J. K-Theory **14** (2014), 103–137
doi:10.1017/is014004013jkt263

The triangulated category of K-motives $DK_{-}^{\mathrm{eff}}(k)$

by

GRIGORY GARKUSHA AND IVAN PANIN*

Abstract

For any perfect field k a triangulated category of K-motives $DK_{-}^{\mathrm{eff}}(k)$ is constructed in the style of Voevodsky's construction of the category $DM_{-}^{\mathrm{eff}}(k)$. To each smooth k-variety X the K-motive $M_{\mathbb{K}}(X)$ is associated in the category $DK_{-}^{\mathrm{eff}}(k)$ and

$$K_n(X) = \mathrm{Hom}_{DK_{-}^{\mathrm{eff}}(k)}(M_{\mathbb{K}}(X)[n], M_{\mathbb{K}}(pt)), \quad n \in \mathbb{Z},$$

where $pt = \mathrm{Spec}(k)$ and $K(X)$ is Quillen's K-theory of X.

Key Words: Motivic homotopy theory, algebraic K-theory, spectral categories.
Mathematics Subject Classification 2010: 14F42, 19E08, 55U35.

1. Introduction

The Voevodsky triangulated category of motives $DM_{-}^{\mathrm{eff}}(k)$ [16] provides a natural framework to study motivic cohomology. In [2] the authors constructed a triangulated category of K-motives providing a natural framework for Grayson's motivic spectral sequence [5]

$$E_2^{pq} = H_{\mathcal{M}}^{p-q,-q}(X,\mathbb{Z}) \Longrightarrow K_{-p-q}(X)$$

that relates the motivic cohomology groups of a smooth variety X to its algebraic K-groups. The main idea was to use a kind of motivic algebra of spectral categories and modules over them.

In this paper an alternative approach to constructing a triangulated category of K-motives is presented. We work in the framework of strict V-spectral categories introduced in the paper (Definition 2.5). The main feature of such a spectral category $\mathcal{O}$ is that it is connective and Nisnevich excisive in the sense of [2], and $\pi_0\mathcal{O}$-(pre)sheaves, where $\pi_0\mathcal{O}$ is a ringoid associated to $\mathcal{O}$, share lots of common properties with (pre)sheaves with transfers (or Cor-(pre)sheaves) in the sense of Voevodsky [15].

*Supported by EPSRC grant EP/J013064/1.

To any strict V-spectral category over k-smooth varieties we associate a triangulated category $D\mathcal{O}_-^{\mathrm{eff}}(k)$, which in spirit is constructed similarly to $DM_-^{\mathrm{eff}}(k)$ (Section 3). For instance, the ringoid of correspondences Cor gives rise to a strict V-spectral category $\mathcal{O} = \mathcal{O}_{cor}$ whenever the base field k is perfect. In this case the Voevodsky category $DM_-^{\mathrm{eff}}(k)$ is recovered as the category $D\mathcal{O}_-^{\mathrm{eff}}(k)$ (Corollary 3.6).

The main V-spectral category $\mathbb{K}$ is constructed in Section 4 (see Theorem 4.16). It is strict over perfect fields. The associated triangulated category $D\mathcal{O}_-^{\mathrm{eff}}(k)$ is denoted by $DK_-^{\mathrm{eff}}(k)$. The spectral category $\mathbb{K}$ is a priori different from spectral categories constructed by the authors in [2]. But we expect that associated motivic model categories of modules over the spectral categories are equivalent.

To each smooth k-variety X we associate its K-motive $M_{\mathbb{K}}(X)$. By definition, it is an object of the category $DK_-^{\mathrm{eff}}(k)$. We prove in Theorem 5.11 that

$$K_n(X) = \mathrm{Hom}_{DK_-^{\mathrm{eff}}(k)}(M_{\mathbb{K}}(X)[n], M_{\mathbb{K}}(pt)), \quad n \in \mathbb{Z},$$

where $pt = \mathrm{Spec}(k)$ and $K(X)$ is Quillen's K-theory of X. Thus Quillen's K-theory is represented by the K-motive of the point.

The spectral category $\mathbb{K}$ is of great utility in authors' paper [3], in which they solve some problems related to the motivic spectral sequence. In fact, the problems were the main motivation for constructing the spectral category $\mathbb{K}$ and developing the machinery of K-motives.

Throughout the paper we denote by Sm/k the category of smooth separated schemes of finite type over the base field k.

2. Preliminaries

We work in the framework of spectral categories and modules over them in the sense of Schwede–Shipley [12]. We start with preparations.

Recall that symmetric spectra have two sorts of homotopy groups which we shall refer to as *naive* and *true homotopy groups* respectively following terminology of [11]. Precisely, the kth naive homotopy group of a symmetric spectrum X is defined as the colimit

$$\hat{\pi}_k(X) = \mathrm{colim}_n \, \pi_{k+n} X_n.$$

Denote by γX a stably fibrant model of X in Sp^{Σ}. The k-th true homotopy group of X is given by

$$\pi_k X = \hat{\pi}_k(\gamma X),$$

the naive homotopy groups of the symmetric spectrum γX.

