{R} \to \operatorname{Mod}\mathcal{O}, \quad M \mapsto M \circ \Psi$$

has a left adjoint functor Ψ_*, also denoted $- \wedge_{\mathcal{O}} \mathcal{R}$, which we refer to as extension of scalars. It is given by an enriched coend, i.e., for an $\mathcal{O}$-module N the $\mathcal{R}$-module $\Psi_* N = N \wedge_{\mathcal{O}} \mathcal{R}$ is

$$\int^{o \in \operatorname{Ob}\mathcal{O}} N(o) \wedge \mathcal{R}(-, \Psi(o)).$$

Recall that the *underlying category* $\mathcal{UO}$ of a spectral category $\mathcal{O}$ has the same objects as $\mathcal{O}$ and the Hom-sets are defined as $\operatorname{Hom}_{\mathcal{UO}}(o, o') = \operatorname{Hom}_{Sp^\Sigma}(S, \mathcal{O}(o, o'))$. Suppose $\mathcal{C}$ is a small category and

$$f\colon \mathcal{C} \to \mathcal{UO}$$

a functor. Denote by $Pre^\Sigma(\mathcal{C})$ the category of presheaves of symmetric spectra on $\mathcal{C}$. Let $U\colon \operatorname{Mod}\mathcal{O} \to Pre^\Sigma(\mathcal{C})$ be the forgetful functor. It can be proved similar to [**21**, X.4.1] that U has a left adjoint $F\colon Pre^\Sigma(\mathcal{C}) \to \operatorname{Mod}\mathcal{O}$ defined as

$$F(M) = \int^{c \in \operatorname{Ob}\mathcal{C}} M(c) \wedge \mathcal{O}(-, f(c)), \quad M \in Pre^\Sigma(\mathcal{C}).$$

4. Model category structures for $\operatorname{Mod}\mathcal{O}$

Let $\mathcal{O}$ be a spectral category and let $\operatorname{Mod}\mathcal{O}$ be the category of $\mathcal{O}$-modules. We refer the reader to [**5**] for basic facts about model categories enriched over symmetric spectra.

Definition 4.1. (1) A morphism f in $\operatorname{Mod}\mathcal{O}$ is a

- ◇ *level weak equivalence* if $f(c)$ is a level weak equivalence in Sp^Σ for all $c \in \operatorname{Ob}\mathcal{O}$.
- ◇ *projective (flat) level fibration* if $f(c)$ is a projective (flat) level fibration in Sp^Σ for all $c \in \operatorname{Ob}\mathcal{O}$.
- ◇ *projective (flat) cofibration* if f has the left lifting property with respect to all projective (flat) level acyclic fibrations.

(2) A morphism f in $\operatorname{Mod}\mathcal{O}$ is a

- ◇ *stable weak equivalence* if $f(c)$ is a stable weak equivalence in Sp^Σ for all $c \in \operatorname{Ob}\mathcal{O}$.
- ◇ *stable projective (flat) level fibration* if $f(c)$ is a stable projective (flat) fibration in Sp^Σ for all $c \in \operatorname{Ob}\mathcal{O}$.
- ◇ *stable projective (flat) cofibration* if f has the left lifting property with respect to all stable projective (flat) acyclic fibrations.

Theorem 4.2 ([**6, 27**])**.** *The category* $\operatorname{Mod}\mathcal{O}$ *admits the following four model structures:*

- ◇ *In the projective (respectively, flat) level model structure the weak equivalences are the level weak equivalences and fibrations are the projective (respectively, flat) level fibrations.*

- ◇ *In the projective (respectively, flat) stable model structure the weak equivalences are the stable weak equivalences and fibrations are the stable projective (respectively, flat) fibrations.*

The four model structures are cellular, proper, spectral and weakly finitely generated. Moreover, if $\mathcal{O}$ is a symmetric monoidal spectral category then each of the four model structures on $\operatorname{Mod}\mathcal{O}$ *is symmetric monoidal with respect to the smash product* (3) *of $\mathcal{O}$-modules and satisfies the monoid axiom.*

Proof. Let us consider one of the four model structures of symmetric spectra stated in Theorems 2.3-2.4. By those theorems Sp^{Σ} is a weakly finitely generated monoidal model category and the monoid axiom holds in Sp^{Σ}. It follows from [**6**, 4.2] that $\operatorname{Mod}\mathcal{O}$ is a cofibrantly generated, weakly finitely generated model category. Let I and J be the family of generating cofibrations and trivial cofibrations, respectively. Then the sets of maps in $\operatorname{Mod}\mathcal{O}$

$$\mathcal{P}_I = \{\mathcal{O}(-,o)\wedge si \xrightarrow{\mathcal{O}(-,o)\wedge i} \mathcal{O}(-,o)\wedge ti \mid i\in I, o\in \operatorname{Ob}\mathcal{O}\}$$

and

$$\mathcal{P}_J = \{\mathcal{O}(-,o)\wedge sj \xrightarrow{\mathcal{O}(-,o)\wedge j} \mathcal{O}(-,o)\wedge tj \mid j\in J, o\in \operatorname{Ob}\mathcal{O}\}$$

are families of generating cofibrations and trivial cofibrations, respectively. Since limits of $\mathcal{O}$-modules are formed objectwise [**6**, 2.2] and Sp^{Σ} is cellular by Theorems 2.3-2.4, then $\operatorname{Mod}\mathcal{O}$ is cellular. It is a spectral model category by [**27**, A.1.1] and [**6**, 4.4]. It is right proper by [**6**, 4.8]. The model structure is also left proper because cofibrations are in particular objectwise monomorphisms, and pushouts along monomorphisms preserve level/stable weak equivalences of symmetric spectra.

Finally, if $\mathcal{O}$ is a symmetric monoidal spectral category then by [**6**, 4.4] $\operatorname{Mod}\mathcal{O}$ is symmetric monoidal with respect to the smash product (3) of $\mathcal{O}$-modules and satisfies the monoid axiom. □

5. Motivic model category structures

In what follows, if otherwise is specified, we work with spectral categories $\mathcal{O}$ over Sm/F such that there is a functor of categories

$$u\colon Sm/F \to \mathcal{U}\mathcal{O}, \tag{4}$$

which is identical on objects. One has a bifunctor

$$\mathcal{O}(-,-)\colon Sm/F^{\mathrm{op}}\times Sm/F \to Sp^{\Sigma}.$$

$\mathcal{O}_{K^{\oplus}}, \mathcal{O}_K, \mathcal{O}_{K^{Gr}}, \mathcal{O}_{K_0^{\oplus}}, \mathcal{O}_{K_0}, \mathcal{O}_{cor}$ are examples of such spectral categories. Note that $\mathcal{O}$ is a $\mathcal{O}_{naive}$-algebra in the sense that there is a spectral functor $\mathcal{O}_{naive}\to\mathcal{O}$ induced by the functor u. The spectral category $\mathcal{O}_{naive}$ plays the same role as the ring of integers for abelian groups or the sphere spectrum for symmetric spectra.

Regarding $\mathcal{O}_{naive}$-modules as presheaves (see Example 3.3) of symmetric spectra

$Pre^{\Sigma}(Sm/F)$, we get a pair of adjoint functors

$$\Psi_*\colon Pre^{\Sigma}(Sm/F) \rightleftarrows \operatorname{Mod}\mathcal{O} : \Psi^*. \tag{5}$$

One has for all $X \in Sm/F$,

$$\Psi_*(\mathcal{O}_{naive}(-,X)) = \mathcal{O}_{naive}(-,X) \wedge_{\mathcal{O}_{naive}} \mathcal{O} \cong \mathcal{O}(-,X). \tag{6}$$

For simplicity we work with the stable projective model structure on $\operatorname{Mod}\mathcal{O}$ from now on. The interested reader can also consider the stable flat model structure as well.

Recall that the Nisnevich topology is generated by the elementary distinguished squares, i.e., pullback squares

$$\begin{array}{ccc} U' & \longrightarrow & X' \\ \downarrow & Q & \downarrow{\scriptstyle\varphi} \\ U & \underset{\psi}{\longrightarrow} & X \end{array} \tag{7}$$

where φ is etale, ψ is an open embedding and $\varphi^{-1}(X \setminus U) \to (X \setminus U)$ is an isomorphism of schemes (with the reduced structure). Let $\mathcal{Q}$ denote the set of elementary distinguished squares in Sm/F. By $\mathcal{Q}_{\mathcal{O}}$ denote the set of squares

$$\begin{array}{ccc} \mathcal{O}(-,U') & \longrightarrow & \mathcal{O}(-,X') \\ \downarrow & \mathcal{O}Q & \downarrow{\scriptstyle\varphi} \\ \mathcal{O}(-,U) & \underset{\psi}{\longrightarrow} & \mathcal{O}(-,X) \end{array} \tag{8}$$

which are obtained from the squares in $\mathcal{Q}$ by taking $X \in Sm/F$ to $\mathcal{O}(-,X)$. The arrow $\mathcal{O}(-,U') \to \mathcal{O}(-,X')$ can be factored as a cofibration $\mathcal{O}(-,U') \rightarrowtail Cyl$ followed by a simplicial homotopy equivalence $Cyl \to \mathcal{O}(-,X')$. There is a canonical morphism $A_{\mathcal{O}Q} := \mathcal{O}(-,U) \coprod_{\mathcal{O}(-,U')} Cyl \to \mathcal{O}(-,X)$.

Definition 5.1. (1) The *Nisnevich local model structure* on $\operatorname{Mod}\mathcal{O}$ is the Bousfield localization of the stable projective model structure (see Theorem 4.2) with respect to the set of projective cofibrations

$$\mathcal{N}_{\mathcal{O}} = \{\mathrm{cyl}(A_{\mathcal{O}Q} \to \mathcal{O}(-,X))\}_{\mathcal{Q}_{\mathcal{O}}}.$$

The homotopy category for the Nisnevich local model structure will be denoted by $SH^{\mathrm{nis}}\mathcal{O}$. If $\mathcal{O} = \mathcal{O}_{naive}$ then we shall write $SH^{\mathrm{nis}}(F)$ to denote $SH^{\mathrm{nis}}\mathcal{O}_{naive}$.

(2) The *motivic model structure* on $\operatorname{Mod}\mathcal{O}$ is the Bousfield localization of the Nisnevich local model structure with respect to the set of projective cofibrations

$$\mathcal{A}_{\mathcal{O}} = \{\mathrm{cyl}(\mathcal{O}(-,X \times \mathbb{A}^1) \to \mathcal{O}(-,X))\}_{X \in Sm/F}.$$

The homotopy category for the motivic model structure will be denoted by $SH^{\mathrm{mot}}\mathcal{O}$. If $\mathcal{O} = \mathcal{O}_{naive}$ then we shall write $SH^{\mathrm{mot}}(F)$ to denote $SH^{\mathrm{mot}}\mathcal{O}_{naive}$.

Remark 5.2. A stably fibrant $\mathcal{O}$-module M is Nisnevich local if and only if it is *flasque* in the sense that for each elementary distinguished square $Q \in \mathcal{Q}$ the square of symmetric spectra $M(Q)$ is homotopy pullback.

Before collecting properties for the Nisnevich local and motivic model structures we need to recall some facts from unstable $\mathbb{A}^1$-topology.

Let $Pre^{\mathbb{N}}(Sm/F)$ be the category of presheaves of ordinary simplicial spectra $Sp^{\mathbb{N}}$. Then $Sp^{\mathbb{N}}$ and $Pre^{\mathbb{N}}(Sm/F)$ enjoy the stable projective model structure defined similar to Sp^{Σ} and $Pre^{\Sigma}(Sm/F)$ (see [**15**]). By [**15**, 3.5] the stable projective model structure on $Sp^{\mathbb{N}}$ coincides with the stable model structure of Bousfield–Friedlander [**3**]. By [**16**, 4.2.5] the forgetful functor $U\colon Sp^{\Sigma} \to Sp^{\mathbb{N}}$ has a left adjoint V and the pair (U, V) forms a Quillen equivalence of the stable model categories. The Nisnevich local and motivic model structures on $Pre^{\mathbb{N}}(Sm/F)$ are defined similar to those on $Pre^{\Sigma}(Sm/F)$ by means of the Bousfield localization.

Lemma 5.3. *The adjoint pair $(V, U)\colon Sp^{\mathbb{N}} \leftrightarrows Sp^{\Sigma}$ can be extended to an adjoint pair $(V, U)\colon Pre^{\mathbb{N}}(Sm/F) \leftrightarrows Pre^{\Sigma}(Sm/F)$. It forms a Quillen equivalence for the stable projective, Nisnevich local and motivic model structures, respectively.*

Proof. See [**15**, section 10] and [**18**, section 4.5]. □

Let $Pre(Sm/F)$ denote the category of pointed simplicial presheaves on Sm/F. Then $Pre(Sm/F)$ enjoys the projective model structure in which fibrations and weak equivalences are defined schemewise. Projective cofibrations are those maps which have the corresponding lifting property. As above, one defines the Nisnevich local projective and motivic model structures by means of the Bousfield localization. By [**1**] both model structures are proper simplicial cellular and the Nisnevich local projective model structure coincides with the model structure in which weak equivalences are the local weak equivalences with respect to Nisnevich topology and cofibrations are the projective cofibrations.

In [**15**] Hovey constructed stable (symmetric) model structures out of certain model categories with certain Quillen endofunctors. For example, one can apply Hovey's constructions to Nisnevich local and motivic projective model structures on $Pre(Sm/F)$ with $- \wedge S^1$ a Quillen endofunctor. Denote the resulting model categories by $Sp^{\mathbb{N}}_{nis}(Pre(Sm/F))$ and $Sp^{\mathbb{N}}_{mot}(Pre(Sm/F))$ (respectively, $Sp^{\Sigma}_{nis}(Pre(Sm/F))$ and $Sp^{\Sigma}_{mot}(Pre(Sm/F))$). As categories $Sp^{\mathbb{N}}_{nis}(Pre(Sm/F))$ and $Sp^{\mathbb{N}}_{mot}(Pre(Sm/F))$ (respectively $Sp^{\Sigma}_{nis}(Pre(Sm/F))$ and $Sp^{\Sigma}_{mot}(Pre(Sm/F))$) coincide with $Pre^{\mathbb{N}}(Sm/F)$ (respectively, $Pre^{\Sigma}(Sm/F)$). The following proposition states that the corresponding model category structures on these coincide as well.

Proposition 5.4. *The model categories $Sp^{\mathbb{N}}_{nis}(Pre(Sm/F))$ and $Sp^{\mathbb{N}}_{mot}(Pre(Sm/F))$ (respectively, $Sp^{\Sigma}_{nis}(Pre(Sm/F))$ and $Sp^{\Sigma}_{mot}(Pre(Sm/F))$) coincide with the Nisnevich local projective and motivic model category structures on $Pre^{\mathbb{N}}(Sm/F)$ ($Pre^{\Sigma}(Sm/F)$), respectively.*

Proof. We prove the statement for the Nisnevich local projective model structure. The statement for the motivic model structure is proved in a similar way.

By Hovey's construction of the model category $Sp^{\mathbb{N}}_{nis}(Pre(Sm/F))$ a map of spectra $i\colon A \to B$ is a cofibration if and only if the induced maps $i_0\colon A_0 \to B_0$ and $j_n\colon A_n \coprod_{A_{n-1}\wedge S^1} B_{n-1} \wedge S^1 \to B_n$, $n \geqslant 1$, are projective cofibrations in $Pre(Sm/F)$. It follows that cofibrations in $Sp^{\mathbb{N}}_{nis}(Pre(Sm/F))$ coincide with cofibrations in the stable projective model structure on $Pre^{\mathbb{N}}(Sm/F)$, and hence with cofibrations in the

Nisnevich local model structure on $Pre^{\mathbb{N}}(Sm/F)$ because the Bousfield localization preserves cofibrations.

An object X in $Sp^{\mathbb{N}}_{nis}(Pre(Sm/F))$ is fibrant if and only if it is levelwise fibrant, i.e., levelwise projective fibrant and flasque, and each map $X_n \to \Omega X_{n+1}$, $n \geqslant 0$, which is adjoint to the structure map $X_n \wedge S^1 \to X_{n+1}$, is a weak equivalence. It follows that each map $X_n \to \Omega X_{n+1}$, $n \geqslant 0$, is a schemewise weak equivalence, and hence X is stable fibrant in the stable projective model structure on $Pre^{\mathbb{N}}(Sm/F)$. Since X is levelwise flasque, we see that $X(Q)$ is homotopy cartesian for every elementary distinguished square $Q \in \mathcal{Q}$. Therefore X is fibrant in the Nisnevich local model structure on $Pre^{\mathbb{N}}(Sm/F)$. One easily sees that every fibrant object in the Nisnevich local model structure on $Pre^{\mathbb{N}}(Sm/F)$ is fibrant in $Sp^{\mathbb{N}}_{nis}(Pre(Sm/F))$. We have shown that cofibrations and fibrant objects in both model categories coincide.

By [**13**, 9.7.4(4)] a map g in a simplicial model category is a weak equivalence if and only if for some cofibrant approximation $\tilde{g}\colon \widetilde{X} \to \widetilde{Y}$ to g and every fibrant object Z the map of simplicial sets $\tilde{g}^*\colon \operatorname{Map}(Y,Z) \to \operatorname{Map}(X,Z)$ is a weak equivalence. We infer that weak equivalences in $Sp^{\mathbb{N}}_{nis}(Pre(Sm/F))$ and in the Nisnevich local model structure on $Pre^{\mathbb{N}}(Sm/F)$ coincide. Thus the model structure on $Sp^{\mathbb{N}}_{nis}(Pre(Sm/F))$ coincides with the Nisnevich local model structure on $Pre^{\mathbb{N}}(Sm/F)$. The fact that $Sp^{\Sigma}_{nis}(Pre(Sm/F))$ and the Nisnevich local model structure on $Pre^{\Sigma}(Sm/F)$ coincide is checked similar to presheaves of ordinary spectra. □

In order to show some homotopically important properties for $Pre^{\Sigma}(Sm/F)$, we need to discuss Jardine's model structures. $Pre(Sm/F)$ enjoys the injective model structure in which cofibrations are monomorphisms and weak equivalences are the local weak equivalences. Global fibrations are those maps which have the right lifting property with respect to all maps which are cofibrations and local weak equivalences. As above, one defines the motivic injective model structure by means of the Bousfield localization. Then both the Nisnevich local injective and motivic injective model structures satisfy the axioms for a proper simplicial model category [**17, 18**]. One easily sees that the Nisnevich local projective (respectively, motivic projective) model structure on $Pre(Sm/F)$ is Quillen equivalent to the Nisnevich local injective (respectively, motivic injective) model structure.

Denote by $Sp^{\Sigma}_{nis,J}(Pre(Sm/F))$ and $Sp^{\Sigma}_{mot,J}(Pre(Sm/F))$ ("J" for Jardine) stable symmetric model structures corresponding to the Nisnevich local injective (respectively, motivic injective) model structure on $Pre(Sm/F)$. By [**18**, 4.32] and [**19**, Thm. 12] these are proper simplicial model categories. It follows from [**15**, 9.3] that the natural functors

$$Sp^{\Sigma}_{nis}(Pre(Sm/F)) \to Sp^{\Sigma}_{nis,J}(Pre(Sm/F))$$

and

$$Sp^{\Sigma}_{mot}(Pre(Sm/F)) \to Sp^{\Sigma}_{mot,J}(Pre(Sm/F))$$

are Quillen equivalences.

The category $Pre^{\Sigma}(Sm/F)$ of presheaves of symmetric spectra also enjoys the following model structures (see [**18, 19**] for details). A map $f\colon X \to Y$ is a *level equivalence* if each of the component maps $f\colon X_n \to Y_n$ is a local weak equivalence (respectively, motivic equivalence). The map f is a *level cofibration* if each of the maps $X_n \to Y_n$ is a monomorphism of simplicial presheaves. Denote these model categories by $Pre^{\Sigma}_{nis}(Sm/F)$ and $Pre^{\Sigma}_{mot}(Sm/F)$, respectively. These are proper simplicial model categories.

For every $X \in Pre^{\Sigma}(Sm/F)$ we obtain a natural construction

$$X \xrightarrow{i_1} X_s \xrightarrow{i_2} X_{si}$$

of an *injective stably fibrant model* X_{si}, where i_1 is a trivial stable cofibration in the model category $Sp^{\Sigma}_{nis,J}(Pre(Sm/F))$ (respectively, in $Sp^{\Sigma}_{mot,J}(Pre(Sm/F))$) and i_2 is a level cofibration and a level equivalence in $Pre^{\Sigma}_{nis}(Sm/F)$ (respectively, in $Pre^{\Sigma}_{mot}(Sm/F)$).