Naive and true homotopy groups of X can be considerably different in general (see, e.g., [6, 11]). The true homotopy groups detect stable equivalences, and are thus more important than the naive homotopy groups. There is an important class of *semistable* symmetric spectra within which $\hat{\pi}_*$-isomorphisms coincide with π_*-isomorphisms. Recall that a symmetric spectrum is semistable if some (hence any) stably fibrant replacement is a π_*-isomorphism. Suspension spectra, Eilenberg–Mac Lane spectra, Ω-spectra or Ω-spectra from some point X_n on are examples of semistable symmetric spectra (see [11]). So Waldhausen's algebraic K-theory symmetric spectrum, which we shall use later, is semistable. Semistability is preserved under suspension, loop, wedges and shift.

A symmetric spectrum X is *n-connected* if the true homotopy groups of X are trivial for $k \leqslant n$. The spectrum X is *connective* if it is (-1)-connected, i.e., its true homotopy groups vanish in negative dimensions. X is *bounded below* if $\pi_i(X) = 0$ for $i \ll 0$.

Definition 2.1 (1) Following [12] a *spectral category* is a category $\mathcal{O}$ which is enriched over the category Sp^{Σ} of symmetric spectra (with respect to smash product, i.e., the monoidal closed structure of [6, 2.2.10]). In other words, for every pair of objects $o,o' \in \mathcal{O}$ there is a morphism symmetric spectrum $\mathcal{O}(o,o')$, for every object o of $\mathcal{O}$ there is a map from the sphere spectrum S to $\mathcal{O}(o,o)$ (the "identity element" of o), and for each triple of objects there is an associative and unital composition map of symmetric spectra $\mathcal{O}(o',o'') \wedge \mathcal{O}(o,o') \to \mathcal{O}(o,o'')$. An $\mathcal{O}$-module M is a contravariant spectral functor to the category Sp^{Σ} of symmetric spectra, i.e., a symmetric spectrum $M(o)$ for each object of $\mathcal{O}$ together with coherently associative and unital maps of symmetric spectra $M(o) \wedge \mathcal{O}(o',o) \to M(o')$ for pairs of objects $o,o' \in \mathcal{O}$. A morphism of $\mathcal{O}$-modules $M \to N$ consists of maps of symmetric spectra $M(o) \to N(o)$ strictly compatible with the action of $\mathcal{O}$. The category of $\mathcal{O}$-modules will be denoted by $\mathrm{Mod}\,\mathcal{O}$.

(2) A *spectral functor* or a *spectral homomorphism* F from a spectral category $\mathcal{O}$ to a spectral category $\mathcal{O}'$ is an assignment from $\mathrm{Ob}\,\mathcal{O}$ to $\mathrm{Ob}\,\mathcal{O}'$ together with morphisms $\mathcal{O}(a,b) \to \mathcal{O}'(F(a),F(b))$ in Sp^{Σ} which preserve composition and identities.

(3) The *monoidal product* $\mathcal{O} \wedge \mathcal{O}'$ of two spectral categories $\mathcal{O}$ and $\mathcal{O}'$ is the spectral category where $\mathrm{Ob}(\mathcal{O} \wedge \mathcal{O}') := \mathrm{Ob}\,\mathcal{O} \times \mathrm{Ob}\,\mathcal{O}'$ and $\mathcal{O} \wedge \mathcal{O}'((a,x),(b,y)) := \mathcal{O}(a,b) \wedge \mathcal{O}'(x,y)$.

(4) A spectral category $\mathcal{O}$ is said to be *connective* if for any objects a,b of $\mathcal{O}$ the spectrum $\mathcal{O}(a,b)$ is connective.

(5) By a *ringoid* over Sm/k we mean a preadditive category $\mathcal{R}$ (i.e., a category enriched over abelian groups) whose objects are those of Sm/k together with a

functor

$$\rho : \mathrm{Sm}/k \to \mathcal{R},$$

which is identity on objects. Every such ringoid gives rise to a spectral category $\mathcal{O}_{\mathcal{R}}$ whose objects are those of Sm/k and the morphisms spectrum $\mathcal{O}_{\mathcal{R}}(X,Y)$, $X,Y \in \mathrm{Sm}/k$, is the Eilenberg–Mac Lane spectrum $H\mathcal{R}(X,Y)$ associated with the abelian group $\mathcal{R}(X,Y)$. Given a map of schemes α, its image $\rho(\alpha)$ will also be denoted by α, dropping ρ from notation.

(6) Let $\mathcal{O}_{naive}$ be the spectral category whose objects are those of Sm/k and morphism spectra are defined as

$$\mathcal{O}_{naive}(X,Y)_p = \mathrm{Hom}_{\mathrm{Sm}/k}(X,Y)_+ \wedge S^p$$

for all $p \geqslant 0$ and $X,Y \in \mathrm{Sm}/k$. By a *spectral category over* Sm/k we mean a pair $(\mathcal{O},\sigma)$, where $\mathcal{O}$ is a spectral category whose objects are those of Sm/k and

$$\sigma : \mathcal{O}_{naive} \to \mathcal{O}$$

is a spectral functor which is identity on objects. If there is no likelihood of confusion, we shall drop σ from notation.

Remark 2.2 It is straightforward to verify that the category of $\mathcal{O}_{naive}$-modules can be regarded as the category of presheaves $Pre^{\Sigma}(\mathrm{Sm}/k)$ of symmetric spectra on Sm/k. This is used in the sequel without further comment.

Let $\mathcal{O}$ be a spectral category and let $\mathrm{Mod}\mathcal{O}$ be the category of $\mathcal{O}$-modules. Recall that the projective stable model structure on $\mathrm{Mod}\mathcal{O}$ is defined as follows (see [12]). The weak equivalences are the objectwise stable weak equivalences and fibrations are the objectwise stable projective fibrations. The stable projective cofibrations are defined by the left lifting property with respect to all stable projective acyclic fibrations.

Let $\mathcal{Q}$ denote the set of elementary distinguished squares in Sm/k (see [10, 3.1.3])

$$\begin{tikzcd} U' \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathcal{Q}"] & X' \arrow[d, "\varphi"] \\ U \arrow[r, "\psi"'] & X \end{tikzcd}$$

and let $\mathcal{O}$ be a spectral category over Sm/k in the sense of Definition 2.1(6). By $\mathcal{Q}_{\mathcal{O}}$ denote the set of squares

$$\begin{tikzcd} \mathcal{O}(-,U') \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathcal{O}\mathcal{Q}"] & \mathcal{O}(-,X') \arrow[d, "\varphi"] \\ \mathcal{O}(-,U) \arrow[r, "\psi"'] & \mathcal{O}(-,X) \end{tikzcd}$$

which are obtained from the squares in $\mathcal{Q}$ by taking $X \in \mathrm{Sm}/k$ to $\mathcal{O}(-,X)$. The arrow $\mathcal{O}(-,U') \to \mathcal{O}(-,X')$ can be factored as a cofibration $\mathcal{O}(-,U') \rightarrowtail Cyl$ followed by a simplicial homotopy equivalence $Cyl \to \mathcal{O}(-,X')$. There is a canonical morphism $A_{\mathcal{O}Q} := \mathcal{O}(-,U)\coprod_{\mathcal{O}(-,U')} Cyl \to \mathcal{O}(-,X)$.

Definition 2.3 (see [2]) I. The *Nisnevich local model structure* on $\mathrm{Mod}\mathcal{O}$ is the Bousfield localization of the stable projective model structure with respect to the family of projective cofibrations

$$\mathcal{N}_{\mathcal{O}} = \{\mathrm{cyl}(A_{\mathcal{O}Q} \to \mathcal{O}(-,X))\}_{\mathcal{Q}_{\mathcal{O}}}.$$

The homotopy category for the Nisnevich local model structure will be denoted by $SH_{S^1}^{\mathrm{nis}}\mathcal{O}$. In particular, if $\mathcal{O} = \mathcal{O}_{naive}$ then we have the Nisnevich local model structure on $Pre^{\Sigma}(\mathrm{Sm}/k) = \mathrm{Mod}\mathcal{O}_{naive}$ and we shall write $SH_{S^1}^{\mathrm{nis}}(k)$ to denote $SH_{S^1}^{\mathrm{nis}}\mathcal{O}_{naive}$.

II. The *motivic model structure* on $\mathrm{Mod}\mathcal{O}$ is the Bousfield localization of the Nisnevich local model structure with respect to the family of projective cofibrations

$$\mathcal{A}_{\mathcal{O}} = \{\mathrm{cyl}(\mathcal{O}(-,X \times \mathbb{A}^1) \to \mathcal{O}(-,X))\}_{X \in \mathrm{Sm}/k}.$$

The homotopy category for the motivic model structure will be denoted by $SH_{S^1}^{\mathrm{mot}}\mathcal{O}$. In particular, if $\mathcal{O} = \mathcal{O}_{naive}$ then we have the motivic model structure on $Pre^{\Sigma}(\mathrm{Sm}/k) = \mathrm{Mod}\mathcal{O}_{naive}$ and we shall write write $SH_{S^1}^{\mathrm{mot}}(k)$ to denote $SH_{S^1}^{\mathrm{mot}}\mathcal{O}_{naive}$.

We refer the reader to [2, Definition 5.7] for the notions of *Nisnevich excisive* and *motivically excisive* spectral categories. These basically mean that $\mathcal{O}$ converts elementary distinguished squares to homotopy pushouts with respect to the appropriate model structure.

Let AffSm/k be the full subcategory of Sm/k whose objects are the smooth affine varieties. AffSm/k gives rise to a spectral category $\mathcal{O}_{\mathrm{Aff}}$ whose objects are those of AffSm/k and morphisms spectra are defined as

$$\mathcal{O}_{\mathrm{Aff}}(X,Y) := \mathrm{Hom}_{\mathrm{AffSm}/k}(X,Y)_+ \wedge \mathbb{S},$$

where $\mathbb{S}$ is the sphere spectrum and $X,Y \in \mathrm{AffSm}/k$.

Recall that a sheaf $\mathcal{F}$ of abelian groups in the Nisnevich topology on Sm/k is *strictly* $\mathbb{A}^1$*-invariant* if for any $X \in \mathrm{Sm}/k$, the canonical morphism

$$H^*_{\mathrm{nis}}(X,\mathcal{F}) \to H^*_{\mathrm{nis}}(X \times \mathbb{A}^1,\mathcal{F})$$

is an isomorphism.