By [**18, 19**] a map $f\colon X \to Y$ of $Pre^{\Sigma}(Sm/F)$ is a stable weak equivalence if and only if it induces a weak equivalence of simplicial mapping Kan complexes

$$f\colon \operatorname{Map}(Y,W) \to \operatorname{Map}(X,W)$$

for each injective stably fibrant object W. Observe that the maps i_1 and i_2 are both stable weak equivalences.

Proposition 5.5. ([**18**, 4.41]) *Suppose that $i\colon A \to B$ is a stable cofibration in $Sp^{\Sigma}_{nis,J}(Pre(Sm/F))$ or in $Sp^{\Sigma}_{mot,J}(Pre(Sm/F))$ and that $j\colon C \to D$ is a level cofibration. Then the map*

$$(i,j)_*\colon (B \wedge_{\mathcal{O}_{naive}} C) \cup_{(A\wedge_{\mathcal{O}_{naive}} C)} (A \wedge_{\mathcal{O}_{naive}} D) \to B \wedge_{\mathcal{O}_{naive}} D$$

is a level cofibration. If either i or j is a stable equivalence in $Sp^{\Sigma}_{nis,J}(Pre(Sm/F))$ (respectively, in $Sp^{\Sigma}_{mot,J}(Pre(Sm/F))$), then so is $(i,j)_*$.

The proposition was actually shown for $Sp^{\Sigma}_{mot,J}(Pre(Sm/F))$. However the proof of this result is really quite generic, and holds essentially anywhere that one succeeds in generating the usual machinery of symmetric spectrum. This includes the present discussion of symmetric S^1-spectra in the Nisnevich local case $Sp^{\Sigma}_{nis,J}(Pre(Sm/F))$.

Theorem 5.6 (Jardine)**.** *The Nisnevich local projective and motivic projective model structures on the category $Pre^{\Sigma}(Sm/F)$ of presheaves of symmetric spectra are cellular, proper, spectral, weakly finitely generated, symmetric monoidal and satisfy the monoid axiom.*

Proof. Since Bousfield localization respects cellularity and left properness, then both model structures are cellular and left proper. Right properness is proved similar to [**18**, 4.15] and [**19**, Thm. 12].

It follows from [**16**, 3.2.13] that every object of $Pre^{\Sigma}(Sm/F)$ is small. By Lemma 2.2 both model structures are weakly finitely generated, because domains and codomains of morphisms from $\mathcal{N} \cup \mathcal{A}$ are finitely presentable. These are plainly spectral as well. Proposition 5.5 and Theorem 4.2 imply that both model structures are symmetric monoidal.

It remains to verify the monoid axiom. Let $i\colon A \to B$ be a trivial stable cofibration and let $C \in Pre^{\Sigma}(Sm/F)$. By Proposition 5.5 the map

$$i \wedge 1\colon A \wedge_{\mathcal{O}_{naive}} C \to B \wedge_{\mathcal{O}_{naive}} C$$

is a level cofibration and a stable equivalence. Therefore for any injective stably fibrant object W the map $\operatorname{Map}(i \wedge 1, W)$ is a trivial fibration of simplicial sets. Let j be a pushout of $i \wedge 1$ along some map. It follows that $\operatorname{Map}(j, W)$ is a trivial fibration of simplicial sets, and hence j is a stable equivalence. Now the monoid axiom follows from the fact that any transfinite composition of weak equivalences is a weak equivalence in a weakly finitely generated model category. □

Given a spectral category $\mathcal{O}$ over Sm/F, we want to establish the same properties for the Nisnevich local and motivic model structures on $\operatorname{Mod}\mathcal{O}$ as for $Pre^{\Sigma}(Sm/F)$ from the preceding theorem. For this we have to give the following

Definition 5.7. (1) We say that $\mathcal{O}$ is *Nisnevich excisive* if for every elementary distinguished square Q

$$\begin{array}{ccc} U' & \longrightarrow & X' \\ \downarrow & Q & \downarrow{\scriptstyle\varphi} \\ U & \xrightarrow[\psi]{} & X \end{array}$$

the square $\mathcal{O}Q$ (8) is homotopy pushout in the Nisnevich local model structure on $Pre^{\Sigma}(Sm/F)$.

(2) $\mathcal{O}$ is *motivically excisive* if:

(A) for every elementary distinguished square Q the square $\mathcal{O}Q$ (8) is homotopy pushout in the motivic model structure on $Pre^{\Sigma}(Sm/F)$ and

(B) for every $X \in Sm/F$ the natural map

$$\mathcal{O}(-, X \times \mathbb{A}^1) \to \mathcal{O}(-, X)$$

is a weak equivalence in the motivic model structure on $Pre^{\Sigma}(Sm/F)$.

The following lemma says that property (B) is redundant for symmetric monoidal spectral categories.

Lemma 5.8. *Let $\mathcal{O}$ be a symmetric monoidal spectral category on Sm/F such that the monoidal product is given by cartesian product of schemes. Then the map*

$$f\colon \mathcal{O}(-, X \times \mathbb{A}^1) \to \mathcal{O}(-, X)$$

is a weak equivalence in the motivic model structure on $Pre^{\Sigma}(Sm/F)$.

Proof. We follow an argument of [**24**, p. 694]. As in classical algebraic topology, an inclusion of motivic spaces $g\colon A \to B$ is an $\mathbb{A}^1$-deformation retract if there exist a map $r\colon B \to A$ such that $rg = \mathrm{id}_A$ and an $\mathbb{A}^1$-homotopy $H\colon B \times \mathbb{A}^1 \to B$ between gr and id_B which is constant on A. Then $\mathbb{A}^1$-deformation retracts are motivic weak equivalences.

There is an obvious map $r\colon \mathcal{O}(-, X) \to \mathcal{O}(-, X \times \mathbb{A}^1)$ such that $fr = 1$. Since $\mathcal{O}$ is a symmetric monoidal spectral category, it follows that

$$\mathcal{O}(- \times \mathbb{A}^1, X \times \mathbb{A}^1) \cong \underline{\operatorname{Mod}}\mathcal{O}(\mathcal{O}(-, \mathbb{A}^1), \mathcal{O}(-, X \times \mathbb{A}^1))$$

is an $\mathcal{O}$-module.

There is a natural isomorphism of symmetric spectra

$$Sp^{\Sigma}(\mathcal{O}(-, X \times \mathbb{A}^1), \mathcal{O}(- \times \mathbb{A}^1, X \times \mathbb{A}^1)) \cong \mathcal{O}(X \times \mathbb{A}^1 \times \mathbb{A}^1, X \times \mathbb{A}^1).$$

Consider the functor (4) of categories $u\colon Sm/F \to \mathcal{UO}$. Denote by α the obvious map $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$. We set $h = u(1_X \times \alpha)$; then h uniquely determines a morphism of $\mathcal{O}$-modules

$$h'\colon \mathcal{O}(-, X \times \mathbb{A}^1) \to \mathcal{O}(- \times \mathbb{A}^1, X \times \mathbb{A}^1).$$

This morphism can be regarded as a morphism of $\mathcal{O}_{naive}$-modules, denoted by the same letter. By adjointness h' uniquely determines a map of $\mathcal{O}_{naive}$-modules

$$H\colon \mathcal{O}(-, X \times \mathbb{A}^1) \wedge_{\mathcal{O}_{naive}} \mathcal{O}_{naive}(-, \mathbb{A}^1) \to \mathcal{O}(-, X \times \mathbb{A}^1).$$

Then H yields a level $\mathbb{A}^1$-homotopy between the identity map and rf. We see that f is a level motivic equivalence, and hence it is a weak equivalence in $Sp^{\Sigma}_{mot}(Pre(Sm/F))$. □

Theorem 5.9. *$\mathcal{O}_{K^{\oplus}}, \mathcal{O}_K, \mathcal{O}_{K^{Gr}}, \mathcal{O}_{K_0^{\oplus}}, \mathcal{O}_{K_0}, \mathcal{O}_{cor}$ are Nisnevich excisive spectral categories.*

Proof. We want to prove the statement first for $\mathcal{O}_K$. The cases $\mathcal{O}_{K^{\oplus}}, \mathcal{O}_{K^{Gr}}$ are checked in a similar way. Given $q \in \mathbb{Z}$ and a smooth scheme X, let $\mathcal{K}_q(-, X)$ denote the sheaf associated to the presheaf $W \mapsto K_q(W, X) = \pi_q(\mathcal{O}_K(W, X))$. If we show that $\mathcal{O}_K(Q)$ is homotopy pushout in $Sp^{\mathbb{N}}_{\mathrm{nis}}(Pre(Sm/F))$ for every elementary distinguished square Q then it will follow from [**19**, Lemma 10] that $\mathcal{O}_K$ is a Nisnevich excisive spectral category.

We have to verify that for any elementary distinguished square Q the sequence of sheaves

$$\cdots \to \mathcal{K}_{q+1}(-, X) \to \mathcal{K}_q(-, U') \to \mathcal{K}_q(-, U) \oplus \mathcal{K}_q(-, X') \to \mathcal{K}_q(-, X) \to \cdots$$

is exact. Because the Nisnevich topology has enough points, the sequence

$$\cdots \to K_{q+1}(-, X) \to K_q(-, U') \to K_q(-, U) \oplus K_q(-, X') \to K_q(-, X) \to \cdots$$

will become exact after sheafifying precisely if it becomes exact whenever one applies the presheaves to the Henselization W of a smooth scheme T at a point t. Thus it is enough to show that for any such W the square

$$\begin{array}{ccc} \mathcal{O}_K(W, U') & \longrightarrow & \mathcal{O}_K(W, X') \\ \downarrow & & \downarrow \\ \mathcal{O}_K(W, U) & \longrightarrow & \mathcal{O}_K(W, X) \end{array} \qquad (9)$$

is homotopy pushout (=homotopy pullback) of spectra. We shall actually show that (9) gives a split exact sequence

$$\mathcal{O}_K(W, U') \hookrightarrow \mathcal{O}_K(W, U) \oplus \mathcal{O}_K(W, X') \twoheadrightarrow \mathcal{O}_K(W, U') \qquad (10)$$

in the (triangulated) homotopy category of spectra $\mathrm{Ho}\, Sp$.

Given a triangulated category $\mathcal{T}$, consider a commutative diagram

$$\begin{array}{ccccccc} A'_1 & \xrightarrow{i'} & C' & \xrightarrow{\Phi'\times\Psi'} & A'\times B' & \xrightarrow{p'} & A' \\ {\scriptstyle \pi_1}\downarrow & & {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \pi\times\rho} & & \downarrow{\scriptstyle \pi} \\ A_1 & \xrightarrow{i} & C & \xrightarrow{\Phi\times\Psi} & A\times B & \xrightarrow{p} & A, \end{array} \qquad \begin{array}{l} \alpha'\times 0\colon A'_1 \to A'\times B',\ \ \alpha'\colon A'_1 \to A' \\ \alpha\times 0\colon A_1 \to A\times B,\ \ \alpha\colon A_1 \to A \end{array} \tag{11}$$

where $\rho\colon B'\to B$, $\Phi\times\Psi$, $\Phi'\times\Psi'$ are isomorphisms, $\alpha=\Phi i$ and $\alpha'=\Phi' i'$ are isomorphisms. Clearly, the right and the middle squares are cartesian, as well as so is the outer square with vertices (A'_1, A', A_1, A). Therefore the left square is cartesian and, moreover, the induced sequence

$$A'_1 \to A_1 \oplus C' \to C$$

is split exact in $\mathcal{T}$. Below we shall be constructing such a diagram in the triangulated category $\operatorname{Ho} Sp$.

Consider an elementary distinguished square Q

$$\begin{array}{ccc} U' & \xrightarrow{i'} & X' \\ {\scriptstyle \pi_1}\downarrow & Q & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow{i} & X. \end{array}$$

Denote by $\mathcal{P}(W,X)^U$ (respectively, $\mathcal{P}(W,X)^{\neg U}$) the full subcategory of $\mathcal{P}(W,X)$ consisting of those bimodules $P\in\mathcal{P}(W,X)$ for which $\operatorname{Supp}(P)\subseteq W\times U$ (respectively, $\operatorname{Supp}(P)\subsetneq W\times U$). There exists a functor

$$\Phi\colon \mathcal{P}(W,X)\to\mathcal{P}(W,X)^U,$$

constructed as follows. Given $P\in\mathcal{P}(W,X)$ set $S:=\operatorname{Supp}(P)$. Then $S=S'\sqcup S''$ with $S'\subseteq W\times U$ and S'' such that its each connected component is not contained in $W\times U$. The sheaf P is canonically equal to $P'\oplus P''$ with $\operatorname{Supp}(P')=S'$ and $\operatorname{Supp}(P'')=S''$. Moreover, if $P_1\in\mathcal{P}(W,X)$ and $P_1=P'_1\oplus P''_1$ is a similar decomposition, then every morphism $f\colon P\to P_1$ is of the form $f=\begin{pmatrix} f' & 0 \\ 0 & f''\end{pmatrix}$. This is because $\operatorname{Hom}(P',P''_1)=\operatorname{Hom}(P'',P'_1)=0$ and the corresponding supports are disjoint. So we set $\Phi(P)=P',\Phi(f)=f'$. There is also a functor

$$\Psi\colon \mathcal{P}(W,X)\to\mathcal{P}(W,X)^{\neg U},$$

defined as $\Psi(P)=P'',\Psi(f)=f''$. The full subcategories $\mathcal{P}(W,X')^{U'},\mathcal{P}(W,X')^{\neg U'}\subset\mathcal{P}(W,X')$ and functors $\Phi'\colon\mathcal{P}(W,X')\to\mathcal{P}(W,X')^{U'},\Psi'\colon\mathcal{P}(W,X')\to\mathcal{P}(W,X')^{\neg U'}$ are defined in a similar way.

The map $\pi\colon X'\to X$ induces two functors

$$\pi_*^U\colon\mathcal{P}(W,X')^{U'}\to\mathcal{P}(W,X)^U,\quad \pi_*^{\neg U}\colon\mathcal{P}(W,X')^{\neg U'}\to\mathcal{P}(W,X)^{\neg U}.$$

Indeed, if $Q \in \mathcal{P}(W, X')^{U'}$ then $\mathrm{Supp}(Q) \subset W \times U'$. It follows that $\mathrm{Supp}(\pi_*(Q)) = \pi(\mathrm{Supp}(Q)) \subset W \times U$, and hence $\pi_*(Q) \in \mathcal{P}(W, X)^U$. One sets $\pi_*^U := \pi_*|_{\mathcal{P}(W,X')^{U'}}$. If $Q \in \mathcal{P}(W, X')^{\neg U'}$ then no connected component of $\mathrm{Supp}(Q)$ is contained in $W \times U'$. Since $\pi^{-1}(U) = U'$, it follows that no connected component of $\mathrm{Supp}(\pi_*(Q)) = \pi(\mathrm{Supp}(Q))$ is contained in $W \times U$. We see that $\pi_*(Q) \in \mathcal{P}(W, X)^{\neg U}$ and one puts $\pi_*^{\neg U} := \pi_*|_{\mathcal{P}(W,X')^{\neg U'}}$.

Consider a diagram of categories

$$\begin{array}{ccccccc}
\mathcal{P}(W,U') & \xrightarrow{i'_*} & \mathcal{P}(W,X') & \xrightarrow{\Phi'\times\Psi'} & \mathcal{P}(W,X')^{U'} \times \mathcal{P}(W,X')^{\neg U'} & \xrightarrow{p'} & \mathcal{P}(W,X')^{U'} \\
\downarrow{\scriptstyle \pi_{1,*}} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*^U \times \pi_*^{\neg U}} & & \downarrow{\scriptstyle \pi_*^U} \\
\mathcal{P}(W,U) & \xrightarrow[i_*]{} & \mathcal{P}(W,X) & \xrightarrow[\Phi\times\Psi]{} & \mathcal{P}(W,X)^{U} \times \mathcal{P}(W,X)^{\neg U} & \xrightarrow[p]{} & \mathcal{P}(W,X)^{U},
\end{array} \tag{12}$$

We claim that:

1. $\Phi \times \Psi$ and $\Phi' \times \Psi'$ are equivalences of categories;
2. $\Psi \circ i_*$ is the zero functor;
3. $\Psi' \circ i'_*$ is the zero functor;
4. $\Phi' \circ i'_*$ is an equivalence of categories;
5. $\Phi \circ i_*$ is an equivalence of categories;
6. $\pi_*^{\neg U}$ is an equivalence of categories;
7. the diagram is commutative up to natural isomorphisms of functors.

Statements (1)-(7) together with [**35**, 1.3.1] will imply that the diagram of spectra

$$\begin{array}{ccccccc}
\mathcal{O}_K(W,U') & \xrightarrow{i'_*} & \mathcal{O}_K(W,X') & \xrightarrow{\Phi'\times\Psi'} & \mathcal{O}_K(W,X')^{U'} \times \mathcal{O}_K(W,X')^{\neg U'} & \xrightarrow{p'} & \mathcal{O}_K(W,X')^{U'} \\
\downarrow{\scriptstyle \pi_{1,*}} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*^U \times \pi_*^{\neg U}} & & \downarrow{\scriptstyle \pi_*^U} \\
\mathcal{O}_K(W,U) & \xrightarrow[i_*]{} & \mathcal{O}_K(W,X) & \xrightarrow[\Phi\times\Psi]{} & \mathcal{O}_K(W,X)^{U} \times \mathcal{O}_K(W,X)^{\neg U} & \xrightarrow[p]{} & \mathcal{O}_K(W,X)^{U},
\end{array}$$

obtained from (12) by taking realizations, is commutative in $\mathrm{Ho}\, Sp$. So we shall obtain a diagram of the form (11), which will yield a split exact sequence (10), and hence square (9) is homotopy pushout, as required.

So it remains to show (1)-(7). Let us show that $\Phi \times \Psi$ is an equivalence of categories (the same fact for $\Phi' \times \Psi'$ is checked in a similar way). Consider a functor

$$\Theta \colon \mathcal{P}(W, X)^U \times \mathcal{P}(W, X)^{\neg U} \to \mathcal{P}(W, X),$$

defined as $\Theta(P', P'') = P' \oplus P''$, $\Theta(f', f'') = f' \oplus f''$. Clearly, the canonical morphism $can_P \colon P \to \Theta \circ (\Phi \times \Psi)(P)$ is an isomorphism for every $P \in \mathcal{P}(W, X)$

$$\begin{array}{ccc}
P & \xrightarrow{\Phi\times\Psi} & (P', P'') \\
& {\scriptstyle can_P}\searrow^{\cong} & \downarrow{\scriptstyle \Theta} \\
& & P' \oplus P''.
\end{array}$$

Given a morphism $f\colon P \to P_1$ in $\mathcal{P}(W, X)$, we have that

$$\operatorname{Hom}(P'', P_1') = \operatorname{Hom}(P', P_1'')= 0$$

and the diagram

$$\begin{array}{ccc} P & \xrightarrow{can_P} & P' \oplus P'' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f' \oplus f''} \\ P_1 & \xrightarrow[can_{P_1}]{} & P_1' \oplus P_1'' \end{array}$$

is commutative. The latter shows that there is a natural transformation of functors $can\colon \mathrm{id} \to \Theta \circ (\Phi \times \Psi)$. Since can_P is an isomorphism for all P, then can is an isomorphism of functors.

The composition $(\Phi \times \Psi) \circ \Theta$ is just the identity functor. Indeed,

$$[(\Phi, \Psi) \circ \Theta](P', P'') = (\Phi \times \Psi)(P' \oplus P'') = (P', P'').$$

So (1) is verified.

For any $P \in \mathcal{P}(W, U)$ one has $\operatorname{Supp}(i_*(P)) = i(\operatorname{Supp}(P)) \subset W \times U$. We see that $\Psi(i_*(P)) = 0$. So (2) is verified. Property (3) is checked in a similar way.

Let us prove (5). Consider the functor

$$i^*|_{\mathcal{P}(W,X)^U}\colon \mathcal{P}(W, X)^U \to \mathcal{P}(W, U).$$

First, it is well defined. Indeed, $\operatorname{Supp}(i^*(P)) = i^{-1}(\operatorname{Supp}(P)) = \operatorname{Supp}(P)$, because $\operatorname{Supp}(P) \subset W \times U$ for all $P \in \mathcal{P}(W, X)^U$. For brevity we shall write i^* instead of $i^*|_{\mathcal{P}(W,X)^U}$. Second, one has adjunction morphisms

$$adj_P\colon i^* i_*(P) \to P, \quad P \in \mathcal{P}(W, U),$$

and

$$adj_{P'}\colon P' \to i_* i^*(P'), \quad P' \in \mathcal{P}(W, X)^U.$$

Clearly, these determine natural transformations of functors $i^* i_* \to \mathrm{id}$ and $\mathrm{id} \to i_* i^*$.