Definition 2.4 Let $\mathcal{R}$ be a ringoid over Sm/k together with the structure functor $\rho : \mathrm{Sm}/k \to \mathcal{R}$. We say that $\mathcal{R}$ is a *V-ringoid* (“V" for Voevodsky) if

1. for any elementary distinguished square Q the sequence of Nisnevich sheaves associated to representable presheaves

$$0 \to \mathcal{R}_{\mathrm{nis}}(-,U') \to \mathcal{R}_{\mathrm{nis}}(-,U) \oplus \mathcal{R}_{\mathrm{nis}}(-,X') \to \mathcal{R}_{\mathrm{nis}}(-,X) \to 0$$

   is exact;

2. there is a functor

$$\boxtimes : \mathcal{R} \times \mathrm{AffSm}/k \to \mathcal{R}$$

   sending $(X,U) \in \mathrm{Sm}/k \times \mathrm{AffSm}/k$ to $X \times U \in Sm/k$ and such that $1_X \boxtimes \alpha = \rho(1_X \times \alpha)$, $(u+v) \boxtimes \alpha = u \boxtimes \alpha + v \boxtimes \alpha$ for all $\alpha \in \mathrm{Mor}(\mathrm{AffSm}/k)$ and $u,v \in \mathrm{Mor}(\mathcal{R})$.

3. for any $\mathcal{R}$-presheaf of abelian groups $\mathcal{F}$, i.e. $\mathcal{F}$ is a contravariant functor from $\mathcal{R}$ to abelian groups, the associated Nisnevich sheaf $\mathcal{F}_{\mathrm{nis}}$ has a unique structure of a $\mathcal{R}$-presheaf for which the canonical homomorphism $\mathcal{F} \to \mathcal{F}_{\mathrm{nis}}$ is a homomorphism of $\mathcal{R}$-presheaves. Moreover, if $\mathcal{F}$ is homotopy invariant then so is $\mathcal{F}_{\mathrm{nis}}$;

We refer to $\mathcal{R}$ as a *strict $\mathbb{A}^1$-invariant V-ringoid* if every $\mathbb{A}^1$-invariant Nisnevich $\mathcal{R}$-sheaf is strictly $\mathbb{A}^1$-invariant.

We want to make several remarks regarding the definition. Condition (1) implies the spectral category $\mathcal{O}_{\mathcal{R}}$ associated to the ringoid $\mathcal{R}$ is Nisnevich excisive. Condition (2) implies that for any $\mathcal{R}$-presheaf $\mathcal{F}$ and any affine scheme $U \in \mathrm{AffSm}/k$ the presheaf

$$\underline{\mathrm{Hom}}(U,\mathcal{F}) := \mathcal{F}(- \times U)$$

is an $\mathcal{R}$-presheaf. Moreover, it is functorial in U.

Definition 2.5 Let $(\mathcal{O},\sigma)$ be a spectral category over Sm/k in the sense of Definition 2.1(6). We say that $\mathcal{O}$ is a *V-spectral category* if

1. $\mathcal{O}$ is connective and Nisnevich excisive;

2. there is a spectral functor

$$\boxdot : \mathcal{O} \wedge \mathcal{O}_{\mathrm{Aff}} \to \mathcal{O}$$

   sending $(X,U) \in \mathrm{Sm}/k \times \mathrm{AffSm}/k$ to $X \times U \in \mathrm{Sm}/k$ and such that $1_X \boxdot \alpha = \sigma(1_X \times \alpha)$ for all $\alpha \in \mathrm{Mor}(\mathrm{AffSm}/k)$;

3. $\pi_0\mathcal{O}$ is a V-ringoid such that the structure map $\rho : \mathrm{Sm}/k \to \pi_0\mathcal{O}$ equals the composite map
$$\mathrm{Sm}/k \to \pi_0\mathcal{O}_{naive} \xrightarrow{\pi_0(\sigma)} \pi_0\mathcal{O}.$$
We also require the structure pairing $\boxtimes : \pi_0\mathcal{O} \times \mathrm{AffSm}/k \to \pi_0\mathcal{O}$ to be the composite functor
$$\pi_0\mathcal{O} \times \mathrm{AffSm}/k \to \pi_0\mathcal{O} \times \pi_0\mathcal{O}_{\mathrm{Aff}} \to \pi_0(\mathcal{O} \wedge \mathcal{O}_{\mathrm{Aff}}) \xrightarrow{\pi_0(\boxdot)} \pi_0\mathcal{O}.$$

We refer to $\mathcal{O}$ as a *strict V-spectral category* if $\pi_0\mathcal{O}$ is a strict $\mathbb{A}^1$-invariant V-ringoid.

Since the main category $D\mathcal{O}_-^{\mathrm{eff}}(k)$ we shall work with consists of bounded below $\mathcal{O}$-modules (see section 3 for precise definitions), we assume $\mathcal{O}$ to be connective in Definition 2.5.

We note that if $\mathcal{O}$ is a V-spectral category, then for every $\mathcal{O}$-module M and any affine smooth scheme U, the presheaf of symmetric spectra
$$\underline{\mathrm{Hom}}(U, M) := M(- \times U)$$
is an $\mathcal{O}$-module. Moreover, $M(- \times U)$ is functorial in U.

Lemma 2.6 *Every V-spectral category $\mathcal{O}$ is motivically excisive.*

Proof: Every V-spectral category is, by definition, Nisnevich excisive. Since there is an action of affine smooth schemes on $\mathcal{O}$, the fact that $\mathcal{O}$ is motivically excisive is proved similar to [2, 5.8]. □

Let $\mathcal{O}$ be a V-spectral category. Since it is both Nisnevich and motivically excisive, it follows from [2, 5.13] that the pair of natural adjoint fuctors
$$\Psi_* : Pre^{\Sigma}(\mathrm{Sm}/k) \rightleftarrows \mathrm{Mod}\mathcal{O} : \Psi^*$$
induces a Quillen pair for the Nisnevich local projective (respectively motivic) model structures on $Pre^{\Sigma}(\mathrm{Sm}/k)$ and $\mathrm{Mod}\mathcal{O}$. In particular, one has adjoint functors between triangulated categories
$$\Psi_* : SH_{S^1}^{\mathrm{nis}}(k) \leftrightarrows SH_{S^1}^{\mathrm{nis}}\mathcal{O} : \Psi^* \quad \text{and} \quad \Psi_* : SH_{S^1}^{\mathrm{mot}}(k) \leftrightarrows SH_{S^1}^{\mathrm{mot}}\mathcal{O} : \Psi^*. \qquad (1)$$

3. The triangulated category $D\mathcal{O}_-^{\mathrm{eff}}(k)$

Throughout this section we work with a strict V-spectral category $\mathcal{O}$. We shall often work with simplicial $\mathcal{O}$-modules $M[\bullet]$. The *realization* of $M[\bullet]$ is the $\mathcal{O}$-module $|M|$ defined as the coend
$$|M| = \Delta[\bullet]_+ \wedge_{\Delta} M[\bullet]$$

of the functor $\Delta[\bullet]_+ \wedge M[\bullet] : \Delta \times \Delta^{\mathrm{op}} \to \mathrm{Mod}\mathcal{O}$. Here $\Delta[n]$ is the standard simplicial n-simplex.

Recall that the simplicial ring $k[\Delta]$ is defined as

$$k[\Delta]_n = k[x_0,\dots,x_n]/(x_0+\cdots+x_n-1).$$

By $\Delta^{\cdot}$ we denote the cosimplicial affine scheme $\mathrm{Spec}(k[\Delta])$. Let

$$M \in \mathrm{Mod}\mathcal{O} \mapsto M_f \in \mathrm{Mod}\mathcal{O}$$

be a fibrant replacement functor in the Nisnevich local model structure on $\mathrm{Mod}\mathcal{O}$. Given an $\mathcal{O}$-module M, we set

$$C_*(M) := |\underline{\mathrm{Hom}}(\Delta^{\cdot}, M_f)|.$$

Note that $C_*(M)$ is an $\mathcal{O}$-module and is functorial in M. If we regard M_f as a constant simplicial $\mathcal{O}$-module, the map of cosimplicial schemes $\Delta^{\cdot} \to pt$ induces a map of $\mathcal{O}$-modules

$$M \to C_*(M).$$

Lemma 3.1 *The functor C_* respects Nisnevich local weak equivalences. In particular, it induces a triangulated endofunctor*

$$C_* : SH^{\mathrm{nis}}_{S^1}\mathcal{O} \to SH^{\mathrm{nis}}_{S^1}\mathcal{O}.$$

Proof: Let $\alpha : L \to M$ be a Nisnevich local weak equivalence of $\mathcal{O}$-modules. By [2, 5.12] the forgetful functor $\Psi^* : \mathrm{Mod}\mathcal{O} \to Pre^{\Sigma}(\mathrm{Sm}/k)$ respects Nisnevich local weak equivalences and Nisnevich local fibrant objects. It follows that the fibrant replacement

$$\alpha_f : L_f \to M_f$$

of α is a level equivalence of presheaves of ordinary symmetric spectra, and hence so is each map

$$\underline{\mathrm{Hom}}(\Delta^n, \alpha_f) : \underline{\mathrm{Hom}}(\Delta^n, L_f) \to \underline{\mathrm{Hom}}(\Delta^n, M_f), \quad n \geqslant 0.$$

Since the realization functor respects level equivalences, our assertion follows. □

One of advantages of strict V-spectral categories is that we can construct an $\mathbb{A}^1$-local replacement of an $\mathcal{O}$-module M in two steps. We first take $C_*(M)$ and then its Nisnevich local replacement $C_*(M)_f$.

Theorem 3.2 *The natural map $M \to C_*(M)_f$ is an $\mathbb{A}^1$-local replacement of M in the motivic model structure of $\mathcal{O}$-modules.*

Proof: The presheaves $\pi_i(C_*(M))$, $i \in \mathbb{Z}$, are homotopy invariant and have $\pi_0\mathcal{O}$-transfers. Since $\mathcal{O}$ is a strict V-spectral category then each Nisnevich sheaf $\pi_i^{\mathrm{nis}}(C_*(M)_f)$ is strictly homotopy invariant and has $\pi_0\mathcal{O}$-transfers. By [9, 6.2.7] $C_*(M)_f$ is $\mathbb{A}^1$-local in the motivic model category structure on $Pre^{\Sigma}(\mathrm{Sm}/k)$. By Lemma 2.6 $\mathcal{O}$ is motivically excisive, hence [2, 5.12] implies $C_*(M)_f$ is $\mathbb{A}^1$-local in the motivic model category structure on $\mathrm{Mod}\mathcal{O}$.