We want to show that these are isomorphisms. If $P \in \mathcal{P}(W, U)$ then

$$\operatorname{Supp}(i^* i_*(P)) = i^{-1} i(\operatorname{Supp}(P)) = \operatorname{Supp}(P).$$

Moreover, for every $x \in \operatorname{Supp}(P)$ the induced morphism of stalks

$$(i^* i_*(P))_x \to P_x$$

is an isomorphism. Therefore adj_P is an isomorphism. If $P' \in \mathcal{P}(W, X)^U$ it follows that $\operatorname{Supp}(i_* i^*(P')) = i(i^{-1}(\operatorname{Supp}(P'))) = i(\operatorname{Supp}(P')) = \operatorname{Supp}(P')$. Moreover, $adj_{P'}$ induces an isomorphism of stalks

$$P'_x \to (i_* i^*(P'))_x.$$

Therefore $adj_{P'}$ is an isomorphism. So (5) is verified. Property (4) is verified in a similar way.

Next, let us check (7). If $Q \in \mathcal{P}(W, X')$ and $Q = Q' \oplus Q''$ is its canonical decomposition with $Q' = \Phi'(Q), Q'' = \Psi'(Q)$. Then $\pi_*(Q) = \pi_*(Q') \oplus \pi_*(Q'')$. On the other hand, $\pi_*(Q) = \pi_*(Q)' \oplus \pi_*(Q)''$. Comparing supports, one gets that

$$\pi_*^U(Q') := \pi_*(Q') = \pi_*(Q') \qquad \text{and} \qquad and \pi_*^{\neg U}(Q'') := \pi_*(Q'') = \pi_*(Q)''.$$

So,

$$(\pi_*(Q)', \pi_*(Q)'') = (\pi_*^U(Q'), \pi_*^{\neg U}(Q'')).$$

We see that $\pi_*^U \circ \Phi' = \Phi \circ \pi_*$ and $\pi_*^{\neg U} \circ \Psi' = \Psi \circ \pi_*$. So (7) is verified.

To prove (6), we shall need some notation. Let Q be a coherent $\mathcal{O}_{W\times X'}$-module such that $S(Q) := \mathrm{Supp}(Q)$ is quasi-finite over W. Let $S(Q)'' \subset S(Q)$ denote the union of those connected components in $S(Q)$, each of which is not contained in $W \times U'$. Let $S(Q)' \subset S(Q)$ denote the union of the other connected components in $S(Q)$. Clearly, $S(Q)' \subset W \times U'$ and $S(Q) = S(Q)' \sqcup S(Q)''$. Then one has a decomposition

$$Q = Q' \oplus Q''$$

such that $\mathrm{Supp}(Q') = S(Q)'$ and $\mathrm{Supp}(Q'') = S(Q)''$.

Given a coherent $\mathcal{O}_{W\times X}$-module $\mathcal{M}$ which is coherent as an $\mathcal{O}_W$-module, we set $\pi^*_{\neg U}(\mathcal{M}) := \pi^*(\mathcal{M})''$.

Sublemma. *The following statements are true:*

(a) If $P \in \mathcal{P}(W,X)^{\neg U}$ then $\pi^*_{\neg U}(P) \in \mathcal{P}(W,X')^{\neg U'}$. In particular, $\mathrm{Supp}(\pi_*^{\neg U}(P))$ *is finite and surjective over W.*

(b) The composition $P \xrightarrow{adj} \pi^*\pi_*(P) \to \pi_*(\pi^*_{\neg U}(P))$ is an isomorphism for all $P \in \mathcal{P}(W,X)^{\neg U}$.

(c) The composition $\pi^*_{\neg U}(\pi_*(Q)) \to \pi^*\pi_*(Q) \xrightarrow{adj} Q$ is an isomorphism for all $Q \in \mathcal{P}(W,X')^{\neg U'}$.

Proof. Firstly prove (a). It is easy to check that $\pi_*^{\neg U}(P)$ is coherent as an $\mathcal{O}_W$-module. Since π is étale, it is flat as well. Thus $\pi_*^{\neg U}(P)$ is a flat $\mathcal{O}_W$-module. A coherent flat $\mathcal{O}_W$-module is necessarily a locally free $\mathcal{O}_W$-module. Whence $\pi_*^{\neg U}(P) \in \mathcal{P}(W,X)^{\neg U}$. Assertion (a) is proven.

Prove now assertion (b). Let $Z = X \setminus U$ be the complement of U and let $Z' = X' \setminus U'$ and both regarded as reduced schemes. The square Q is an elementary Nisnevich square, so π induces a scheme isomorphism $Z' \to Z$. Let $y \in W \times X$. In this case $\pi^{-1}(y) = \{y'\}$ as sets for a unique point y' and $y' \in W \times Z'$. Moreover, $\pi^{-1}(y) = Spec(k(y'))$ as schemes and the map $\pi^*\colon k(y) \to k(y')$ is an isomorphism.

By (a) one has $\pi_*^{\neg U}(P) \in \mathcal{P}(W,X)^{\neg U}$. So the composite morphism

$$P \xrightarrow{adj} \pi^*\pi_*(P) \to \pi_*(\pi^*_{\neg U}(P))$$

is a morphism of coherent locally free $\mathcal{O}_W$-modules. Thus to check that it is an isomorphism it suffices to check that for each closed point $y \in Supp(P)$ the induced morphism $P(y) \to \pi_*(\pi^*_{\neg U}(P))(y)$ is an isomorphism. Since $P \in \mathcal{P}(W,X)^{\neg U}$ one has $y \in W \times Z$. In that case the composite map

$$k(y) \xrightarrow{adj} \pi^*\pi_*(k(y)) \to \pi_*(\pi^*_{\neg U}(k(y))) = \pi_*(k(y')) = k(y')$$

is an isomorphism. It follows that the induced morphism $P(y) \to \pi_*(\pi^*_{\neg U}(P))(y)$ is an isomorphism as well. So the morphism $P \to \pi_*(\pi^*_{\neg U}(P))$ from (b) is indeed an isomorphism.

Finally prove (c). Since $Q \in P(W, X')^{\neg U'}$ then each closed point $y' \in Supp(Q)$ belongs to $W \times Z'$, where $Z' \subset X'$ is from the proof of (b). We already know that $\pi_*(Q) \in P(W, X)^{\neg U}$. Thus (a) implies $\pi^*_{\neg U}(\pi_*(Q)) \in P(W, X')^{\neg U'}$. So the map $\pi^*_{\neg U}(\pi_*(Q)) \to \pi^*\pi_*(Q) \xrightarrow{adj} Q$ is a morphism of coherent locally free $\mathcal{O}_W$-modules. Therefore it suffices to check that for each closed point $y' \in Supp(Q)$ the map

$$\pi^*_{\neg U}(\pi_*(Q))(y') \to \pi^*\pi_*(Q)(y') \xrightarrow{adj} Q(y') \tag{13}$$

is an isomorphism. The latter follows from the fact that $y' \in W \times Z'$ and the map

$$k(y') = \pi^*_{\neg U}(\pi_*(k(y'))) \to \pi^*(\pi_*(k(y'))) \xrightarrow{adj} k(y')$$

is identity. Assertion (c) is proven. □

The sublemma shows that

$$\pi^*_{\neg U} \colon \mathcal{P}(W, X)^{\neg U} \leftrightarrows \mathcal{P}(W, X')^{\neg U'} : \pi^{\neg U}_*$$

are mutually inverse equivalences of categories. So we have shown that $\mathcal{O}_K$ is Nisnevich excisive. The fact that $\mathcal{O}_{K^\oplus}, \mathcal{O}_{K^{Gr}}$ are Nisnevich excisive is proved in a similar way.

By above arguments it follows that for any elementary distinguished square Q the sequence of Nisnevich sheaves

$$0 \to \mathcal{K}_0(-, U') \to \mathcal{K}_0(-, U) \oplus \mathcal{K}_0(-, X') \to \mathcal{K}_0(-, X) \to 0$$

is exact showing that $\mathcal{O}_{K_0}$ is Nisnevich excisive. The fact that $\mathcal{O}_{K_0^\oplus}$ is Nisnevich excisive is proved in a similar way.

Now the sequence of Nisnevich sheaves

$$0 \to Cor(-, U') \to Cor(-, U) \oplus Cor(-, X') \to Cor(-, X) \to 0$$

is exact by [**29**, 4.3.9]. We conclude that $\mathcal{O}_{cor}$ is Nisnevich excisive as well. The theorem is proved. □

Corollary 5.10. $\mathcal{O}_{K_0^\oplus}, \mathcal{O}_{K_0}, \mathcal{O}_{cor}$ *are symmetric monoidal spectral categories, and hence motivically excisive.*

Proof. We have shown above that $\mathcal{O}_{cor}$ is symmetric monoidal (see p. 219). Let X, Y, X', Y' be four smooth schemes and let $P \in \mathcal{P}(X, Y), P' \in \mathcal{P}(X', Y')$. In this case, the external tensor product $P \boxtimes P'$ is obviously finite and flat over $X \times X'$. Thus, we get a bifunctor

$$\boxtimes \colon \mathcal{P}(X, Y) \times \mathcal{P}(X', Y') \to \mathcal{P}(X \times X', Y \times Y'),$$

which is obviously additive and biexact. This gives a canonical operation – an external tensor product

$$\boxtimes \colon K_0^\oplus(X, Y) \otimes K_0^\oplus(X', Y') \to K_0^\oplus(X \times X', Y \times Y')$$

and

$$\boxtimes \colon K_0(X, Y) \otimes K_0(X', Y') \to K_0(X \times X', Y \times Y').$$

This external tensor product determines symmetric monoidal spectral category structures for $\mathcal{O}_{K_0^\oplus}$ and $\mathcal{O}_{K_0}$. Since $\mathcal{O}_{K_0^\oplus}, \mathcal{O}_{K_0}, \mathcal{O}_{cor}$ are Nisnevich excisive by the preceding theorem, it follows from Lemma 5.8 that these are motivically excisive as well. □

Remark 5.11. We shall show below that $\mathcal{O}_{K^{Gr}}$, $\mathcal{O}_{K^{\oplus}}$, $\mathcal{O}_K$ are motivically excisive spectral categories. For this we first need to construct the bivariant motivic spectral sequence relating bivariant K-theory to motivic cohomology.

The next theorem is the main result of the section.

Theorem 5.12. *Suppose $\mathcal{O}$ is a Nisnevich (respectively, motivic) excisive spectral category. Then the Nisnevich local (motivic) model structure on* $\operatorname{Mod}\mathcal{O}$ *is cellular, proper, spectral and weakly finitely generated. Moreover, a map of $\mathcal{O}$-modules is a weak equivalence in the Nisnevich local (respectively, motivic) model structure if and only if it is a weak equivalence in the Nisnevich local (respectively, motivic) model structure on $Pre^{\Sigma}(Sm/F)$. If $\mathcal{O}$ is a symmetric monoidal spectral category then each of the model structures on* $\operatorname{Mod}\mathcal{O}$ *is symmetric monoidal with respect to the smash product* (3) *of $\mathcal{O}$-modules.*

Proof. Since Bousfield localization respects cellularity and left properness, then both model structures are cellular and left proper. It follows from [**16**, 3.2.13] that every object of $\operatorname{Mod}\mathcal{O}$ is small. By Lemma 2.2 both model structures are weakly finitely generated, because domains and codomains of morphisms from $\mathcal{N}\cup\mathcal{A}$ are finitely presentable. These are plainly spectral as well.

We are going to show the statement first for the Nisnevich local model structure.

Let J be a family of generating trivial stable projective cofibrations for Sp^{Σ}. Notice that J can be chosen in such a way that domains and codomains of the maps in J are finitely presentable. Recall that the set of maps in $\operatorname{Mod}\mathcal{O}$

$$\mathcal{P}_J=\{\mathcal{O}(-,X)\wedge sj\xrightarrow{\mathcal{O}(-,X)\wedge j}O(-,X)\wedge tj\mid j\in J,X\in Sm/F\}$$

is a family of generating trivial cofibrations for the stable projective model structure on $\operatorname{Mod}\mathcal{O}$.

We set

$$\widehat{\mathcal{N}}_{\mathcal{O}}:=\{A\wedge\Delta[n]_+\coprod_{A\wedge\partial\Delta[n]_+}B\wedge\partial\Delta[n]_+\to B\wedge\Delta[n]_+\mid(A\to B)\in\mathcal{N}_{\mathcal{O}},n\geqslant 0\}.$$

Following terminology of [**13**, section 4.2] an augmented family of $\mathcal{N}_{\mathcal{O}}$-horns is the following family of trivial cofibrations:

$$\Lambda(\mathcal{N}_{\mathcal{O}})=\mathcal{P}_J\cup\widehat{\mathcal{N}}_{\mathcal{O}}.$$

Observe that domains and codomains of the maps in $\Lambda(\mathcal{N}_{\mathcal{O}})$ are finitely presentable. It can be proven similar to [**15**, 4.2] that a map $f\colon A\to B$ is a fibration in the Nisnevich local model structure with fibrant codomain if and only if it has the right lifting property with respect to $\Lambda(\mathcal{N}_{\mathcal{O}})$.

Consider the adjoint functors (5)

$$\Psi_*\colon Pre^{\Sigma}(Sm/F)\rightleftarrows\operatorname{Mod}\mathcal{O}:\Psi^*.$$

We first observe that Ψ^* takes every map in $\Lambda(\mathcal{N}_{\mathcal{O}})$ to a weak equivalence in $Sp^{\Sigma}_{nis}(Pre(Sm/F))$. Indeed, if $f\in\mathcal{P}_J$ then $\Psi^*(f)$ is a stable projective weak equivalence, because Ψ^* preserves stable projective weak equivalences. Suppose $f\in\widehat{\mathcal{N}}_{\mathcal{O}}$; then $\Psi^*(f)$ is a weak equivalence in $Sp^{\Sigma}_{nis}(Pre(Sm/F))$, because $\mathcal{O}$ is Nisnevich excisive by assumption.

We want next to check that Ψ^* maps elements of $\Lambda(\mathcal{N}_{\mathcal{O}})$-cell to weak equivalences in $Sp^{\Sigma}_{nis}(Pre(Sm/F))$. Here $\Lambda(\mathcal{N}_{\mathcal{O}})$-cell denotes the class of maps of sequential compositions of cobase changes of coproducts of maps in $\Lambda(\mathcal{N}_{\mathcal{O}})$. Since Ψ^* preserves filtered colimits and weak equivalences are closed under filtered colimits, it suffices to prove that Ψ^* sends the cobase change of a map in $\Lambda(\mathcal{N}_{\mathcal{O}})$ to a weak equivalence.

Clearly, Ψ^* maps the cobase change of a map in $\mathcal{P}_J$ to a stable weak equivalence in $Pre^{\Sigma}(Sm/F)$. Note that every element in $\mathcal{N}_{\mathcal{O}}$ is a trivial cofibration in $Sp^{\Sigma}_{nis}(Pre(Sm/F))$. Therefore every map in $\widehat{\mathcal{N}}_{\mathcal{O}}$ is a trivial cofibration in $Sp^{\Sigma}_{nis}(Pre(Sm/F))$, and hence so is the cobase change of every map in $\widehat{\mathcal{N}}_{\mathcal{O}}$.

In order to show that $\Psi^*(f)$ is a weak equivalence in $Sp^{\Sigma}_{nis}(Pre(Sm/F))$ for any weak equivalence f in $\operatorname{Mod}\mathcal{O}$, we use the small object argument (see [**13**, 10.5.16] or [**14**, 2.1.14]). We construct a fibrant replacement $\alpha\colon X \to L_{\mathcal{N}}X$ for $X \in \operatorname{Mod}\mathcal{O}$, where α is the transfinite composition of a $\aleph_0$-sequence

$$X = E^0 \xrightarrow{\alpha_0} E^1 \xrightarrow{\alpha_1} E^2 \xrightarrow{\alpha_2} \cdots$$

in which each $E^n \to E^{n+1}$ is constructed as follows. Let S be the set of all commutative squares

$$\begin{array}{ccc} A & \longrightarrow & E^n \\ {\scriptstyle g}\downarrow & & \downarrow \\ B & \longrightarrow & * \end{array}$$

where $g \in \Lambda(\mathcal{N}_{\mathcal{O}})$. Then α_n a pushout

$$\begin{array}{ccc} \coprod_{s\in S} A_s & \longrightarrow & E^n \\ {\scriptstyle \coprod g_s}\downarrow & & \downarrow{\scriptstyle \alpha_n} \\ \coprod_{s\in S} B_s & \longrightarrow & E^{n+1} \end{array}$$

This construction is functorial in X. We have verified that α is a weak equivalence in $Sp^{\Sigma}_{nis}(Pre(Sm/F))$.

Now let $f\colon X \to Y$ be a weak equivalence in the Nisnevich local model structure on $\operatorname{Mod}\mathcal{O}$. Then the diagram

$$\begin{array}{ccc} \Psi^*(X) & \longrightarrow & \Psi^*(L_{\mathcal{N}}X) \\ {\scriptstyle \Psi^*(f)}\downarrow & & \downarrow{\scriptstyle \Psi^*(L_{\mathcal{N}}f)} \\ \Psi^*(Y) & \longrightarrow & \Psi^*(L_{\mathcal{N}}Y) \end{array}$$

is commutative. The horizontal arrows are weak equivalences in $Sp^{\Sigma}_{nis}(Pre(Sm/F))$, the right arrow is a stable projective weak equivalence. We infer that the left arrow is a weak equivalence in $Sp^{\Sigma}_{nis}(Pre(Sm/F))$.

On the other hand, if $\Psi^*(f)$ is a weak equivalence in $Sp^{\Sigma}_{nis}(Pre(Sm/F))$, then so is $\Psi^*(L_{\mathcal{N}}f)$. It is, moreover, a stable projective weak equivalence by [**13**, 3.2.13], because $\Psi^*(L_{\mathcal{N}}X), \Psi^*(L_{\mathcal{N}}Y)$ are fibrant in $Sp^{\Sigma}_{nis}(Pre(Sm/F))$. It follows that f is a weak equivalence in the Nisnevich local model structure on $\operatorname{Mod}\mathcal{O}$. We have proved that a map of $\mathcal{O}$-modules is a weak equivalence in the Nisnevich local model structure if and only if it is a weak equivalence in $Sp^{\Sigma}_{nis}(Pre(Sm/F))$.

We claim that Ψ^* respects fibrations. For this we shall apply a theorem of Bousfield [**2**]. Consider the stable model structure for $\mathcal{O}$-modules and a commutative diagram

$$\begin{array}{ccccc}
L_{\mathcal{N}}V & \xrightarrow{L_{\mathcal{N}}k} & L_{\mathcal{N}}X & & \\
\uparrow & & \uparrow & & \\
V & \xrightarrow{k} & X & \xrightarrow{\alpha_X} & L_{\mathcal{N}}X \\
\downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\
W & \xrightarrow{h} & Y & \xrightarrow{\alpha_Y} & L_{\mathcal{N}}Y \\
\downarrow & & \downarrow & & \\
L_{\mathcal{N}}W & \xrightarrow{L_{\mathcal{N}}h} & L_{\mathcal{N}}Y & &
\end{array}$$

with the central square pullback, f a fibration between stably fibrant objects and $\alpha_X, \alpha_Y, L_{\mathcal{N}}h$ stable projective weak equivalences. Note that $\Psi^*(h)$ is a weak equivalence in $Sp_{nis}^{\Sigma}(Pre(Sm/F))$. Since $\Psi^*\colon \operatorname{Mod}\mathcal{O} \to Pre^{\Sigma}(Sm/F)$ is a right Quillen functor with respect to the stable model structure, it follows that $\Psi^*(f)$ is a fibration in $Pre^{\Sigma}(Sm/F)$. Since α_X, α_Y are stable projective weak equivalences and the stable model structure on $Pre^{\Sigma}(Sm/F)$ is right proper, one easily sees that Ψ^* takes the right square of the diagram to a homotopy pullback square. Now [**13**, 3.4.7] implies $\Psi^*(f)$ is a fibration in $Sp_{nis}^{\Sigma}(Pre(Sm/F))$. Since $Sp_{nis}^{\Sigma}(Pre(Sm/F))$ is right proper by Theorem 5.6, we see that $\Psi^*(k)$ is a weak equivalence in $Sp_{nis}^{\Sigma}(Pre(Sm/F))$. Thus $\Psi^*(L_{\mathcal{N}}k)$ is a stable projective weak equivalence, and hence so is $L_{\mathcal{N}}k$.

By [**2**, 9.3, 9.7] the following notions define a proper simplicial model structure on $\operatorname{Mod}\mathcal{O}$: a morphism $f\colon X \to Y$ is a cofibration if and only if it is a stable projective cofibration, a weak equivalence if and only if $L_{\mathcal{N}}f\colon L_{\mathcal{N}}X \to L_{\mathcal{N}}Y$ is a stable projective weak equivalence, and fibration if and only if f is a stable projective fibration and the commutative square

$$\begin{array}{ccc}
X & \longrightarrow & L_{\mathcal{N}}X \\
{\scriptstyle f}\downarrow & & \downarrow \\
Y & \longrightarrow & L_{\mathcal{N}}Y
\end{array}$$

is homotopy cartesian. This model structure plainly coincides with the Nisnevich local model structure on $\operatorname{Mod}\mathcal{O}$, because cofibrations and weak equivalences are the same. As a consequence, Ψ^* respects fibrations and is a right Quillen functor from $\operatorname{Mod}\mathcal{O}$ to $Sp_{nis}^{\Sigma}(Pre(Sm/F))$. By [**25**, A.1.4] if $\mathcal{I}$ ($\mathcal{J}$) is a generating family of (trivial) cofibrations in $Sp_{nis}^{\Sigma}(Pre(Sm/F))$, then $\Psi_*(\mathcal{I})$ ($\Psi_*(\mathcal{J})$) is a generating family of (trivial) cofibrations.

Suppose $\mathcal{O}$ is a symmetric monoidal spectral category. Given two cofibrations $i\colon A \to B, j\colon C \to D$, the map

$$(i,j)_*\colon (B \wedge_{\mathcal{O}} C) \cup_{(A\wedge_{\mathcal{O}}C)} (A \wedge_{\mathcal{O}} D) \to B \wedge_{\mathcal{O}} D$$

is a cofibration, because cofibrations in the Nisnevich local and stable projective model structures are the same and the latter model structure is symmetric monoidal by Theorem 4.2.

Let $i \in \Psi_*(\mathcal{I})$ and $j \in \Psi_*(\mathcal{J})$. Then $i = \Psi_*(i')$ and $j = \Psi_*(j')$ for some $(i' \colon A' \to B') \in \mathcal{I}, (j' \colon C' \to D') \in \mathcal{J}$. The functor Ψ_* is strong symmetric monoidal. Therefore the map

$$(i,j)_* \colon (B \wedge_{\mathcal{O}} C) \cup_{(A \wedge_{\mathcal{O}} C)} (A \wedge_{\mathcal{O}} D) \to B \wedge_{\mathcal{O}} D$$

is isomorphic to

$$\Psi_*((i',j')_*) \colon \Psi_*(B' \wedge_{\mathcal{O}_{naive}} C') \cup_{\Psi_*(A' \wedge_{\mathcal{O}_{naive}} C')} \Psi_*(A' \wedge_{\mathcal{O}_{naive}} D') \\ \to \Psi_*(B' \wedge_{\mathcal{O}_{naive}} D'),$$

which is, in turn, isomorphic to

$$\Psi_*((B' \wedge_{\mathcal{O}_{naive}} C') \cup_{(A' \wedge_{\mathcal{O}_{naive}} C')} (A' \wedge_{\mathcal{O}_{naive}} D')) \to \Psi_*(B' \wedge_{\mathcal{O}_{naive}} D').$$

The map

$$(B' \wedge_{\mathcal{O}_{naive}} C') \cup_{(A' \wedge_{\mathcal{O}_{naive}} C')} (A' \wedge_{\mathcal{O}_{naive}} D') \to (B' \wedge_{\mathcal{O}_{naive}} D')$$

is a trivial cofibration in $Sp^{\Sigma}_{nis}(Pre(Sm/F))$ by Theorem 5.6. Since Ψ_* respects trivial cofibrations, then $(i,j)_*$ is a trivial cofibration in the Nisnevich local model structure on $\operatorname{Mod}\mathcal{O}$. Therefore the Nisnevich local model structure on $\operatorname{Mod}\mathcal{O}$ is symmetric monoidal by [**14**, 4.2.5].

To prove the statement for the motivic model structure on $\operatorname{Mod}\mathcal{O}$, it is enough to verify that each map

$$f \colon \mathcal{O}(-, X \times \mathbb{A}^1) \to \mathcal{O}(-, X), \quad X \in Sm/F,$$

is a weak equivalence in $Sp^{\Sigma}_{mot}(Pre(Sm/F))$, because the rest of the proof is verified similar to the Nisnevich local model structure. It is the case because we assume $\mathcal{O}$ to be motivically excisive. □

Corollary 5.13. *Suppose a spectral category $\mathcal{O}$ is Nisnevich excisive (respectively, motivically excisive). Then the pair of adjoint fuctors*

$$\Psi_* \colon Pre^{\Sigma}(Sm/F) \rightleftarrows \operatorname{Mod}\mathcal{O} : \Psi^*$$

induces a Quillen pair for the Nisnevich local projective (respectively, motivic) model structures on $Pre^{\Sigma}(Sm/F)$ and $\operatorname{Mod}\mathcal{O}$. In particular, one has adjoint functors between triangulated categories $\Psi_* \colon SH^{\mathrm{nis}}(F) \leftrightarrows SH^{\mathrm{nis}}\mathcal{O} : \Psi^*$ (respectively, $\Psi_* \colon SH^{\mathrm{mot}}(F) \leftrightarrows SH^{\mathrm{mot}}\mathcal{O} : \Psi^*$).

To show that the map

$$\mathcal{O}_K(-, X \times \mathbb{A}^1) \to \mathcal{O}_K(-, X)$$

is a motivic weak equivalence in $Pre^{\Sigma}(Sm/F)$ (and similarly for $\mathcal{O}_{K^{\oplus}}, \mathcal{O}_{K^{Gr}}$), one has to construct the bivariant motivic spectral sequence. It will follow then from Theorem 5.9 that $\mathcal{O}_{K^{Gr}}$, $\mathcal{O}_{K^{\oplus}}$, $\mathcal{O}_K$ are motivically excisive spectral categories.

6. Bivariant motivic cohomology groups

Consider a ringoid $\mathcal{A}$ and its Eilenberg-Mac Lane spectral category $H\mathcal{A}$. We denote by $Ch(\mathcal{A})$ the category of unbounded chain complexes of $\mathcal{A}$-modules. By an $\mathcal{A}$-module we just mean a contravariant additive functor from $\mathcal{A}$ to abelian groups. For instance, if $\mathcal{A}$ is Cor (respectively, $K_0^{\oplus}$ or K_0) then $\mathcal{A}$-modules are presheaves with transfers (respectively, with $K_0^{\oplus}$- or K_0-transfers).

By [**27**, section B.1] there is a chain of Quillen equivalences relating the category of $H\mathcal{A}$-modules $\operatorname{Mod} H\mathcal{A}$ with respect to the stable projective model structure and $Ch(\mathcal{A})$ with respect to the usual model structure (the quasi-isomorphisms and epimorphisms are weak equivalences and fibrations, respectively):

$$\operatorname{Mod} H\mathcal{A} \underset{L}{\overset{U}{\rightleftarrows}} \operatorname{Nvmod} H\mathcal{A} \underset{\mathcal{H}}{\overset{\Lambda}{\rightleftarrows}} Ch(\mathcal{A}).$$

Here L, Λ are left adjoint and the intermediate model category of naive $H\mathcal{A}$-modules is defined as follows.

Definition 6.1. Let $\mathcal{O}$ be a spectral category. A *naive* $\mathcal{O}$-module M consists of a collection $\{M(o)\}_{o\in\mathcal{O}}$ of $\mathbb{Z}_{\geqslant 0}$-graded, pointed simplicial sets together with associative and unital action maps $M(o)_p \wedge O(o', o)_q \to M(o')_{p+q}$ for pairs of objects o, o' in $\mathcal{O}$ and for $p, q \geqslant 0$. A morphism of naive $\mathcal{O}$-modules $M \to N$ consists of maps of graded spaces $M(o) \to N(o)$ strictly compatible with the action of $\mathcal{O}$. We denote the category of naive $\mathcal{O}$-modules by $\operatorname{Nvmod}\mathcal{O}$.

The *free* naive $\mathcal{O}$-module F_o at an object $o \in \mathcal{O}$ is given by the graded spaces $F_o(o') = \mathcal{O}(o', o)$ with action maps

$$F_o(o')_p \wedge \mathcal{O}(o'', o')_q = \mathcal{O}(o', o)_p \wedge \mathcal{O}(o'', o')_q \to \mathcal{O}(o'', o)_{p+q} = F_o(o'')_{p+q}$$

given by composition in $\mathcal{O}$.

One defines a model structure for naive $H\mathcal{A}$-modules as follows [**27**, B.1.3]. A morphism of naive $H\mathcal{A}$-modules $f\colon M \to N$ is a weak equivalence if it is an objectwise π_*-isomorphism, i.e., if for all $a \in \mathcal{A}$ the map $f(a)\colon M(a) \to N(a)$ induces an isomorphism of stable homotopy groups. The map f is an objectwise stable fibration if each $f(a)$ is a stable fibration of spectra in the sense of [**3**, 2.3]. A morphism of naive $H\mathcal{A}$–modules is a cofibration if it has the left lifting properties for maps which are objectwise π_*-isomorphisms and objectwise stable firations.

The forgetful functor U takes the free, genuine $H\mathcal{A}$-module to the free, naive $H\mathcal{A}$-module. The left adjoint L sends the naive free modules F_a to the genuine free modules. If we consider the free $\mathcal{A}$-module $\mathcal{A}(-, a)$, as a complex in dimension 0, then it is naturally isomorphic to $\Lambda(F_a)$ (see [**27**, section B.1] for details).

The notions for fibrant naive $H\mathcal{A}$-modules and complexes of $\mathcal{A}$-modules to be flasque are defined similar to $\mathcal{O}$-modules. We say that a morphism $f\colon M \to N$ of $H\mathcal{A}$-modules (respectively, complexes of $\mathcal{A}$-modules) is a *Nisnevich local weak equivalence* if for every flasque $H\mathcal{A}$-module Q the morphism

$$f^*\colon \operatorname{Ho}(\operatorname{Nvmod} H\mathcal{A})(N[i], Q) \to \operatorname{Ho}(\operatorname{Nvmod} H\mathcal{A})(M[i], Q)$$

(respectively, $f^*\colon \operatorname{Ho}(Ch\mathcal{A})(N[i], Q) \to \operatorname{Ho}(Ch\mathcal{A})(M[i], Q)$) is an isomorphism for every integer i. Note that f is a Nisnevich local weak equivalence if and only if sheafification with respect to Nisnevich topology of the graded morphism of graded presheaves $\pi_*(f)\colon \pi_*(M) \to \pi_*(N)$ (respectively, $H^*(f)\colon H^*(M) \to H^*(N)$) is an isomorphism.

Definition 6.2. (1) A flasque $H\mathcal{A}$-module (respectively, a complex of $\mathcal{A}$-modules) Q is said to be $\mathbb{A}^1$-*homotopy invariant* if for every $X \in Sm/F$ the map $Q(X) \to Q(X \times \mathbb{A}^1)$ is a π_*-isomorphism (respectively, quasi-isomorphism).

(2) We say that a morphism $f\colon M \to N$ of $H\mathcal{A}$-modules (respectively, complexes of $\mathcal{A}$-modules) is a *motivic equivalence* if for every $\mathbb{A}^1$-homotopy invariant $H\mathcal{A}$-module Q the morphism $f^*\colon \operatorname{Ho}(\operatorname{Nvmod} H\mathcal{A})(N[i], Q) \to \operatorname{Ho}(\operatorname{Nvmod} H\mathcal{A})(M[i], Q)$ (respectively, $f^*\colon \operatorname{Ho}(Ch\mathcal{A})(N[i], Q) \to \operatorname{Ho}(Ch\mathcal{A})(M[i], Q)$) is an isomorphism.

We define the Nisnevich local and motivic model structures for $H\mathcal{A}$-modules or complexes of $\mathcal{A}$-modules as follows. The cofibrations remain the same but the weak equivalences are the Nisnevich local weak equivalences and motivic equivalences, respectively. Fibrations are defined by the corresponding lifting property. It follows that the chain of adjoint functors

$$\operatorname{Mod} H\mathcal{A} \underset{L}{\overset{U}{\rightleftarrows}} \operatorname{Nvmod} H\mathcal{A} \underset{\mathcal{H}}{\overset{\Lambda}{\rightleftarrows}} Ch(\mathcal{A})$$

yields Quillen equivalences between model categories with respect to Nisnevich local and motivic model structures. Denote by $D_{\mathrm{nis}}(\mathcal{A})$ and $D_{\mathrm{mot}}(\mathcal{A})$ the triangulated homotopy categories of $Ch(\mathcal{A})$ with respect to the Nisnevich local and motivic model structures.

We get a pair of triangulated equivalences between triangulated categories

$$SH^{\mathrm{nis}} H\mathcal{A} \rightleftarrows D_{\mathrm{nis}}(\mathcal{A}).$$

We want to give another description of $D_{\mathrm{nis}}(\mathcal{A})$ for the case when $\mathcal{A}$ is either $K_0^{\oplus}$ or K_0 or Cor. One refers to $K_0^{\oplus}$- and K_0-modules as $K_0^{\oplus}$- and K_0-presheaves, respectively. Cor-modules are called in the literature presheaves with transfers.

Proposition 6.3 ([**28, 33, 36**]). *Let $\mathcal{F}$ be a $K_0^{\oplus}$-presheaf (respectively, K_0-presheaf, presheaf with transfers). Then, the associated Nisnevich sheaf $\mathcal{F}_{\mathrm{nis}}$ has a unique structure of a $K_0^{\oplus}$-presheaf (respectively, K_0-presheaf, presheaf with transfers) for which the canonical homomorphism $\mathcal{F} \to \mathcal{F}_{\mathrm{nis}}$ is a homomorphism of $K_0^{\oplus}$-presheaves (respectively, of K_0-presheaves, presheaves with transfers).*

Denote by $Sh(K_0^{\oplus}), Sh(K_0), ShTr$ the categories of Nisnevich $K_0^{\oplus}$-sheaves, K_0-sheaves and sheaves with transfers, respectively. Their derived categories are denoted by $D(Sh(K_0^{\oplus}))$, $D(Sh(K_0))$, $D(ShTr)$.

Corollary 6.4. *$Sh(K_0^{\oplus}), Sh(K_0), ShTr$ are Grothendieck categories, and hence have enough injectives.*

Proof. We prove the claim for $Sh(K_0^{\oplus})$, because the other two cases are similarly checked. $Sh(K_0^{\oplus})$ has filtered direct limits which are exact, because this is the case for $K_0^{\oplus}$-presheaves and for Nisnevich sheaves. So $Sh(K_0^{\oplus})$ satisfies axiom (Ab5). The category of $K_0^{\oplus}$-presheaves is a Grothendieck category with $\{K_0^{\oplus}(-, X)\}_{X \in Sm/F}$ the family of projective generators. It follows that the family of sheaves $\{K_0^{\oplus}(-, X)_{\mathrm{nis}}\}_{X \in Sm/F}$ is a family of generators of $Sh(K_0^{\oplus})$. □

Using the fact that Nisnevich local weak equivalences between complexes of Nisnevich $K_0^{\oplus}$-sheaves (respectively, K_0-sheaves and sheaves with transfers) coincide with quasi-isomorphisms between such complexes, we infer that the functor of sheafification induces triangulated equivalences of triangulated categories

$$D_{\mathrm{nis}}(K_0^{\oplus}) \xrightarrow{\sim} D(Sh(K_0^{\oplus})), \quad D_{\mathrm{nis}}(K_0) \xrightarrow{\sim} D(Sh(K_0)), \quad D_{\mathrm{nis}}(Cor) \xrightarrow{\sim} D(ShTr).$$

Note that $D_{\mathrm{nis}}(K_0^{\oplus})$, $D_{\mathrm{nis}}(K_0)$ and $D_{\mathrm{nis}}(Cor)$ are compactly generated triangulated categories with compact generators given by representable presheaves, and hence so are $D(Sh(K_0^{\oplus})), D(Sh(K_0)), D(ShTr)$.

Given a smooth scheme X, we define $\mathbb{Z}_{K_0^{\oplus}}(X)$ (respectively, $\mathbb{Z}_{K_0}(X)$ and $\mathbb{Z}_{tr}(X)$) as the complex having $K_0^{\oplus}(-,X)_{\mathrm{nis}}$ (respectively, $K_0(-,X)_{\mathrm{nis}}$ and $Cor(-,X)$) in degree zero and zero in other degrees. Here $K_0^{\oplus}(-,X)_{\mathrm{nis}}$ stands for the Nisnevich sheaf associated to the presheaf $U \mapsto K_0^{\oplus}(U,X)$.

Definition 6.5. (1) We say that a complex of $K_0^{\oplus}$-sheaves (respectively, K_0-sheaves and sheaves with transfers) Q is $\mathbb{A}^1$-*local* if for every scheme $X \in Sm/F$ and every integer n the natural map

$$D(Sh(K_0^{\oplus}))(\mathbb{Z}_{K_0^{\oplus}}(X)[n],Q) \to D(Sh(K_0^{\oplus}))(\mathbb{Z}_{K_0^{\oplus}}(X\times\mathbb{A}^1)[n],Q)$$

(respectively, the maps

$$D(Sh(K_0))(\mathbb{Z}_{K_0}(X)[n],Q) \to D(Sh(K_0))(\mathbb{Z}_{K_0}(X\times\mathbb{A}^1)[n],Q)$$

and $D(ShTr)(\mathbb{Z}_{tr}(X)[n],Q) \to D(ShTr)(\mathbb{Z}_{tr}(X\times\mathbb{A}^1)[n],Q))$ is an isomorphism.

(2) A morphism $M \to N$ of complexes of $K_0^{\oplus}$-sheaves (respectively, K_0-sheaves and sheaves with transfers) is called an $\mathbb{A}^1$-*weak equivalence* if for every $\mathbb{A}^1$-local complex Q the map

$$D(Sh(K_0^{\oplus}))(N,Q) \to D(Sh(K_0^{\oplus}))(M,Q)$$

(respectively, the maps

$$D(Sh(K_0))(N,Q) \to D(Sh(K_0))(M,Q)$$

and $D(ShTr)(N,Q) \to D(ShTr)(\mathbb{Z}_{tr}(M,Q))$ is an isomorphism.

(3) The $\mathbb{A}^1$-*derived category* of $K_0^{\oplus}$-sheaves (respectively, K_0-sheaves and sheaves with transfers) is the one, obtained from $D(Sh(K_0^{\oplus}))$ (respectively, $D(Sh(K_0))$ and $D(ShTr)$) by inverting the $\mathbb{A}^1$-weak equivalences. The corresponding $\mathbb{A}^1$-derived categories will be denoted by $D_{\mathbb{A}^1}(Sh(K_0^{\oplus}))$, $D_{\mathbb{A}^1}(Sh(K_0))$ and $D_{\mathbb{A}^1}(ShTr)$, respectively.

By the general localization theory of compactly generated triangulated categories (see, e.g., [**23**]) $D_{\mathbb{A}^1}(Sh(K_0^{\oplus}))$ (respectively, $D_{\mathbb{A}^1}(Sh(K_0))$ and $D_{\mathbb{A}^1}(ShTr)$) is the localization of $D(Sh(K_0^{\oplus}))$ (respectively, $D(Sh(K_0))$ and $D(ShTr)$) with respect to the localizing subcategory generated by cochain complexes of the form

$$\mathbb{Z}_{K_0^{\oplus}}(X\times\mathbb{A}^1) \to \mathbb{Z}_{K_0^{\oplus}}(X)$$

(respectively, $\mathbb{Z}_{K_0}(X\times\mathbb{A}^1) \to \mathbb{Z}_{K_0}(X)$ and $\mathbb{Z}_{tr}(X\times\mathbb{A}^1) \to \mathbb{Z}_{tr}(X)$).

$D_{\mathbb{A}^1}(Sh(K_0^{\oplus}))$ (respectively, $D_{\mathbb{A}^1}(Sh(K_0))$ and $D_{\mathbb{A}^1}(ShTr)$) can be identified with the full subcategory of $D(Sh(K_0^{\oplus}))$ of $\mathbb{A}^1$-local complexes. The inclusion functor

$$D_{\mathbb{A}^1}(Sh(K_0^{\oplus})) \to D(Sh(K_0^{\oplus}))$$

admits a left adjoint

$$L_{\mathbb{A}^1}\colon D(Sh(K_0^{\oplus})) \to D_{\mathbb{A}^1}(Sh(K_0^{\oplus}))$$

which is also called the $\mathbb{A}^1$-*localization functor*. The same $\mathbb{A}^1$-localization functor exists for $D_{\mathbb{A}^1}(Sh(K_0))$ and $D_{\mathbb{A}^1}(ShTr)$.