The map $M \to C_*(M)_f$ is the composite

$$M \to M_f \to C_*(M) \to C_*(M)_f.$$

The left and right arrows are Nisnevich local trivial cofibrations. The middle arrow is a level $\mathbb{A}^1$-weak equivalence in $Pre^{\Sigma}(\mathrm{Sm}/k)$ by [10, 3.8]. By Lemma 2.6 $\mathcal{O}$ is motivically excisive, hence [2, 5.12] implies the middle arrow is an $\mathbb{A}^1$-weak equivalence in $\mathrm{Mod}\mathcal{O}$. □

Definition 3.3 The $\mathcal{O}$-motive $M_{\mathcal{O}}(X)$ of a smooth algebraic variety $X \in \mathrm{Sm}/k$ is the $\mathcal{O}$-module $C_*(\mathcal{O}(-,X))$. We say that an $\mathcal{O}$-module M is *bounded below* if for $i \ll 0$ the Nisnevich sheaf $\pi_i^{\mathrm{nis}}(M)$ is zero. M is *n-connected* if $\pi_i^{\mathrm{nis}}(M)$ are trivial for $i \leqslant n$. M is *connective* is it is (-1)-connected, i.e., $\pi_i^{\mathrm{nis}}(M)$ vanish in negative dimensions.

Corollary 3.4 *If an $\mathcal{O}$-module M is bounded below (respectively n-connected) then so is $C_*(M)$. In particular, the $\mathcal{O}$-motive $M_{\mathcal{O}}(X)$ of any smooth algebraic variety $X \in Sm/k$ is connective.*

Proof: This follows from the preceding theorem and Morel's Connectivity Theorem [9]. □

Denote by $D\mathcal{O}_{-}(k)$ the full triangulated subcategory of $SH_{S^1}^{\mathrm{nis}}\mathcal{O}$ of bounded below $\mathcal{O}$-modules. We also denote by $D\mathcal{O}_{-}^{\mathrm{eff}}(k)$ the full triangulated subcategory of $D\mathcal{O}_{-}(k)$ of those $\mathcal{O}$-modules M such that each Nisnevich sheaf $\pi_i^{\mathrm{nis}}(M)$ is homotopy invariant. Note that for any smooth algebraic variety $X \in \mathrm{Sm}/k$ its $\mathcal{O}$-motive $M_{\mathcal{O}}(X)$ belongs to $D\mathcal{O}_{-}^{\mathrm{eff}}(k)$. To see this, just apply Corollary 3.4 and Theorem 3.5(2) below.

The category $D\mathcal{O}_{-}^{\mathrm{eff}}(k)$ is an analog of Voevodsky's triangulated category $DM_{-}^{\mathrm{eff}}(k)$ [16]. Let $\mathcal{O}_{cor}$ be the Eilenberg–Mac Lane spectral category associated with the ringoid Cor. We shall show below that $DM_{-}^{\mathrm{eff}}(k)$ is equivalent to $D\mathcal{O}_{-}^{\mathrm{eff}}(k)$ if $\mathcal{O} = \mathcal{O}_{cor}$.

Theorem 3.5 *Let $\mathcal{O}$ be a strict V-spectral category. Then the following statements are true:*

(1) *The kernel of C_* is the full triangulated subcategory $\mathcal{T}$ of $SH_{S^1}^{\mathrm{nis}}\mathcal{O}$ generated by the compact objects*

$$\mathrm{cone}(\mathcal{O}(-,X \times \mathbb{A}^1) \to \mathcal{O}(-,X)), \quad X \in Sm/k.$$

Moreover, the triangulated functor C_* induces an equivalence of triangulated categories

$$SH^{\mathrm{nis}}_{S^1}\mathcal{O}/\mathcal{T} \xrightarrow{\sim} SH^{\mathrm{mot}}_{S^1}\mathcal{O}.$$

(2) *The functor*

$$C_* : D\mathcal{O}_-(k) \to D\mathcal{O}_-(k)$$

lands in $D\mathcal{O}^{\mathrm{eff}}_-(k)$. The kernel of C_* is $\mathcal{T}_- := \mathcal{T} \cap D\mathcal{O}_-(k)$. Moreover, C_* is left adjoint to the inclusion functor

$$i : D\mathcal{O}^{\mathrm{eff}}_-(k) \to D\mathcal{O}_-(k)$$

and $D\mathcal{O}^{\mathrm{eff}}_-(k)$ is equivalent to the quotient category $D\mathcal{O}_-(k)/\mathcal{T}_-$.

Proof: (1). The localization theory of compactly generated triangulated categories implies the quotient category $SH^{\mathrm{nis}}_{S^1}\mathcal{O}/\mathcal{T}$ is equivalent to the full triangulated subcategory

$$\mathcal{T}^{\perp} = \{M \in SH^{\mathrm{nis}}_{S^1}\mathcal{O} \mid \mathrm{Hom}_{SH^{\mathrm{nis}}_{S^1}\mathcal{O}}(T,M) = 0 \text{ for all } T \in \mathcal{T}\}.$$

Moreover,

$$\mathcal{T} = {}^{\perp}(\mathcal{T}^{\perp}) = \{X \in SH^{\mathrm{nis}}_{S^1}\mathcal{O} \mid \mathrm{Hom}_{SH^{\mathrm{nis}}_{S^1}\mathcal{O}}(X,M) = 0 \text{ for all } M \in \mathcal{T}^{\perp}\}.$$

By construction, $\mathcal{T}^{\perp}$ can be identified up to natural equivalence of triangulated categories with the full triangulated subcategory of $\mathbb{A}^1$-local $\mathcal{O}$-modules. The latter subcategory is naturally equivalent to $SH^{\mathrm{mot}}_{S^1}\mathcal{O}$, because the motivic model structure on $\mathcal{O}$-modules is obtained from the Nisnevich local model structure by Bousfield localization with respect to the maps

$$\mathcal{O}(-,X \times \mathbb{A}^1) \to \mathcal{O}(-,X), \quad X \in \mathrm{Sm}/k.$$