The above arguments may be summarized as follows.

Proposition 6.6. *There are natural equivalences*

$$D_{\mathrm{mot}}(K_0^{\oplus}) \xrightarrow{\sim} D_{\mathbb{A}^1}(Sh(K_0^{\oplus})), \qquad D_{\mathrm{mot}}(K_0) \xrightarrow{\sim} D_{\mathbb{A}^1}(Sh(K_0)),$$
$$D_{\mathrm{mot}}(Cor) \xrightarrow{\sim} D_{\mathbb{A}^1}(ShTr).$$

of triangulated categories.

Recall that a sheaf $\mathcal{F}$ of abelian groups in the Nisnevich topology on Sm/F is *strictly* $\mathbb{A}^1$-*invariant* if for any $X \in Sm/F$, the canonical morphism

$$H^*_{\mathrm{nis}}(X, \mathcal{F}) \to H^*_{\mathrm{nis}}(X \times \mathbb{A}^1, \mathcal{F})$$

is an isomorphism. It follows from [**22**, 6.2.7] that a complex of $K_0^{\oplus}$-sheaves (respectively, K_0-sheaves and sheaves with transfers) Q is $\mathbb{A}^1$-local if and only if each cohomology sheaf $H^n(Q)$ is strictly $\mathbb{A}^1$-invariant.

Recall that for any presheaf of abelian groups $\mathcal{F}$ on Sm/F we get a simplicial presheaf $C_n(\mathcal{F})$, by setting $C_n(\mathcal{F})(U) = \mathcal{F}(U \times \Delta^n)$. We shall write $C^*(\mathcal{F})$ to denote the corresponding cochain complex (of degree $+1$) of abelian presheaves. Namely,

$$C^i(\mathcal{F}) := C_{-i}(\mathcal{F}).$$

By [**28, 33, 36**] strictly $\mathbb{A}^1$-invariant $K_0^{\oplus}$-, K_0-sheaves and sheaves with transfers coincide with $\mathbb{A}^1$-invariant ones whenever the field F is perfect. Therefore over perfect fields the categories $D_{\mathbb{A}^1}(Sh(K_0^{\oplus}))$, $D_{\mathbb{A}^1}(Sh(K_0))$ and $D_{\mathbb{A}^1}(ShTr)$ can be identified with the full subcategories of complexes having $\mathbb{A}^1$-invariant cohomology sheaves. Moreover, the functor $L_{\mathbb{A}^1}$ equals the functor C^* in all three cases. A detailed proof for sheaves with transfers is given in [**33**]. The triangulated category of Voevodsky DM_-^{eff} [**33**] is a full subcategory of $D_{\mathbb{A}^1}(ShTr)$ in this case.

We want to introduce bivariant motivic cohomology for smooth schemes, but first recall some facts for motivic cohomology. Let $X \in Sm/F$, $\star \in \{K_0^{\oplus}, K_0, tr\}$, and $n \geqslant 0$; we define the Nisnevich sheaf $\mathbb{Z}_\star(X)(\mathbb{G}_m^{\wedge n})$ as follows. Let $\mathcal{D}_n$ be the sum of images of homomorphisms

$$\mathbb{Z}_\star(X \times \mathbb{G}_m^{\times n-1}) \to \mathbb{Z}_\star(X \times \mathbb{G}_m^{\times n})$$

induced by the embeddings of the form

$$(a_1, \dots, a_{n-1}) \in \mathbb{G}_m^{\times n-1} \mapsto (a_1, \dots, 1, \dots, a_{n-1}) \in \mathbb{G}_m^{\times n}.$$

The sheaf $\mathbb{Z}_\star(X)(\mathbb{G}_m^{\wedge n})$ is, by definition, $\mathbb{Z}_\star(X)(\mathbb{G}_m^{\times n})/\mathcal{D}_n$. In what follows we shall denote the sheaf $\mathbb{Z}_\star(pt)(\mathbb{G}_m^{\wedge n})$ by $\mathbb{Z}_\star(\mathbb{G}_m^{\wedge n})$.

Definition 6.7. The $K_0^{\oplus}$-*motive* $M_{K_0^{\oplus}}(X)$ of a smooth scheme X over F (respectively, the K_0-*motive* $M_{K_0}(X)$ and the *motive* $M_{tr}(X)$) is the image of $\mathbb{Z}_{K_0^{\oplus}}(X)$ in $D_{\mathbb{A}^1}(Sh(K_0^{\oplus}))$ (respectively, the image of $\mathbb{Z}_{K_0}(X)$ in $D_{\mathbb{A}^1}(Sh(K_0))$ and the image of $\mathbb{Z}_{tr}(X)$ in $D_{\mathbb{A}^1}(ShTr)$). If $\star \in \{K_0^{\oplus}, K_0, tr\}$ then by $M_\star(X)(n)$, $n \geqslant 0$, we denote the corresponding image of $\mathbb{Z}_\star(X)(\mathbb{G}_m^{\wedge n})[-n]$.

Let $X \in Sm/F$; the *complex* $\mathbb{Z}_\star(X)(n)$ *of weight* n on Sm/F is the complex $C^*\mathbb{Z}_\star(X)(\mathbb{G}_m^{\wedge n})[-n]$, where the degree shift refers to the cohomological indexing of complexes. If $X = pt$ we shall just write $\mathbb{Z}_\star(n)$ for the motivic complex dropping X from notation.

Definition 6.8. For smooth schemes $U, X \in Sm/F$ we define their *bivariant motivic cohomology groups* $H^{i,n}(U, X, \mathbb{Z})$ as $H^i_{\mathrm{nis}}(U, \mathbb{Z}(X)_{K_0^{\oplus}}(n))$. We shall write $H^{i,n}_{\mathcal{M}}(U, \mathbb{Z})$ to denote $H^i_{\mathrm{nis}}(U, \mathbb{Z}_{K_0^{\oplus}}(n))$.

By a theorem of Suslin [**28**] for any $n \geqslant 0$ and any field F, the canonical homomorphism of complexes of Nisnevich sheaves

$$f_n \colon \mathbb{Z}_{K_0^{\oplus}}(n) \to \mathbb{Z}_{tr}(n)$$

is a quasi-isomorphism. Hence, for any smooth scheme $X \in Sm/F$, cohomology $H^*_{\mathrm{nis}}(X, \mathbb{Z}_{K_0^{\oplus}}(n))$ coincides with motivic cohomology $H^*_{\mathrm{nis}}(X, \mathbb{Z}_{tr}(n))$ of Suslin–Voevodsky [**30**]. By [**34**] for any field F, any smooth scheme X over F and any $i, n \in \mathbb{Z}$, there is a natural isomorphism

$$H^i_{\mathrm{nis}}(X, \mathbb{Z}_{tr}(n)) \cong CH^n(X, 2n - i),$$

where the right hand side groups are higher Chow groups. Since both groups are homotopy invariant, then $H^i_{\mathrm{nis}}(X, \mathbb{Z}_{K_0^{\oplus}}(n))$ are homotopy invariant $K_0^{\oplus}$-presheaves.

Proposition 6.9. *For any Nisnevich* $K_0^{\oplus}$*-sheaf* $\mathcal{F}$ *we have natural identifications*

$$\mathrm{Ext}^i_{Sh(K_0^{\oplus})}(\mathbb{Z}_{K_0^{\oplus}}(X), \mathcal{F}) = H^i_{\mathrm{nis}}(X, \mathcal{F}).$$

In particular $\mathrm{Ext}^i_{Sh(K_0^{\oplus})}(\mathbb{Z}_{K_0^{\oplus}}(X), -) = 0$ *for* $i > \dim X$.

Proof. Since the category $Sh(K_0^{\oplus})$ has sufficiently many injective objects by Corollary 6.4 and for any $K_0^{\oplus}$-sheaf $\mathcal{G}$ one has $\mathrm{Hom}(K_0^{\oplus}(-, X)_{\mathrm{nis}}, \mathcal{G}) = \mathcal{G}(X)$ we only have to show that for any injective $K_0^{\oplus}$-sheaf $\mathcal{I}$ one has $H^i_{\mathrm{nis}}(X, \mathcal{I}) = 0$ for $i > 0$.

It follows from the proof of Theorem 5.9 that for any elementary distinguished square (7) the sequence of $K_0^{\oplus}$-sheaves

$$0 \to K_0^{\oplus}(-, U')_{\mathrm{nis}} \to K_0^{\oplus}(-, U)_{\mathrm{nis}} \oplus K_0^{\oplus}(-, X')_{\mathrm{nis}} \to K_0^{\oplus}(-, X)_{\mathrm{nis}} \to 0$$

is exact. Therefore the sequence of abelian groups

$$0 \to \mathcal{I}(X) \to \mathcal{I}(U) \oplus \mathcal{I}(X') \to \mathcal{I}(U') \to 0$$

is exact, because $\mathcal{I}$ is injective. Let $H\mathcal{I}$ be the Eilenberg–Mac Lane sheaf of S^1-spectra associated with $\mathcal{I}$ (see [**22**, p. 23]). It follows that $H\mathcal{I}$ is Nisnevich excisive.

By [**22**, 3.2.3] there is a natural isomorphism

$$H^n_{\mathrm{nis}}(X,\mathcal{I}) \to [(X_+), H\mathcal{I}[n]], \quad n \in \mathbb{Z},$$

where the right hand side is the Hom-set in the stable homotopy category of S^1-spectra on Sm/F (see [**22**]). It follows from [**22**, 3.1.7] that $[(X_+), H\mathcal{I}[n]]$ is isomorphic to $\pi_{-n}(H\mathcal{I}(X))$ which is zero for $n \neq 0$. □

Corollary 6.10. *Let X be a smooth scheme over a field F and C be a complex of Nisnevich $K_0^{\oplus}$-sheaves bounded above. Then for any $i \in \mathbb{Z}$ there is a canonical isomorphism*

$$\mathrm{Hom}_{D(Sh(K_0^{\oplus}))}(\mathbb{Z}_{K_0^{\oplus}}(X), C[i]) \cong H^i_{\mathrm{nis}}(X, C).$$

Proof. The proof is like that of [**30**, 1.8] (one should use the preceding proposition as well). □

Corollary 6.11. *For every $n \geqslant 0$ the complex $\mathbb{Z}_{K_0^{\oplus}}(n)$ is $\mathbb{A}^1$-local in $D(Sh(K_0^{\oplus}))$.*

Proof. This follows from the previous corollary and the fact that $H^i_{\mathrm{nis}}(X, \mathbb{Z}_{K_0^{\oplus}}(n))$ are homotopy invariant $K_0^{\oplus}$-presheaves. □

Theorem 6.12. *Let F be any field, then for every $X \in Sm/F$ there is a natural isomorphism*

$$H^{i,n}_{\mathcal{M}}(X,\mathbb{Z}) \cong D_{\mathbb{A}^1}(Sh(K_0^{\oplus}))(M_{K_0^{\oplus}}(X), M_{K_0^{\oplus}}(pt)(n)[i]).$$

If the field F is perfect then there is also a natural isomorphism

$$H^{i,n}(U,X,\mathbb{Z}) \cong D_{\mathbb{A}^1}(Sh(K_0^{\oplus}))(M_{K_0^{\oplus}}(U), M_{K_0^{\oplus}}(X)(n)[i])$$

for any $U, X \in Sm/F$.

Proof. The preceding corollary implies $\mathbb{Z}_{K_0^{\oplus}}(n)$ is $\mathbb{A}^1$-local in $D(Sh(K_0^{\oplus}))$. It is proved similar to [**33**, section 3.2] that for any Nisnevich $K_0^{\oplus}$-sheaf $\mathcal{F}$, the natural morphism of complexes

$$\mathcal{F} \to C^*(\mathcal{F})$$

is an $\mathbb{A}^1$-weak equivalence. By Corollary 6.10 there is a canonical isomorphism

$$\mathrm{Hom}_{D(Sh(K_0^{\oplus}))}(\mathbb{Z}_{K_0^{\oplus}}(X), \mathbb{Z}_{K_0^{\oplus}}(n)[i]) \cong H^{i,n}_{\mathcal{M}}(X,\mathbb{Z}).$$

for any $i \in \mathbb{Z}$. On the other hand one has,

$$\mathrm{Hom}_{D(Sh(K_0^{\oplus}))}(\mathbb{Z}_{K_0^{\oplus}}(X), \mathbb{Z}_{K_0^{\oplus}}(n)[i]) \cong \mathrm{Hom}_{D(Sh(K_0^{\oplus}))}(\mathbb{Z}_{K_0^{\oplus}}(X)(0), \mathbb{Z}_{K_0^{\oplus}}(n)[i]).$$

Thus,

$$\begin{aligned} H^{i,n}_{\mathcal{M}}(X,\mathbb{Z}) &\cong \mathrm{Hom}_{D(Sh(K_0^{\oplus}))}(\mathbb{Z}_{K_0^{\oplus}}(X)(0), \mathbb{Z}_{K_0^{\oplus}}(n)[i]) \\ &\cong D_{\mathbb{A}^1}(Sh(K_0^{\oplus}))(M_{K_0^{\oplus}}(X), M_{K_0^{\oplus}}(pt)(n)[i]). \end{aligned}$$

is an isomorphism for all $i \in \mathbb{Z}$.

Assume now that F is perfect. Then all cohomology sheaves for $\mathbb{Z}_{K_0^{\oplus}}(X)(n)$ are homotopy invariant, hence strictly homotopy invariant because F is perfect (see above). We see that each complex $\mathbb{Z}_{K_0^{\oplus}}(X)(n)$ is $\mathbb{A}^1$-local. The second assertion is now checked similar to the first one. □

7. The Grayson tower

Recall that *Ord* denotes the category of finite nonempty ordered sets, and for each $d \geqslant 0$ we introduce the object $[d] = \{0 < 1 < \cdots < d\}$ of *Ord*. Given $A, B \in Ord$ we let $AB \in Ord$ denote the ordered set obtained by concatenating A and B, with the elements of A smaller than the elements of B. Given a simplicial set Y with base point $y_0 \in Y_0$, the natural inclusion maps $A \to AB \leftarrow B$ provide natural face maps $Y(A) \to Y(AB) \leftarrow Y(B)$. Let PY be the simplicial path space of edges in Y with initial endpoint at y_0; it can be defined for $A \in Ord$ by

$$(PY)(A) = \lim(\{y_0\} \to Y([0]) \leftarrow Y([0]A)).$$

The space $|PY|$ is contractible. The face maps $Y([0]A) \to Y(A)$ provide a projection map $PY \to Y$. We define

$$\omega Y = \lim(PY \to Y \leftarrow PY).$$

The commutative square

$$\begin{array}{ccc} \omega Y & \longrightarrow & PY \\ \downarrow & & \downarrow \\ PY & \longrightarrow & Y \end{array}$$

together with the contractibility of the space $|PY|$ provides a natural map $|wY| \to \Omega|Y|$.

For instance, let Y be the nerve of a category $\mathcal{B}$ with $b_0 \in \operatorname{Ob}\mathcal{B}$ the base point. A vertex of ωY is nothing more than a pair of morphisms in $\mathcal{B}$

$$b_0 \to b_1 \leftarrow b_0, \quad b_1 \in \operatorname{Ob}\mathcal{B}.$$

An edge of ωY is a pair of commutative triangles

$$\begin{array}{ccccc} b_0 & \xrightarrow{u} & b_1 & \xleftarrow{v} & b_0 \\ & \searrow & \downarrow & \swarrow & \\ & & b_2 & & \end{array}$$

The map $|\omega Y| \to \Omega|Y|$ is uniquely determined by a map $|\omega Y| \wedge S^1 \to |Y|$. The latter map comes from two maps $|\omega Y| \times |\Delta^1| \to |Y|$, which are uniquely determined by two functors of categories $U, V : \mathcal{B} \times \{0 \to 1\} \to \mathcal{B}$. These functors can be thought of as two natural transformations from the constant functor sending all objects of $\mathcal{B}$ to b_0 to the identity functor on $\mathcal{B}$. The natural transformations are defined by the arrows u, v on objects and the two triangles define them on morphisms. A vertex of ωY yields a loop in $|Y|$ which starts at b_0, follows the edge u to b_1, and returns along the other edge v to b_0, and an edge of ωY yields a homotopy between two such loops.

If $\mathcal{M}$ is an exact category, and S is the S-construction of Waldhausen [**35**] then $\omega S\mathcal{M}$ is the simplicial set $G\mathcal{M}$ of Gillet–Grayson [**9**] and $|\omega S\mathcal{M}| \to \Omega|SM|$ is a homotopy equivalence as it was shown in [**9**]. Also, if $\mathcal{M}$ is an additive category, then $|\omega S^{\oplus}\mathcal{M}| \to \Omega|S^{\oplus}\mathcal{M}|$ is a homotopy equivalence [**12**], where $S^{\oplus}$ is the Grayson $S^{\oplus}$-construction. Note that the latter equivalence is functorial in $\mathcal{M}$.

Given an additive category $\mathcal{M}$, Quillen defines a new category $S^{-1}S\mathcal{M}$ whose objects are pairs (A,B) of objects of $\mathcal{M}$. A morphism $(A,B)\to(C,D)$ in $S^{-1}S\mathcal{M}$ is given by a pair of split monomorphisms

$$f\colon A \overset{\leftarrow}{\rightarrowtail} C, \quad g\colon B \overset{\leftarrow}{\rightarrowtail} D$$

together with an isomorphism $h\colon \operatorname{Coker} f \to \operatorname{Coker} g$. By a split monomorphism we mean a monomorphism together with a chosen splitting. The nerve of the category $S^{-1}S\mathcal{M}$ which is also denoted by $S^{-1}S\mathcal{M}$ is homotopy equivalent to Quillen's K-theory space of $\mathcal{M}$ by [**10**]. There is an obvious map

$$u\colon \omega S^{\oplus}\mathcal{M} \to S^{-1}S\mathcal{M}.$$

It is a homotopy equivalence by [**12**, section 4].

We can regard $\omega S^{\oplus}\mathcal{M}$ and $S^{-1}S\mathcal{M}$ as simplicial additive categories. The equivalence u above can be extended to a stable equivalence of symmetric spectra

$$u\colon K^{Gr}(\omega S^{\oplus}\mathcal{M}) \to K^{Gr}(S^{-1}S\mathcal{M}).$$

Though the space $|\omega S^{-1}S\mathcal{M}|$ is not the loop space of $S^{-1}S\mathcal{M}$ [**12**, section 4], it is worth to consider $|\omega S^{-1}S\mathcal{M}|$ by replacing $\mathcal{M}$ by a simplicial additive category over a connected simplicial ring.

Suppose now that X is a pointed simplicial space. We let I_*X_d denote the connected component of X_d containing the base point. Grayson [**12**] has shown that there is an essential obstruction to the natural map $|d\mapsto \Omega X_d| \to \Omega|X|$ being an equivalence. The following theorem says what this obstruction is when X is a simplicial group-like H-space.

Theorem 7.1 (Grayson [**12**]). *If X is a simplicial group-like H-space then the natural sequence*

$$|d\mapsto \Omega X_d| \to \Omega|X| \to \Omega|d\mapsto \pi_0(X_d)| \tag{14}$$

is a fibration sequence. Moreover, the map

$$|X| \to |d\mapsto \pi_0(X_d)|$$

induces an isomorphism on π_0, and its homotopy fiber is connected.

The standard diagonalization technique allows us to generalize (14) to multisimplicial spaces. For example, if X is a bisimplicial group-like H-space, then

$$|(d,e)\mapsto \Omega X_{d,e}| \to \Omega|X| \to \Omega|(d,e)\mapsto \pi_0(X_{d,e})|$$

is a fibration sequence.

If R is a simplicial ring, and $\mathcal{M}$ is a simplicial additive category, we say that $\mathcal{M}$ is *R-linear* if, for each $d\geqslant 0$, M_d is an R_d-linear category, and for each map $\varphi\colon [e]\to[d]$, each $r\in R_d$, and each arrow f in $\mathcal{M}_d$, we have the equation $\varphi^*(rf)=\varphi^*(r)\varphi^*(f)$. A typical example of a contractible ring is $F[\Delta]$ whose n-simplices are defined as

$$F[\Delta]_n = F[x_0,\dots,x_n]/(x_0+\cdots+x_n-1).$$

By $\Delta^{\cdot}$ we denote the cosimplicial affine scheme $\operatorname{Spec}(F[\Delta])$.

Theorem 7.2 (Grayson [**12**])**.** *If R is a contractible simplicial ring and $\mathcal{M}$ is a simplicial R-linear additive category, then the natural map*

$$|d \mapsto |\omega S^{-1} S\mathcal{M}_d|| \to |d \mapsto \Omega |S^{-1} S\mathcal{M}_d||$$

is a homotopy equivalence of spaces.

Here is some convenient notation for cubes. We let $[1]$ denote the ordered set $\{0 < 1\}$ regarded as a category, and we use ε as notation for an object of $[1]$. By an n-dimensional cube in a category $\mathcal{C}$ we will mean a functor from $[1]^n$ to $\mathcal{C}$. An object C in $\mathcal{C}$ gives a 0-dimensional cube denoted by $[C]$, and an arrow $C \to C'$ in $\mathcal{C}$ gives a 1-dimensional cube denoted by $[C \to C']$. If the category $\mathcal{C}$ has products, we may define an external product of cubes as follows. Given an n-dimensional cube X and an n'-dimensional cube Y in $\mathcal{C}$, we let $X \boxtimes Y$ denote the $n + n'$-dimensional cube defined by $(X \boxtimes Y)(\varepsilon_1, \dots, \varepsilon_{n+n'}) = X(\varepsilon_l, \dots, \varepsilon_n) \times Y(\varepsilon_{n+1}, \dots, \varepsilon_{n+n'})$. Let $\mathbb{G}_m^{\wedge n}$ denote the external product of n copies of $[1 \to \mathbb{G}_m]$. For example, $\mathbb{G}_m^{\wedge 2}$ is the square of schemes

$$\begin{array}{ccc}
\operatorname{Spec} F & \longrightarrow & \mathbb{G}_m \\
\downarrow & & \downarrow \\
\mathbb{G}_m & \longrightarrow & \mathbb{G}_m \times \mathbb{G}_m.
\end{array}$$

Denote by $\mathcal{M}\langle \mathbb{G}_m^{\times n} \rangle$ the exact category where an object is a tuple $(P, \theta_1, \dots, \theta_n)$ consisting of an object P of $\mathcal{M}$ and commuting automorphisms $\theta_1, \dots, \theta_n \in Aut(P)$. Note that $\mathcal{M}\langle \mathbb{G}_m^{\times n} \rangle = (\mathcal{M}\langle \mathbb{G}_m^{\times (n-1)} \rangle)\langle \mathbb{G}_m \rangle$. For instance, if $\mathcal{M} = \mathcal{P}(U, X)$, $U, X \in Sm/F$, then we can identify $\mathcal{P}(U, X)\langle \mathbb{G}_m^{\times n} \rangle$ with $\mathcal{P}(U, X \times \mathbb{G}_m^{\times n})$. The cube of affine schemes $\mathbb{G}_m^{\wedge n}$ gives rise to a cube of exact categories $\mathcal{M}\langle \mathbb{G}_m^{\wedge n} \rangle$ with vertices being $\mathcal{M}\langle \mathbb{G}_m^{\times k} \rangle$, $0 \leqslant k \leqslant n$. The edges of the cube are given by the natural exact functors $i_s \colon \mathcal{M}\langle \mathbb{G}_m^{\times (k-1)} \rangle \to \mathcal{M}\langle \mathbb{G}_m^{\times k} \rangle$ defined as

$$(P, (\theta_1, \dots, \theta_{k-1})) \longmapsto (P, (\theta_1, \dots, 1, \dots, \theta_{k-1})),$$

where 1 is the sth coordinate.

In [**11**, §4] is presented a construction called C which can be applied to a cube of exact categories to convert it into a multisimplicial exact category, the K-theory of which serves as the iterated cofiber space of the corresponding cube of K-theory spaces/spectra.

Given an exact category $\mathcal{M}$ with a chosen zero object 0 and an ordered set A, we call a functor $F \colon \operatorname{Ar}(A) \to \mathcal{M}$ exact if $F(i, i) = 0$ for all i, and $0 \to F(i, j) \to F(i, k) \to F(j, k) \to 0$ is *exact* for all $i \leqslant j \leqslant k$. The set of such exact functors is denoted by $Exact(\operatorname{Ar}(A), \mathcal{M})$.

Now let L be a symbol, and consider $\{L\}$ to be an ordered set. Given an n-dimensional cube of exact categories $\mathcal{M}$, we define an n-fold multisimplicial exact category $C\mathcal{M}$ as a functor from $(Ord^n)^{\mathrm{op}}$ to the category of exact categories by letting $C\mathcal{M}(A_1, \dots, A_n)$ be the set

$$Exact([\operatorname{Ar}(A_1) \to \operatorname{Ar}(\{L\}A_1)] \boxtimes \cdots \boxtimes [\operatorname{Ar}(A_n) \to \operatorname{Ar}(\{L\}A_n)], \mathcal{M})$$

of multi-exact natural transformations. When $n = 0$, we may identify $C\mathcal{M}$ with $\mathcal{M}$. In the case $n = 1$, $C\mathcal{M}$ is the same as a construction of Waldhausen [**35**] denoted

$S.(\mathcal{M}_0 \to \mathcal{M}_1)$. We define $S.\mathcal{M}$ to be $S.C\mathcal{M}$, the result of applying the $S.$-construction of Waldhausen degreewise. The construction $S.\mathcal{M}$ is a $n+1$-fold multisimplicial set. K-theory of $C\mathcal{M}$ serves as the iterated cofiber space/spectrum of the corresponding cube of K-theory spaces/spectra.

Given an n-dimensional cube of additive categories $\mathcal{M}$, we define an n-fold multisimplicial additive category $C^{\oplus}\mathcal{M}$ as a functor from $(Ord^n)^{\rm op}$ to the category of exact categories by letting $C^{\oplus}\mathcal{M}(A_1,\dots,A_n)$ be the set

$$Add([Sub(A_1) \to Sub(\{L\}A_1)] \boxtimes \cdots \boxtimes [Sub(A_n) \to Sub(\{L\}A_n)], \mathcal{M})$$

of multi-additive natural transformations. When $n = 0$, we may identify $C^{\oplus}\mathcal{M}$ with $\mathcal{M}$. We define $S^{\oplus}\mathcal{M}$ to be $S^{\oplus}C^{\oplus}\mathcal{M}$, the result of applying the $S^{\oplus}$-construction of Grayson degreewise. It is an $n+1$-fold multisimplicial set (see [**12**] for details). Grayson's K-theory $K^{Gr}(C^{\oplus}\mathcal{M})$ of $C^{\oplus}\mathcal{M}$ (respectively, Waldhausen's K-theory $K(C\mathcal{M})$ of $C\mathcal{M}$) serves as the iterated cofiber space/spectrum of the corresponding cube of Grayson's (Waldhausen's) K-theory spaces/spectra. It is easy to see that

$$K_0^{Gr}(C^{\oplus}\mathcal{M}\langle\mathbb{G}_m^{\wedge n}\rangle) = K_0^{Gr}(\mathcal{M}\langle\mathbb{G}_m^{\times n}\rangle)/\sum_{k=1}^{n}(i_k)_*(K_0^{Gr}(\mathcal{M}\langle\mathbb{G}_m^{\times(n-1)}\rangle)).$$

If $\mathcal{M}$ is an additive category, Grayson showed [**12**] that the category $\omega S^{-1}S\mathcal{M}$ is equivalent to the category $\mathcal{C}$ where an object is any pair (P,β) with $P \in \mathcal{M}$ and $\beta \in Aut(P)$, and an arrow $(P,\beta) \to (Q,\gamma)$ is any split monomorphism $P \overset{\twoheadleftarrow}{\rightarrowtail} Q$ with respect to which one has the equation $\gamma = \beta \oplus 1$. The objects of $\mathcal{C}$ are themselves the objects of the exact category $\mathcal{M}\langle\mathbb{G}_m\rangle$, the arrows of $\mathcal{C}$ are also the objects of an exact category, and indeed, the nerve of $\mathcal{C}$ can be interpreted as a simplicial exact category.

There is a map $C^{\oplus}\mathcal{M}\langle\mathbb{G}_m^{\wedge 1}\rangle \to \mathcal{C}$ which amounts to forgetting some choices of cokernels, so the map is a homotopy equivalence. Thus we have a homotopy equivalence of symmetric K-theory spectra

$$v\colon K^{Gr}(C^{\oplus}\mathcal{M}\langle\mathbb{G}_m^{\wedge 1}\rangle) \xrightarrow{\sim} K^{Gr}(\omega S^{-1}S\mathcal{M}).$$

Consider the category of topological symmetric spectra $TopSp^{\Sigma}$ (see [**25**, section I.1]). We can apply adjoint functors "geometric realization", denoted by $|-|$, and "singular complex", denoted by $\mathcal{S}$, levelwise to go back and forth between simplicial and topological symmetric spectra

$$|-|\colon Sp^{\Sigma} \rightleftarrows TopSp^{\Sigma} : \mathcal{S}. \tag{15}$$

Remark 7.3. By the standard abuse of notation $|-|$ denotes both the functor from Sp^{Σ} to $TopSp^{\Sigma}$ and the realization functor from simplicial spectra to spectra. It will always be clear from the context which of either meanings is used.

We have a zig-zag of maps in $TopSp^{\Sigma}$ between semistable symmetric spectra

$$|K^{Gr}(C^{\oplus}\mathcal{M}\langle\mathbb{G}_m^{\wedge 1}\rangle)| \xrightarrow{|v|} |K^{Gr}(\omega S^{-1}S\mathcal{M})| \xrightarrow{w} \Omega|K^{Gr}(S^{-1}S\mathcal{M})| \xleftarrow{\Omega|u|}$$
$$\Omega|K^{Gr}(\omega S^{\oplus}\mathcal{M})| \xrightarrow{\sim} \Omega(\Omega|K^{Gr}(\mathcal{M})|[1])| \xleftarrow{\sim} \Omega|K^{Gr}(\mathcal{M})|.$$

All maps except w are weak equivalences of ordinary spectra. The zig-zag yields an arrow in $\mathrm{Ho}(Sp^\Sigma)$

$$\gamma\colon K^{Gr}(C^\oplus\mathcal{M}\langle\mathbb{G}_m^{\wedge 1}\rangle) \to \Omega K^{Gr}(\mathcal{M}). \tag{16}$$

Observe that the arrow is functorial in $\mathcal{M}$. If $\mathcal{M}$ is a simplicial R-linear additive category over a contractible simplicial ring, then

$$|d \mapsto K^{Gr}(C^\oplus\mathcal{M}_d\langle\mathbb{G}_m^{\wedge 1}\rangle)| \xrightarrow{\gamma} |d \mapsto \Omega K^{Gr}(\mathcal{M}_d)|$$

is an isomorphism in $\mathrm{Ho}(Sp^\Sigma)$ by Theorem 7.2. In fact, γ is an isomorphism in the homotopy category $\mathrm{Ho}(Sp)$ of ordinary spectra Sp, because all arrows in the zig-zag producing γ are weak equivalences in Sp.

Thus Theorem 7.1 implies that the following sequence is a triangle in $\mathrm{Ho}(Sp^\Sigma)$

$$|d \mapsto K^{Gr}(C^\oplus\mathcal{M}_d\langle\mathbb{G}_m^{\wedge 1}\rangle)| \longrightarrow \Omega|d \mapsto K^{Gr}(\mathcal{M}_d)| \longrightarrow \Omega|d \mapsto EM(K_0(\mathcal{M}_d))| \xrightarrow{+},$$

where $EM(K_0(\mathcal{M}_d))$ is the Eilenberg–Mac Lane spectrum of $\mathcal{M}_d$ (see Appendix). This triangle yields a triangle

$$S^1 \wedge |d \mapsto K^{Gr}(C^\oplus\mathcal{M}_d\langle\mathbb{G}_m^{\wedge 1}\rangle)| \longrightarrow |d \mapsto K^{Gr}(\mathcal{M}_d)| \longrightarrow |d \mapsto EM(K_0(\mathcal{M}_d))| \xrightarrow{+} .$$

We call it the *Grayson triangle*. It produces more generally triangles

$$S^1 \wedge |d \mapsto K^{Gr}(C^\oplus\mathcal{M}_d\langle\mathbb{G}_m^{\wedge n+1}\rangle)| \to |d \mapsto K^{Gr}(\mathcal{M}_d)\langle\mathbb{G}_m^{\wedge n}\rangle| \\ \to |d \mapsto EM(K_0(\mathcal{M}_d\langle\mathbb{G}_m^{\wedge n}\rangle))| \xrightarrow{+} .$$

In what follows we denote by $\mathcal{K}_0^\oplus$ and $\mathcal{K}_0$ the Eilenberg–Mac Lane spectral categories associated with ringoids $K_0^\oplus$ and K_0 on Sm/F. We refer the reader to Appendix to read about Eilenberg–Mac Lane spectral categories. There are canonical morphisms of spectral categories

$$\mathcal{O}_{K^\oplus} \to \mathcal{K}_0^\oplus \leftarrow \mathcal{O}_{K_0^\oplus}$$

and

$$\mathcal{O}_K \to \mathcal{K}_0 \leftarrow \mathcal{O}_{K_0},$$

where the arrows on the right are equivalences of spectral categories. It follows from [**27**, A.1.1] that restriction and extension of scalars functors induce spectral Quillen equivalences of modules

$$\mathrm{Mod}\,\mathcal{K}_0^\oplus \rightleftarrows \mathrm{Mod}\,\mathcal{O}_{K_0^\oplus}, \quad \mathrm{Mod}\,\mathcal{K}_0 \rightleftarrows \mathrm{Mod}\,\mathcal{O}_{K_0}.$$

Consider the case when each $\mathcal{M}_d = \mathcal{P}'(U \times \Delta^d, X)$, $U, X \in Sm/F$. Let $S_{K^{Gr}}(X)(n)$ (respectively, $S_{\mathcal{K}_0^\oplus}(X)(n)$) denote the presheaf

$$S^n \wedge |d, U \mapsto K^{Gr}(C^\oplus\mathcal{P}'(U \times \Delta^d, X)\langle\mathbb{G}_m^{\wedge n}\rangle)|$$

(respectively, the presheaf $S^n \wedge |d, U \mapsto EM(K_0^\oplus(C^\oplus\mathcal{P}'(U \times \Delta^d, X))\langle\mathbb{G}_m^{\wedge n}\rangle)|$). Note that $S_{\mathcal{K}_0^\oplus}(X)(n)$ is an $\mathcal{O}_{K^{Gr}}$-module by means of the natural morphism of spectral categories

$$\mathcal{O}_{K^{Gr}} \to \mathcal{K}_0^{\oplus}.$$

We have a canonical morphism of presheaves

$$g_q\colon S_{K^{Gr}}(X)(q) \to S_{\mathcal{K}_0^{\oplus}}(X)(q), \quad q \geqslant 0.$$

The Grayson triangle produces triangles in the homotopy category $\mathrm{Ho}(\mathrm{Mod}\,\mathcal{O}_{naive})$ of $\mathcal{O}_{naive}$-modules regarded as a model category with respect to the stable projective model structure

$$S_{K^{Gr}}(X)(q+1) \xrightarrow{f_{q+1}} S_{K^{Gr}}(X)(q) \xrightarrow{g_q} S_{\mathcal{K}_0^{\oplus}}(X)(q) \xrightarrow{+} .$$

Recall that $Sp_{nis}^{\mathbb{N}}(Pre(Sm/F))$ stands for the model category of S^1-spectra on pointed simplicial presheaves with respect to the Nisnevich local model structure. Below we shall need the following couple of lemmas.

Lemma 7.4 ([**7**])**.** *Consider a sequence of maps of spectra*

$$\cdots \xrightarrow{f_{q+2}} X_{q+1} \xrightarrow{f_{q+1}} X_q \xrightarrow{f_q} \cdots \xrightarrow{f_1} X_0 = X$$

Assume further that for each q we are given a map of spectra $p_q\colon X_q \to B_q$ such that the composition $X_{q+1} \to X_q \to B_q$ is trivial and the associated map from X_{q+1} to the homotopy fiber of $X_q \to B_q$ is a weak equivalence. Assume further that for each $i \geqslant 0$ there exists $n \geqslant 0$ such that X_q is i-connected for $q \geqslant n$. In this case there exists a strongly convergent spectral sequence

$$E^2_{pq} = \pi_{p+q}(B_q) \Longrightarrow \pi_{p+q}(X).$$

We shall write $[E, L]$ to denote $\mathrm{Ho}(Sp_{nis}^{\mathbb{N}}(Pre(Sm/F)))(E, L)$ for any two presheaves of spectra E, L.

Lemma 7.5. *Let E be a presheaf of spectra and $U \in Sm/F$ of Krull dimension d. Assume further that for each $i < q$ the i-th stable homotopy Nisnevich sheaf $\pi_i(E)$ of E is zero. Then for all $n < q - d$ one has:*

$$[U_+[n], E] = 0.$$

Proof. This follows from [**22**, 3.3.3]. □

Definition 7.6. (1) Given a smooth scheme X over F, the *Grayson tower* is the sequence of maps in $\mathrm{Ho}(\mathrm{Mod}\,\mathcal{O}_{naive})$:

$$\cdots \xrightarrow{f_{q+2}} S_{K^{Gr}}(X)(q+1) \xrightarrow{f_{q+1}} S_{K^{Gr}}(X)(q) \xrightarrow{f_q} \cdots \xrightarrow{f_1} S_{K^{Gr}}(X)(0).$$

By construction, the Grayson tower naturally produces a tower in $\mathrm{Ho}(Sp_{nis}^{\mathbb{N}}(Pre(Sm/F)))$.

(2) For any $U, X \in Sm/F$ the *bivariant K-theory groups* are defined as

$$K_i(U, X) = [U_+[i], S_{K^{Gr}}(X)(0)], \quad i \in \mathbb{Z}.$$

Lemma 7.7. *For any $U \in Sm/F$ and any integer i there is a natural isomorphism $K_i(U) \cong K_i(U, pt)$.*

Proof. This follows from Thomason's theorem [**31**] stating that algebraic K-theory satisfies Nisnevich descent and the fact that $K(U)$ is homotopy invariant. □

Denote by $Sp^{\mathbb{N}}_{nis,J}(Pre(Sm/F))$ ("J" for Jardine) the stable model category of presheaves of ordinary spectra corresponding to the Nisnevich local injective model structure on $Pre(Sm/F)$ (see [**19**]). Note that the identity functor

$$Sp^{\mathbb{N}}_{nis}(Pre(Sm/F)) \to Sp^{\mathbb{N}}_{nis,J}(Pre(Sm/F))$$

induces a left Quillen equivalence.