Recall that a map $M \to N$ of $\mathcal{O}$-modules is a motivic equivalence if and only if for any $\mathbb{A}^1$-local $\mathcal{O}$-module L the induced map

$$\mathrm{Hom}_{SH^{\mathrm{nis}}_{S^1}\mathcal{O}}(N,L) \to \mathrm{Hom}_{SH^{\mathrm{nis}}_{S^1}\mathcal{O}}(M,L)$$

is an isomorphism. Given an $\mathcal{O}$-module M, the map $M \to C_*(M)$ is a motivic equivalence by Theorem 3.2. If we fit the arrow into a triangle in $SH^{\mathrm{nis}}_{S^1}\mathcal{O}$

$$X_M \to M \to C_*(M) \to X_M[1], \tag{2}$$

it will follow that $\mathrm{Hom}_{SH^{\mathrm{nis}}_{S^1}\mathcal{O}}(X_M,L) = 0$ for all $L \in \mathcal{T}^{\perp}$. We see that for any $\mathcal{O}$-module M one has $X_M \in {}^{\perp}(\mathcal{T}^{\perp}) = \mathcal{T}$.

If $C_*(M) \cong 0$ in $SH_{S^1}^{\mathrm{nis}}\mathcal{O}$, then $M \cong X_M \in \mathcal{T}$. Thus, $M \in \mathcal{T}$ in this case. On the other hand, if $M \in \mathcal{T}$ then $C_*(M) \in \mathcal{T}$, since $X_M \in \mathcal{T}$ and $\mathcal{T}$ is a thick triangulated subcategory in $SH_{S^1}^{\mathrm{nis}}\mathcal{O}$. On the other hand, Theorem 3.2 implies $C_*(M) \in \mathcal{T}^{\perp}$, and therefore $C_*(M) \in \mathcal{T} \cap \mathcal{T}^{\perp} = 0$. We conclude that $\mathcal{T} = \mathrm{Ker} C_*$.

(2). For any $M \in \mathrm{Mod}\mathcal{O}$ the presheaves $\pi_i(C_*(M))$, $i \in \mathbb{Z}$, are homotopy invariant and have $\pi_0\mathcal{O}$-transfers. Since $\mathcal{O}$ is a strict V-spectral category then each Nisnevich sheaf $\pi_i^{\mathrm{nis}}(C_*(M))$ is homotopy invariant. Therefore the functor

$$C_* : D\mathcal{O}_-(k) \to D\mathcal{O}_-(k)$$

lands in $D\mathcal{O}_-^{\mathrm{eff}}(k)$. It follows from the first part of the theorem that the kernel of C_* is $\mathcal{T}_- := \mathcal{T} \cap D\mathcal{O}_-(k)$.

Let us prove that $D\mathcal{O}_-^{\mathrm{eff}}(k) = \mathcal{T}^{\perp} \cap D\mathcal{O}_-(k)$. Clearly, $\mathcal{T}^{\perp} \cap D\mathcal{O}_-(k) \subset D\mathcal{O}_-^{\mathrm{eff}}(k)$. Suppose $M \in D\mathcal{O}_-^{\mathrm{eff}}(k)$. Then $M_f \in D\mathcal{O}_-^{\mathrm{eff}}(k)$. We have that M_f is a fibrant $\mathcal{O}$-module in the Nisnevich local model structure and each $\pi_i^{\mathrm{nis}}(M_f)$ is a strictly homotopy invariant sheaf, because $\mathcal{O}$ is a strict V-spectral category. By [9, 6.2.7] M_f is $\mathbb{A}^1$-local in the motivic model category structure on $Pre^{\Sigma}(\mathrm{Sm}/k)$. By Lemma 2.6 $\mathcal{O}$ is motivically excisive, hence [2, 5.12] implies M_f is $\mathbb{A}^1$-local in the motivic model category structure on $\mathrm{Mod}\mathcal{O}$. We see that $M \in \mathcal{T}^{\perp} \cap D\mathcal{O}_-(k)$.

Let $E \in D\mathcal{O}_-^{\mathrm{eff}}(k)$ and $M \in D\mathcal{O}_-(k)$. Applying the functor $\mathrm{Hom}_{D\mathcal{O}_-(k)}(-, E)$ to triangle (2), one gets

$$\mathrm{Hom}_{D\mathcal{O}_-(k)}(M, E) \cong \mathrm{Hom}_{D\mathcal{O}_-(k)}(C_*(M), E) = \mathrm{Hom}_{D\mathcal{O}_-^{\mathrm{eff}}(k)}(C_*(M), E).$$

Thus C_* is left adjoint to the inclusion functor $i : D\mathcal{O}_-^{\mathrm{eff}}(k) \to D\mathcal{O}_-(k)$.