Proposition 7.8. *For any $U, X \in Sm/F$ and any p, q there is a natural isomorphism*

$$H^{p,q}(U, X, \mathbb{Z}) \cong [U_+, S_{\mathcal{K}_0^{\oplus}}(X)(q)[p-2q]].$$

Proof. Since the map of spectral categories $\mathcal{O}_{K_0^{\oplus}} \to \mathcal{K}_0^{\oplus}$ is a levelwise weak equivalence, it is enough to show the assertion for the $\mathcal{O}_{K_0^{\oplus}}$-module

$$S'_{K_0^{\oplus}}(X)(q) = S^q \wedge |d, U \mapsto H(K_0^{\oplus}(C^{\oplus}\mathcal{P}'(U \times \Delta^d, X)\langle \mathbb{G}_m^{\wedge q}\rangle))|.$$

Denote by $Sp^{\mathbb{N}}_{nis}(s\,\mathrm{Mod}_{\mathbb{Z}})$ the model category of simplicial $\mathbb{Z}$-module spectra in the sense of Jardine [**20**]. Every such spectrum consists of a sequence of simplicial presheaves of abelian groups A^n, $n \geqslant 0$, together with simplicial homomorphisms $A^n \otimes S^1 \to A^{n+1}$, which are also called bonding maps. The suspension spectrum $\Sigma^{\infty}_{S^1}\mathcal{X}$ in $Sp^{\mathbb{N}}_{nis}(s\,\mathrm{Mod}_{\mathbb{Z}})$ of a simplicial presheaf of abelian groups $\mathcal{X}$ is defined in the usual way. We set,

$$S''_{K_0^{\oplus}}(X)(q) := S^q \otimes |d, U \mapsto \Sigma^{\infty}_{S^1}(K_0^{\oplus}(C^{\oplus}\mathcal{P}'(U \times \Delta^d, X)\langle \mathbb{G}_m^{\wedge q}\rangle))|.$$

There is a natural forgetful functor

$$U\colon Sp^{\mathbb{N}}_{nis}(s\,\mathrm{Mod}_{\mathbb{Z}}) \to Sp^{\mathbb{N}}_{nis,J}(Pre(Sm/F)).$$

A map $f\colon A \to B$ of simplicial $\mathbb{Z}$-module spectra is a weak equivalence if the underlying map of presheaves of spectra $Uf\colon UA \to UB$ is a weak equivalence in $Sp^{\mathbb{N}}_{nis}(Pre(Sm/F))$. In fact, U is a right Quillen functor whose left adjoint is denoted by V. The spectra $U(S''_{K_0^{\oplus}}(X)(q))$ and $S'_{K_0^{\oplus}}(X)(q)$ are stably equivalent by [**20**, 5.5]. Therefore there is an isomorphism

$$\mathrm{Ho}(Sp^{\mathbb{N}}_{nis,J}(Pre(Sm/F)))(U_+, S'_{K_0^{\oplus}}(X)(q)) \cong \mathrm{Ho}(Sp^{\mathbb{N}}_{nis}(s\,\mathrm{Mod}_{\mathbb{Z}}))(U_+, S''_{K_0^{\oplus}}(X)(q)).$$

It follows from [**20**] that there are pairs of functors of triangulated categories

$$\mathrm{Ho}(Sp^{\mathbb{N}}_{nis,J}(Pre(Sm/F))) \underset{U}{\overset{V}{\rightleftarrows}} \mathrm{Ho}(Sp^{\mathbb{N}}_{nis}(s\,\mathrm{Mod}_{\mathbb{Z}})) \underset{\Gamma}{\overset{N}{\rightleftarrows}} D(Sh(Sm/F)).$$

Here $D(Sh(Sm/F))$ is the derived category of Nisnevich sheaves, N, Γ are mutually inverse equivalences, and V, N are left adjoint. Moreover, N takes the cofibrant presheaf of spectra $S''_{K_0^{\oplus}}(X)(q) \in Sp^{\mathbb{N}}_{nis}(s\,\mathrm{Mod}_{\mathbb{Z}})$ to $\mathbb{Z}_{K_0^{\oplus}}(X)(q)[2q]$. Our assertion now follows from the fact that $Sp^{\mathbb{N}}_{nis}(Pre(Sm/F)), Sp^{\mathbb{N}}_{nis,J}(Pre(Sm/F))$ are Quillen equivalent by means of the identity functor. □

We shall say that a presheaf of spectra E is *n-connected* if for all $k \leqslant n$ its k-th homotopy sheaf $\pi_k(E)$ vanishes. We are now in a position to prove the main result of this section.

Theorem 7.9. *For every $q \geqslant 0$ and every $X \in Sm/F$ the sequence of maps*

$$S_{K^{Gr}}(X)(q+1) \xrightarrow{f_{q+1}} S_{K^{Gr}}(X)(q) \xrightarrow{g_q} S_{\mathcal{K}_0^{\oplus}}(X)(q) \xrightarrow{+}$$

is a triangle in $\mathrm{Ho}(Sp^{\mathbb{N}}_{nis}(Pre(Sm/F)))$ *and the Grayson tower produces a strongly convergent spectral sequence which we call the* bivariant motivic spectral sequence,

$$E_2^{pq} = H^{p-q,-q}(U, X, \mathbb{Z}) \Longrightarrow K_{-p-q}(U, X), \quad U \in Sm/F.$$

*Moreover, if $X = pt$ and U is the spectrum of a smooth Henselian F-algebra then Grayson's spectral sequence [**12**] takes the form*

$$E_2^{pq} = H^{p-q,-q}_{\mathcal{M}}(U, \mathbb{Z}) \Longrightarrow K_{-p-q}(U).$$

Proof. The presheaf of spectra $S(X)(q)$, $q \geqslant 0$, is $(q-1)$-connected. Lemmas 7.4-7.5 and Proposition 7.8 imply that the Grayson tower produces a strongly convergent spectral sequence

$$E_2^{pq} = H^{p-q,-q}(U, X, \mathbb{Z}) \Longrightarrow K_{-p-q}(U, X).$$

If U is the spectrum of a smooth Henselian F-algebra then for every presheaf of spectra E one has:

$$[U_+[n], E] \cong \pi_n(E(U)), \quad n \in \mathbb{Z}.$$

For $X = pt$, evaluation of the Grayson tower at U is isomorphic in $\mathrm{Ho}(Sp)$ to the tower constructed by Walker [**36**]. The latter tower produces a spectral sequence which agrees with Grayson's spectral sequence [**12**]. □

We want to construct an $\mathbb{A}^1$-local counterpart for the bivariant motivic spectral sequence. We denote by

$$L_{\mathbb{A}^1} \colon Sp^{\mathbb{N}}_{nis}(Pre(Sm/F)) \to Sp^{\mathbb{N}}_{nis}(Pre(Sm/F))$$

the $\mathbb{A}^1$-localization functor of Morel [**22**]. For any presheaf E of spectra and any integer n, the sheaves

$$\pi_n^{\mathbb{A}^1}(E) := \pi_n(L_{\mathbb{A}^1}(E))$$

are strictly $\mathbb{A}^1$-invariant [**22**].

Definition 7.10. (1) Given $X \in Sm/F$ and $q \geqslant 0$, the *Grayson $\mathcal{O}_{K^{Gr}}$-module $G(X)(q)$ of weight q* is the $\mathcal{O}_{K^{Gr}}$-module $S^q \wedge K^{Gr}(C^{\oplus}\mathcal{P}'(-, X)\langle \mathbb{G}_m^{\wedge q}\rangle)$. We shall also write $G_0(X)(q)$ to denote the $\mathcal{K}_0^{\oplus}$-module $S^q \wedge EM(K_0^{\oplus}(C^{\oplus}\mathcal{P}'(-, X)\langle \mathbb{G}_m^{\wedge q}\rangle))$.

(2) For any $U, X \in Sm/F$ the *$\mathbb{A}^1$-local bivariant K-theory groups* are defined as

$$K_i^{\mathbb{A}^1}(U, X) := [U_+[i], L_{\mathbb{A}^1}(G(X)(0))], \quad i \in \mathbb{Z}.$$

(3) The *$\mathbb{A}^1$-local bivariant motivic cohomology groups* are defined as

$$H^{p,q}_{\mathbb{A}^1}(U, X, \mathbb{Z}) := [U_+, L_{\mathbb{A}^1}(G_0(X)(q))[p-2q]].$$

Lemma 7.11. *Let E be a connective presheaf of spectra and $U \in Sm/F$ of Krull dimension d. Then the group*

$$[U_+, L_{\mathbb{A}^1}(E)[n]]$$

vanishes for $n > d$.

Proof. This follows from [**22**, 4.3.1]. □

The map (16) yields a map in the stable homotopy category $\mathrm{Ho}(\mathrm{Mod}\,\mathcal{O}_{K^{Gr}})$ of $\mathcal{O}_{K^{Gr}}$-modules

$$f_q \colon G(X)(q) \to G(X)(q-1), \quad q > 0.$$

In turn, one has a natural map of $\mathcal{O}_{K^{Gr}}$-modules

$$g_q \colon G(X)(q) \to G_0(X)(q), \quad q \geqslant 0.$$

Definition 7.12. (1) Given a smooth scheme X over F, the *Grayson tower of Grayson's modules* is the sequence of maps in $\mathrm{Ho}(\mathrm{Mod}\,\mathcal{O}_{K^{Gr}})$:

$$\cdots \xrightarrow{f_{q+2}} G(X)(q+1) \xrightarrow{f_{q+1}} G(X)(q) \xrightarrow{f_q} \cdots \xrightarrow{f_1} G(X)(0).$$

(2) We say that a presheaf of spectra E is $\mathbb{A}^1$*-local* if for every scheme $U \in Sm/F$ and every integer n the natural map

$$[(U_+)[n], E] \to [((U \times \mathbb{A}^1)_+)[n], E]$$

is an isomorphism.

We denote by $Sp^{\mathbb{N}}_{mot}(Pre(Sm/F))$ the model category of S^1-spectra associated to the projective motivic model structure on $Pre(Sm/F)$.

Theorem 7.13. *For every $q \geqslant 0$ and every $X \in Sm/F$ the sequence of maps*

$$G(X)(q+1) \xrightarrow{f_{q+1}} G(X)(q) \xrightarrow{g_q} G_0(X)(q) \xrightarrow{+}$$

is a triangle in $\mathrm{Ho}(Sp^{\mathbb{N}}_{mot}(Pre(Sm/F)))$ *and the Grayson tower of Grayson's modules produces a strongly convergent spectral sequence which we shall call the* $\mathbb{A}^1$-local bivariant motivic spectral sequence,

$$E_2^{pq} = H^{p-q,-q}_{\mathbb{A}^1}(U, X, \mathbb{Z}) \Longrightarrow K^{\mathbb{A}^1}_{-p-q}(U, X), \quad U \in Sm/F.$$

Assume further that one of the following conditions is satisfied:

1. $X = pt$;

2. the field F is perfect.

Then $H^{p,q}_{\mathbb{A}^1}(U, X, \mathbb{Z})$ agree with bivariant motivic cohomology groups $H^{p,q}(U, X, \mathbb{Z})$ and the $\mathbb{A}^1$-local bivariant motivic spectral sequence coincides with the bivariant motivic spectral sequence of Theorem 7.9.

Proof. We have a commutative diagram in the stable homotopy category of presheaves of spectra

$$\begin{array}{ccccccc}
G(X)(q+1) & \xrightarrow{f_{q+1}} & G(X)(q) & \xrightarrow{g_q} & G_0(X)(q) & \xrightarrow{+} & \\
\downarrow & & \downarrow & & \downarrow & & \\
S_{K^{Gr}}(X)(q+1) & \xrightarrow{f_{q+1}} & S_{K^{Gr}}(X)(q) & \xrightarrow{g_q} & S_{\mathcal{K}_0^{\oplus}}(X)(q) & \xrightarrow{+} &
\end{array}$$

with vertical arrows level $\mathbb{A}^1$-equivalences. Since the lower sequence is a triangle in $\mathrm{Ho}(Sp_{mot}^{\mathbb{N}}(Pre(Sm/F)))$ then so is the upper one. The presheaf of spectra $G(X)(q)$, $q \geqslant 0$, is $(q-1)$-connected. Lemmas 7.4 and 7.11 imply that the Grayson tower of Grayson's modules produces a strongly convergent spectral sequence

$$E_2^{pq} = H_{\mathbb{A}^1}^{p-q,-q}(U, X, \mathbb{Z}) \Longrightarrow K_{-p-q}^{\mathbb{A}^1}(U, X).$$

Assume now that $X = pt$. By Corollaries 6.10-6.11 and Proposition 7.8 each $S_{\mathcal{K}_0^{\oplus}}(X)(q)$, $q \geqslant 0$, is $\mathbb{A}^1$-local. In turn, if F is perfect but X is any smooth scheme then Theorem 6.12 and Proposition 7.8 imply each $S_{\mathcal{K}_0^{\oplus}}(X)(q)$, $q \geqslant 0$, is $\mathbb{A}^1$-local. In both cases therefore $H_{\mathbb{A}^1}^{p,q}(U, X, \mathbb{Z})$ agree with bivariant motivic cohomology groups $H^{p,q}(U, X, \mathbb{Z})$. The fact that the $\mathbb{A}^1$-local bivariant motivic spectral sequence coincides with the bivariant motivic spectral sequence of Theorem 7.9 is now obvious. □

Corollary 7.14. *There is a natural isomorphism of abelian groups*

$$K_i^{\mathbb{A}^1}(U, pt) \cong K_i(U)$$

for any $i \in \mathbb{Z}$. *If* F *is a perfect field, then there is also a natural isomorphism*

$$K_i^{\mathbb{A}^1}(U, X) \cong K_i(U, X)$$

for any $i \in \mathbb{Z}$ *and* $U, X \in Sm/F$.

Proof. This follows from Lemma 7.7 and the preceding theorem. □

We conclude the section by noting that the presheaves of K-groups

$$K_i^{Gr}(\mathcal{P}'(-, Y)) = \pi_i(\mathcal{O}_{K^{Gr}}(-, Y))$$

are different from both $K_i(-, Y)$ and $K_i^{\mathbb{A}^1}(-, Y)$ in general. Indeed, suppose F is perfect. Then the preceding corollary implies the presheaves $K_i(-, Y)$ and $K_i^{\mathbb{A}^1}(-, Y)$ are isomorphic. Let $SH\mathcal{O}_{K^{Gr}}$ be the homotopy category of $\mathcal{O}_{K^{Gr}}$-modules with respect to the stable projective model structure (see Theorem 4.2). It is a compactly generated triangulated category. If the presheaves $K_i^{Gr}(\mathcal{P}'(-, Y))$ were isomorphic to $K_i^{\mathbb{A}^1}(-, Y)$ then we would have that the natural map

$$SH\mathcal{O}_{K^{Gr}}(\mathcal{O}_{K^{Gr}}(-, X), \mathcal{O}_{K^{Gr}}(-, Y)[n]) \\ \to SH\mathcal{O}_{K^{Gr}}(\mathcal{O}_{K^{Gr}}(-, X \times \mathbb{A}^1), \mathcal{O}_{K^{Gr}}(-, Y)[n])$$

is an isomorphism for every $X, Y \in Sm/k$ and $n \in \mathbb{Z}$. Since

$$\{\mathcal{O}_{K^{Gr}}(-, Y)[n]\}_{n \in \mathbb{Z}, Y \in Sm/k}$$

is a family of compact generators for $SH\mathcal{O}_{K^{Gr}}$, it would follow that the natural map

$$\mathcal{O}_{K^{Gr}}(-, X \times \mathbb{A}^1) \to \mathcal{O}_{K^{Gr}}(-, X)$$

is an isomorphism in $SH\mathcal{O}_{K^{Gr}}$ what is not the case.

8. K-motives

We use the $\mathbb{A}^1$-local bivariant motivic spectral sequence to prove the following

Theorem 8.1. *The spectral categories $\mathcal{O}_{K^{Gr}}, \mathcal{O}_{K^{\oplus}}, \mathcal{O}_K$ are motivically excisive.*

Proof. The spectral categories $\mathcal{O}_{K^{Gr}}$, $\mathcal{O}_{K^{\oplus}}$, $\mathcal{O}_K$ are Nisnevich excisive by Theorem 5.9. We first check that the natural map

$$\mathcal{O}_{K^{Gr}}(-, X \times \mathbb{A}^1) \to \mathcal{O}_{K^{Gr}}(-, X), \quad X \in Sm/F,$$

is a motivic weak equivalence in $Pre^{\Sigma}(Sm/F)$. By [**18**, 4.34] it is enough to show that the map is a motivic weak equivalence of presheaves of ordinary spectra. Since $\mathcal{O}_{K_0^{\oplus}}$ is motivically excisive, then so is $\mathcal{K}_0^{\oplus}$ because these are equivalent spectral categories. In fact, the natural map

$$\mathcal{K}_0^{\oplus}(-, X \times \mathbb{A}^1) \to \mathcal{K}_0^{\oplus}(-, X), \quad X \in Sm/F,$$

is a level motivic weak equivalence of presheaves of ordinary spectra.

Observe that the exact category $\mathcal{P}(U, X)\langle \mathbb{G}_m^{\times q}\rangle$, $U, X \in Sm/F$, can be identified with $\mathcal{P}(U, X \times \mathbb{G}_m^{\times q})$. It follows that the map

$$\mathcal{K}_0^{\oplus}(-, X \times \mathbb{A}^1 \times \mathbb{G}_m^{\times q}) \to \mathcal{K}_0^{\oplus}(-, X \times \mathbb{G}_m^{\times q})$$

is a motivic weak equivalence of presheaves of ordinary spectra, and hence so is the map

$$G_0(X \times \mathbb{A}^1)(q) \to G_0(X)(q).$$

It induces an isomorphism of $\mathbb{A}^1$-local bivariant motivic cohomology groups

$$H^{p,q}_{\mathbb{A}^1}(U, X \times \mathbb{A}^1, \mathbb{Z}) \xrightarrow{\cong} H^{p,q}_{\mathbb{A}^1}(U, X, \mathbb{Z}).$$

We infer that the natural map of Grayson towers

$$\begin{array}{ccccccc}
\cdots \xrightarrow{f_{q+1}} & G(X \times \mathbb{A}^1)(q) & \xrightarrow{f_q} & G(X \times \mathbb{A}^1)(q-1) & \xrightarrow{f_{q-1}} \cdots \\
& \downarrow & & \downarrow & \\
\cdots \xrightarrow{f_{q+1}} & G(X)(q) & \xrightarrow{f_q} & G(X)(q-1) & \xrightarrow{f_{q-1}} \cdots
\end{array}$$

produces an isomorphism of $\mathbb{A}^1$-local bivariant motivic spectral sequences, and hence each map

$$K_i^{\mathbb{A}^1}(U, X \times \mathbb{A}^1) \to K_i^{\mathbb{A}^1}(U, X)$$

is an isomorphism. We see that

$$L_{\mathbb{A}^1}(G(X \times \mathbb{A}^1)(0)) \to L_{\mathbb{A}^1}(G(X)(0))$$

is a motivic weak equivalence, and hence so is

$$\mathcal{O}_{K^{Gr}}(-, X \times \mathbb{A}^1) = G(X \times \mathbb{A}^1)(0) \to \mathcal{O}_{K^{Gr}}(-, X) = G(X)(0).$$

So $\mathcal{O}_{K^{Gr}}$ is motivically excisive. Since $\mathcal{O}_{K^{Gr}}$ and $\mathcal{O}_{K^\oplus}$ are equivalent spectral categories, then also $\mathcal{O}_{K^\oplus}$ is motivically excisive.

It follows from [**12**, 10.5] that the natural map of spectra

$$|n \mapsto K^{\oplus}(\mathcal{P}'(X \times \Delta^n, Y))| \to |n \mapsto K(\mathcal{P}'(X \times \Delta^n, Y))|$$

is a stable equivalence of spectra. Therefore,

$$|n \mapsto \mathcal{O}_{K^\oplus}(X \times \Delta^n, Y)| \to |n \mapsto \mathcal{O}_K(X \times \Delta^n, Y)|$$

is a stable equivalence of spectra. For any presheaf $\mathcal{F}\colon Sm/F \to SSets$, the natural map $\mathcal{F} \to Sing(\mathcal{F})$ is a motivic equivalence. Here $Sing\colon Pre(Sm/F) \to Pre(Sm/F)$ is the singular functor (see, e.g., [**18**, p. 542]). We conclude that for any $X \in Sm/F$ the map of presheaves of spectra

$$\mathcal{O}_{K^\oplus}(-, X) \to \mathcal{O}_K(-, X)$$

is a motivic equivalence, and therefore it is a motivic equivalence of presheaves of symmetric spectra by [**18**, 4.34]. Since $\mathcal{O}_{K^\oplus}$ is motivically excisive, then so is $\mathcal{O}_K$. □

We say that a presheaf of symmetric spectra $E \in Pre^{\Sigma}(Sm/F)$ is *semistable* if $E(U)$ is a semistable symmetric spectrum for every $U \in Sm/F$. We remark that all presheaves of symmetric spectra we work with in practice such as $\mathcal{O}_{K^{Gr}}(-, X)$, $\mathcal{O}_{K^\oplus}(-, X)$, $\mathcal{O}_K(-, X)$, $\mathcal{O}_{K_0^\oplus}(-, X)$, $\mathcal{O}_{K_0}(-, X)$, $\mathcal{K}_0^\oplus(-, X)$, $\mathcal{K}_0(-, X)$ are semistable.

Lemma 8.2. *Let E be a semistable presheaf of symmetric spectra and $U \in Sm/F$. Then there are natural isomorphisms*

$$[U_+[n], E] \cong SH^{\mathrm{nis}}(F)(U_+[n], E), \quad [U_+[n], L_{\mathbb{A}^1}(E)] \cong SH^{\mathrm{mot}}(F)(U_+[n], E)$$

for all integers n.

Proof. Straightforward. □

Definition 8.3. (1) The $K^\oplus$*-motive* $M_{K^\oplus}(X)$ of a smooth scheme X over F (respectively, the K-*,* K_{Gr}-*,* $\mathcal{K}_0^\oplus$-*,* $\mathcal{K}_0$*-motives* $M_K(X)$, $M_{K_{Gr}}(X)$, $M_{\mathcal{K}_0^\oplus}(X)$, $M_{\mathcal{K}_0}(X)$) is the image of $\mathcal{O}_{K^\oplus}(-, X)$ in $SH^{\mathrm{mot}}\mathcal{O}_{K^\oplus}$ (respectively, the corresponding images of $\mathcal{O}_K(-, X)$, $\mathcal{O}_{K_{Gr}}(-, X)$, $\mathcal{O}_{\mathcal{K}_0^\oplus}(-, X)$, $\mathcal{O}_{\mathcal{K}_0}(-, X)$ in $SH^{\mathrm{mot}}\mathcal{O}_K$, $SH^{\mathrm{mot}}\mathcal{O}_{K_{Gr}}$, $SH^{\mathrm{mot}}\mathcal{O}_{\mathcal{K}_0^\oplus}$, $SH^{\mathrm{mot}}\mathcal{O}_{\mathcal{K}_0}$).

(2) The image of each Grayson module $G(X)(q)$ (respectively, $G_0(X)(q)$), $q \geqslant 0$, in $SH^{\mathrm{mot}}\mathcal{O}_{K^{Gr}}$ (respectively, in $SH^{\mathrm{mot}}\mathcal{K}_0^\oplus$) will be denoted by $M_{K^{Gr}}(X)(q)$ (respectively, $M_{\mathcal{K}_0^\oplus}(X)(q)$).

Lemma 8.4. *For any $X, Y \in Sm/F$ there are canonical isomorphisms:*

$$K_i^{\mathbb{A}^1}(X,Y) \cong SH^{\mathrm{mot}}\mathcal{O}_{K^{Gr}}(M_{K^{Gr}}(X)[i], M_{K^{Gr}}(Y)) \cong$$
$$SH^{\mathrm{mot}}\mathcal{O}_{K^{\oplus}}(M_{K^{\oplus}}(X)[i], M_{K^{\oplus}}(Y)) \cong SH^{\mathrm{mot}}\mathcal{O}_{K}(M_{K}(X)[i], M_{K}(Y))$$

and

$$H_{\mathbb{A}^1}^{p,q}(X,Y,\mathbb{Z}) \cong SH^{\mathrm{mot}}\mathcal{O}_{K^{Gr}}(M_{K^{Gr}}(X), M_{\mathcal{K}_0^{\oplus}}(Y)(q)[p-2q]) \cong$$
$$SH^{\mathrm{mot}}\mathcal{O}_{K^{\oplus}}(M_{K^{\oplus}}(X), M_{\mathcal{K}_0^{\oplus}}(Y)(q)[p-2q]).$$

Proof. All presheaves of the statement are semistable. The proof of Theorem 8.1 shows that the natural maps

$$M_{K^{Gr}}(X) \to M_{K^{\oplus}}(X) \to M_K(X)$$

are isomorphisms in $SH^{\mathrm{mot}}(F)$. Now Corollary 5.13 and Lemma 8.2 imply the claim. □

Corollary 8.5. *There is a natural isomorphism*

$$K_i(X) \cong SH^{\mathrm{mot}}\mathcal{O}_{K^{Gr}}(M_{K^{Gr}}(X)[i], M_{K^{Gr}}(pt)) \cong$$
$$SH^{\mathrm{mot}}\mathcal{O}_{K^{\oplus}}(M_{K^{\oplus}}(X)[i], M_{K^{\oplus}}(pt)) \cong SH^{\mathrm{mot}}\mathcal{O}_{K}(M_{K}(X)[i], M_{K}(pt))$$

for any $i \in \mathbb{Z}$ and $X \in Sm/F$.

Proof. This follows from Corollary 7.14 and Lemma 8.4. □

Corollary 8.6. *The maps of spectral categories*

$$\mathcal{O}_{K^{Gr}} \to \mathcal{O}_{K^{\oplus}} \to \mathcal{O}_K$$

induce triangulated equivalences

$$SH^{\mathrm{mot}}\mathcal{O}_{K^{Gr}} \to SH^{\mathrm{mot}}\mathcal{O}_{K^{\oplus}} \to SH^{\mathrm{mot}}\mathcal{O}_K$$

of compactly generated triangulated categories.

Proof. The objects $\{M_{K^{Gr}}(X)[i]\}_{i\in\mathbb{Z}, X\in Sm/F}$ (respectively, $\{M_{K^{\oplus}}(X)[i]\}_{i\in\mathbb{Z}, X\in Sm/F}$ and $\{M_K(X)[i]\}_{i\in\mathbb{Z}, X\in Sm/F}$) are compact generators of the compactly generated triangulated category $SH^{\mathrm{mot}}\mathcal{O}_{K^{Gr}}$ (respectively, $SH^{\mathrm{mot}}\mathcal{O}_{K^{\oplus}}$ and $SH^{\mathrm{mot}}\mathcal{O}_K$). Both functors take the compact generators to compact generators and induce isomorphisms of Hom-sets between them by Lemma 8.4. Now our assertion follows from standard facts about compactly generated triangulated categories. □

We now have all the necessary information to prove the following result saying that the Grayson (bivariant) motivic spectral sequence is realized in a natural way in the triangulated category of K^{Gr}-motives.

Theorem 8.7. *For every $q \geqslant 0$ and every $X \in Sm/F$ the sequence of maps*

$$M_{K^{Gr}}(X)(q+1) \xrightarrow{f_{q+1}} M_{K^{Gr}}(X)(q) \xrightarrow{g_q} M_{\mathcal{K}_0^{\oplus}}(q) \xrightarrow{+}$$

is a triangle in $SH^{\mathrm{mot}}\mathcal{O}_{K^{Gr}}$ and the Grayson tower in $SH^{\mathrm{mot}}\mathcal{O}_{K^{Gr}}$

$$\cdots \xrightarrow{f_{q+2}} M_{K^{Gr}}(X)(q+1) \xrightarrow{f_{q+1}} M_{K^{Gr}}(X)(q) \xrightarrow{f_q} \cdots \xrightarrow{f_1} M_{K^{Gr}}(X)(0)$$

produces a strongly convergent spectral sequence

$$E^2_{pq} = SH^{\mathrm{mot}}\mathcal{O}_{K^{Gr}}(M_{K^{Gr}}(U)[p+q], M_{\mathcal{K}_0^{\oplus}}(X)(q)) \Rightarrow SH^{\mathrm{mot}}\mathcal{O}_{K^{Gr}}(M_{K^{Gr}}(U)[p+q], M_{K^{Gr}}(X)).$$

It agrees with the $\mathbb{A}^1$-local bivariant motivic spectral sequence of Theorem 7.13. Assume further that one of the following conditions is satisfied:

1. *$X = pt$;*
2. *the field F is perfect.*

Then this spectral sequence agrees with the bivariant motivic spectral sequence of Theorem 7.9.

Proof. This follows from Theorem 7.13 and Lemma 8.4. □

9. Concluding remarks

The interested reader may have observed that the authors have not considered monoidal structures on the category of $\mathcal{O}_{K^{\oplus}}$-modules. We believe that there should exist new transfers Cor_{virt} on Sm/F which produce a spectral category $\mathcal{O}_{K^{new}}$ such that:

- ⋄ $\mathcal{O}_{K^{new}}$ is symmetric monoidal and motivically excisive;
- ⋄ for any $X \in Sm/F$, $\mathcal{O}_{K^{new}}(-, X)$ is a sheaf of symmetric spectra;
- ⋄ the motivic model category of (pre-)sheaves of symmetric spectra which are also $\mathcal{O}_{K^{new}}$-modules is zig-zag Quillen equivalent to the motivic model category for $\mathcal{O}_{K^{\oplus}}$-modules.

The use of motivically excisive spectral categories on Sm/F and their modules is a reminiscence of the theory of sheaves of $\mathcal{O}_X$-modules over a ringed space $(X, \mathcal{O}_X)$. The structure sheaf $\mathcal{O}_X$ is replaced with a motivically excisive spectral category $\mathcal{O}$ on Sm/F and sheaves of $\mathcal{O}_X$-modules are replaced with the motivic model category of (pre-)sheaves of symmetric spectra which are also $\mathcal{O}$-modules.

From this point of view the theory of spectral categories over Sm/F and their modules is a sort of "motivic brave new algebra", where the base symmetric monoidal model category is $Pre^{\Sigma}(Sm/F)$.

Appendix A. Eilenberg–Mac Lane spectral categories

In this section we construct the Eilenberg–Mac Lane spectral categories $EM(\mathcal{A})$ associated with ringoids $\mathcal{A}$ which are equivalent to $H\mathcal{A}$. Although the authors have not found such constructions in the literature, they do not have pretensions to originality.

Let $(A,+)$ be an abelian monoid with neutral element 0 and let $\mathrm{Ar}[n]$ be the category of arrows for the poset $\{0<1<\cdots<n\}$. It can be regarded as the partially ordered set of pairs (i,j), $0\leqslant i\leqslant j\leqslant n$, where $(i,j)\leqslant(i',j')$ if and only if $i\leqslant i'$ and $j\leqslant j'$.

We consider the set of functions

$$\begin{aligned} a\colon \operatorname{Ob}\mathrm{Ar}[n] &\longrightarrow A\\ (i,j) &\longmapsto a(i,j)=a_{i,j}\end{aligned}$$

having the property that for every j, $a_{j,j}=0$ and $a_{i,k}=a_{i,j}+a_{j,k}$ whenever $i\leqslant j\leqslant k$. Let us denote it by σ_nA.

Given a map $u\colon [m]\longrightarrow[n]$ in Δ, the function $u^*\colon \sigma_nA\longrightarrow\sigma_mA$ sends the element $(i,j)\mapsto a(i,j)$ to the element $(r,s)\longmapsto a(u(r),u(s))$.

The elements of σ_nA may also be regarded as diagrams of the form

$$\begin{array}{ccccc} & & & & a_{n-1,n}\\ & & & & \vdots\\ & & a_{23} & \cdots & a_{2n}\\ & a_{12} & a_{13} & \cdots & a_{1n}\\ a_{01} & a_{02} & a_{03} & \cdots & a_{0n}\end{array} \tag{17}$$

Then the degeneracy maps are defined as the functions $s_i\colon\sigma_nA\to\sigma_{n+1}A$ by duplicating $a_{0,i}$, and reindexing with the normalization $a_{i,i+1}=0$.

Also, the face map $d_0\colon\sigma_nA\to\sigma_{n-1}A$ is the function which is defined by deleting the bottom row of (17). For $0<i\leqslant n$ we define the face maps as the functions $d_i\colon\sigma_nA\to\sigma_{n-1}A$ by omitting the row $a_{i,*}$ and the column containing a_{0i} in (17), and reindexing the $a_{j,k}$ as needed.

Given two functions $a,b\colon \operatorname{Ob}\mathrm{Ar}[n]\to A$, we define a binary operation on σ_nA as $(a+b)_{i,j}:=a_{i,j}+b_{i,j}$. Then the set σ_nA is an abelian monoid as well. So we arrive at a simplicial abelian monoid $n\mapsto\sigma_nA$ denoted by $\sigma.A$. We can iterate the σ-construction to get a bisimplicial abelian monoid $\sigma^2A=\sigma.\sigma.A$ or, more generally, a multisimplicial abelian monoid σ^nA.

Similar to Waldhausen's S-construction [**35**] there is a natural inclusion $A\wedge S^1\to\sigma.A$, and by adjointness therefore an inclusion of $|A|$ into the loop space of $|\sigma.A|$. There results a spectrum

$$EM(A):=(A,\sigma.A,\sigma.\sigma.A,\ldots)$$

whose structural maps are defined just as the map $|A|\to\Omega|\sigma.A|$ above. One can actually define $EM(A)$ as a symmetric spectrum in a similar way that in [**8**].

Let B,C be two other abelian monoids. A map $f\colon A\times B\to C$ is called a *bilinear pairing* if $f(a+a',b)=f(a,b)+f(a',b)$ and $f(a,b+b')=f(a,b)+f(a,b')$ for all $a,a'\in A$ and $b,b'\in B$. In particular, $f(a,0)=f(0,b)=0$. Every bilinear map induces a map of symmetric spectra

$$f\colon EM(A)\wedge EM(B)\to EM(C).$$

These maps are associative for strictly associative bilinear pairings.

The universal example of an abelian monoid which acts on any other such monoid is the monoid of non-negative integers $\mathbb{Z}_{\geqslant 0}$. Given an abelian monoid A, there is a bilinear pairing $\mathbb{Z}_{\geqslant 0} \times A \to A$ sending (n, a) to na.

Recall that a *semiring* is a set A equipped with two binary operations $+$ and $*$, called addition and multiplication, such that $(A, +)$ is an abelian monoid with identity element 0, $(A, *)$ is a monoid with identity element 1, multiplication distributes over addition, and 0 annihilates A with respect to multiplication. It follows from above that $EM(A)$ is a symmetric ring spectrum for every semiring A such that the structure map from the sphere spectrum to $EM(A)$ is given by the map $S^0 \to EM(A)_0 = A$ sending the basepoint to 0 and the non-basepoint to 1.

The universal example of a semiring which is mapped to any other semiring is $\mathbb{Z}_{\geqslant 0}$. One easily sees that for every abelian monoid A the symmetric spectrum $EM(A)$ is an $EM(\mathbb{Z}_{\geqslant 0})$-module. If A is a semiring then there is a natural map of ring spectra

$$EM(\mathbb{Z}_{\geqslant 0}) \to EM(A).$$

More generally, any *semiringoid* $\mathcal{A}$, that is a category whose Hom-sets are abelian monoids with bilinear composition and whose End-sets are semirings gives rise to a spectral category $EM(\mathcal{A})$ with $EM(\mathcal{A})(x, y) = EM(\operatorname{Hom}_{\mathcal{A}}(x, y))$ for all $x, y \in \operatorname{Ob} \mathcal{A}$.

Let $i\mathcal{C}$ be the Waldhausen category of isomorphisms for a Waldhausen category $\mathcal{C}$ in which all cofibrations split. For instance, $\mathcal{C}$ is an additive category or the category Γ of finite pointed sets $n^+ = \{0, 1, \dots, n\}$ with 0 as basepoint and pointed set maps. Denote by $\pi_0\mathcal{C}$ the abelian monoid of isomorphism classes for objects in $\mathcal{C}$ (e.g., $\pi_0\Gamma = \mathbb{Z}_{\geqslant 0}$). Given two classes $[A], [B] \in \pi_0\mathcal{C}$, the binary operation is defined as usual $[A] + [B] := [A \coprod B]$. There is a natural map of spectra

$$\tau\colon K(\mathcal{C}) \to EM(\pi_0\mathcal{C})$$

sending each diagram $(F\colon \operatorname{Ar}[n_1] \times \cdots \times \operatorname{Ar}[n_d] \to \mathcal{C}) \in S^d\mathcal{C}$ to the composition $(\operatorname{Ob}\operatorname{Ar}[n_1] \times \cdots \times \operatorname{Ob}\operatorname{Ar}[n_d] \xrightarrow{F} \operatorname{Ob}\mathcal{C} \to \pi_0\mathcal{C}) \in \sigma^d(\pi_0\mathcal{C})$.

Let $i\mathcal{A}, i\mathcal{B}$ be other two Waldhausen categories of isomorphisms for Waldhausen categories $\mathcal{A}, \mathcal{B}$ in which all cofibrations split. Suppose $f\colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is a biexact functor between Waldhausen categories. Then the following diagram of maps of symmetric spectra commutes

$$\begin{array}{ccc} K(\mathcal{A}) \wedge K(\mathcal{B}) & \xrightarrow{f} & K(\mathcal{C}) \\ \downarrow{\scriptstyle \tau\wedge\tau} & & \downarrow{\scriptstyle \tau} \\ EM(\pi_0\mathcal{A}) \wedge EM(\pi_0\mathcal{B}) & \xrightarrow{f} & EM(\pi_0\mathcal{C}). \end{array}$$

Let $\mathcal{O}$ be a spectral category generated by Waldhausen categories of isomorphisms $i\mathcal{C}(x, y)$, where $x, y \in \operatorname{Ob}\mathcal{O}$, in which all cofibrations split (i.e., $\mathcal{O}(x, y) = K(\mathcal{C}(x, y))$), and such that the composition law

$$\mathcal{O}(y, z) \wedge \mathcal{O}(x, y) \to \mathcal{O}(x, z)$$

comes from biexact functors $\mathcal{C}(y, z) \times \mathcal{C}(x, y) \to \mathcal{C}(x, z)$. For example, $\mathcal{O} = \mathcal{O}_{K^\oplus}$. Then the collections $\pi_0(\mathcal{C}(x, y))$, $x, y \in \operatorname{Ob}\mathcal{O}$, are abelian monoids and form a semiringoid. It gives rise to a spectral category, denoted by $EM(\pi_0\mathcal{O})$. There is a map of spectral categories

$$\upsilon\colon \mathcal{O} \to EM(\pi_0\mathcal{O}).$$

The additivity theorem for the σ-construction says that the natural map

$$\sigma.(\sigma_2 A) \to \sigma.A \times \sigma.A$$

sending $a\colon \operatorname{Ob}\operatorname{Ar}[2] \to A$ to the couple $(a_{0,1}, a_{1,2})$ is an isomorphism of simplicial sets for every abelian monoid A. Repeating now Waldhausen's results [**35**, section 1.5] for the σ.-construction, we get that the natural map $|\sigma.A| \to \Omega|\sigma.\sigma.A|$ which is adjoint to the map $|\sigma.A| \wedge S^1 \to |\sigma.\sigma.A|$ is a homotopy equivalence and more generally also the map $|\sigma^n A| \to \Omega|\sigma^{n+1}A|$ for every $n > 0$. This proves that the spectrum $EM(A)$ is an Ω-spectrum beyond the first term.

There is an isomorphism of simplicial sets $\sigma.A \cong BA$, where BA stands for the classifying space of A. It takes every diagram (17) to $(a_{0,1}, a_{1,2}, \ldots, a_{n-1,n}) \in BA$. We conclude that if A is an abelian group, then each space $|\sigma^n A|$, $n \geqslant 0$, has the homotopy type of the Eilenberg–Mac Lane space $K(A, n)$. Moreover, $EM(A)$ is a genuine Ω-spectrum.

Consider a spectral category $\mathcal{O}$ generated by Waldhausen categories (see above). Then the collections $K_0(\mathcal{C}(x, y)) = \pi_0(\mathcal{O}(x, y))$, $x, y \in \operatorname{Ob}\mathcal{O}$, are abelian groups and form a ringoid. It gives rise to a spectral category, denoted by $K_0\mathcal{O}$. There are maps of spectral categories

$$\mathcal{O} \xrightarrow{\upsilon} EM(\pi_0\mathcal{O}) \xrightarrow{\varkappa} K_0\mathcal{O},$$

where $\varkappa$ is induced by the universal group completion maps

$$\pi_0(|i\mathcal{O}(x, y)|) \to K_0(\mathcal{C}(x, y)).$$

Consider the symmetric spectrum HA associated with an abelian group A. Recall that $HA_p = A \otimes \widetilde{\mathbb{Z}}[S^p]$ for any $p \geqslant 0$. The identity map of A induces a map of symmetric spectra

$$l\colon \Sigma^\infty A = (A, A \wedge S^1, A \wedge S^2, \ldots) \to EM(A).$$

The maps $l_p\colon A \wedge S^p \to |\sigma^p A|$ induce in a unique way maps $\ell_p\colon A \otimes \widetilde{\mathbb{Z}}[S^p] \to |\sigma^p A|$. These yield a map of symmetric spectra

$$\ell_A\colon HA \to EM(A),$$

functorial in A. Recall that HA_p has homotopy type of the Eilenberg–Mac Lane space $K(A, p)$. We deduce that each ℓ_p is a weak equivalence, and hence ℓ is a levelwise weak equivalence.

For two abelian groups A and B, there is a natural morphism of symmetric spectra (see, e.g., [**25**, Example I.3.11])

$$HA \wedge HB \to H(A \otimes B).$$

It is easily verified that the diagram

$$\begin{array}{ccc} HA \wedge HB & \longrightarrow & H(A \otimes B) \\ {\scriptstyle \ell_A \wedge \ell_B}\downarrow & & \downarrow{\scriptstyle \ell_{A\otimes B}} \\ EM(A) \wedge EM(B) & \longrightarrow & EM(A \otimes B). \end{array}$$

is commutative.

Now let $\mathcal{A}$ be a ringoid and let $H\mathcal{A}$ be the spectral category associated with it. Recall that $H\mathcal{A}(x,y)_p = \mathcal{A}(x,y) \otimes \widetilde{\mathbb{Z}}[S^p]$ for any objects $x, y \in \operatorname{Ob}\mathcal{A}$ and $p \geqslant 0$. It follows from above that there is a map of spectral categories

$$\ell \colon H\mathcal{A} \to EM(\mathcal{A})$$

such that $H\mathcal{A}(x,y) \to EM(\mathcal{A})(x,y)$ is a levelwise equivalence of symmetric spectra for any $x, y \in \operatorname{Ob}\mathcal{A}$.

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