It remains to show that $D\mathcal{O}_-^{\mathrm{eff}}(k)$ is equivalent to the quotient category $D\mathcal{O}_-(k)/\mathcal{T}_-$. By the first part of the theorem it is enough to prove that the natural functor

$$D\mathcal{O}_-(k)/\mathcal{T}_- \to SH_{S^1}^{\mathrm{nis}}\mathcal{O}/\mathcal{T}$$

is fully faithful. Consider an arrow $M \xrightarrow{s} N$ in $SH_{S^1}^{\mathrm{nis}}\mathcal{O}$, where $M \in D\mathcal{O}_-(k)$ and s is such that $\mathrm{cone}(s) \in \mathcal{T}$. There is a commutative diagram in $SH_{S^1}^{\mathrm{nis}}\mathcal{O}$

$$\begin{tikzcd} M \arrow[r, "s"] \arrow[d, "u_M"'] & N \arrow[d, "u_N"] \\ C_*(M) \arrow[r, "C_*(s)"] & C_*(N) \end{tikzcd}$$

in which cones of the vertical arrows are in $\mathcal{T}$. Since $\mathrm{cone}(C_*(s)) \cong C_*(\mathrm{cone}(s)) = 0$ in $SH_{S^1}^{\mathrm{nis}}\mathcal{O}$, we see that $C_*(s)$ is an isomorphism in $SH_{S^1}^{\mathrm{nis}}\mathcal{O}$. Therefore $C_*(N) \in D\mathcal{O}_-(k)$ and $\mathrm{cone}(u_N \circ s) \in \mathcal{T}_-$. By [7, 9.1] $D\mathcal{O}_-(k)/\mathcal{T}_-$ is a full subcategory of $SH_{S^1}^{\mathrm{nis}}\mathcal{O}/\mathcal{T}$. □

Suppose the field k is perfect. Then [15] implies $\mathcal{O}_{cor}$ is a strict V-spectral category. Recall that the Voevodsky triangulated category of motives $DM_{-}^{\mathrm{eff}}(k)$ is the full triangulated subcategory of (cohomologically) bounded above complexes of the derived category $D(ShTr)$ of Nisnevich sheaves with transfers (see [13, 16]). The next result says that $DM_{-}^{\mathrm{eff}}(k)$ can be recovered from $D\mathcal{O}_{-}^{\mathrm{eff}}(k)$ if $\mathcal{O} = \mathcal{O}_{cor}$.

Corollary 3.6 *Let k be a perfect field and $\mathcal{O} = \mathcal{O}_{cor}$, then there is a natural equivalence of triangulated categories*

$$D\mathcal{O}_{-}^{\mathrm{eff}}(k) \xrightarrow{\sim} DM_{-}^{\mathrm{eff}}(k).$$

Proof: By [2, section 6] there is a natural equivalence of triangulated categories $SH_{S^1}^{\mathrm{nis}}\mathcal{O}$ and $D(ShTr)$. Moreover, this equivalence takes bounded below $\mathcal{O}$-modules to (cohomologically) bounded above complexes. Restriction of the equivalence to $D\mathcal{O}_{-}^{\mathrm{eff}}(k)$ yields the desired equivalence between $D\mathcal{O}_{-}^{\mathrm{eff}}(k)$ and $DM_{-}^{\mathrm{eff}}(k)$. □

To conclude the section, it is also worth to mention another way of constructing a motivic fibrant replacement on $\mathcal{O}$-modules. Namely, for any $M \in \mathrm{Mod}\mathcal{O}$ we set

$$\widetilde{C}_*(M) := |d \mapsto (\underline{\mathrm{Hom}}(\Delta^d, M))_f|.$$

Clearly, $\widetilde{C}_*(M)$ is functorial in M. Observe that if M is Nisnevich local then $C_*(M)$ is zigzag level equivalent to $\widetilde{C}_*(M)$, because $\underline{\mathrm{Hom}}(\Delta^d, M)$ and $(\underline{\mathrm{Hom}}(\Delta^d, M))_f$ are Nisnevich local and the arrows

$$\underline{\mathrm{Hom}}(\Delta^d, M_f) \leftarrow \underline{\mathrm{Hom}}(\Delta^d, M) \rightarrow (\underline{\mathrm{Hom}}(\Delta^d, M))_f$$

are level weak equivalences.

Proposition 3.7 *The natural map $M \to \widetilde{C}_*(M)_f$ is an $\mathbb{A}^1$-local replacement of M in the motivic model structure of $\mathcal{O}$-modules.*

Proof: The map $M \to \widetilde{C}_*(M)_f$ is the composite

$$M \to |d \mapsto \underline{\mathrm{Hom}}(\Delta^d, M)| \to |d \mapsto (\underline{\mathrm{Hom}}(\Delta^d, M))_f| \to \widetilde{C}_*(M)_f.$$

The left arrow is a level $\mathbb{A}^1$-weak equivalence in $Pre^{\Sigma}(\mathrm{Sm}/k)$ by [10, 3.8]. The middle arrow is a Nisnevich local weak equivalence, because it is the realization of a simplicial Nisnevich local weak equivalence. The right arrow is plainly a Nisnevich local weak equivalence as well.

The presheaves $\pi_i(|d \mapsto \underline{\mathrm{Hom}}(\Delta^d, M)|)$, $i \in \mathbb{Z}$, are homotopy invariant and have $\pi_0\mathcal{O}$-transfers. Since $\mathcal{O}$ is a strict V-spectral category then each Nisnevich sheaf $\pi_i^{\mathrm{nis}}(\widetilde{C}_*(M)_f)$ is strictly homotopy invariant and has $\pi_0\mathcal{O}$-transfers. By [9, 6.2.7] $\widetilde{C}_*(M)_f$ is $\mathbb{A}^1$-local in the motivic model category structure on $Pre^{\Sigma}(\mathrm{Sm}/k)$. By Lemma 2.6 $\mathcal{O}$ is motivically excisive, hence [2, 5.12] implies the arrow of the proposition is an $\mathbb{A}^1$-weak equivalence in $\mathrm{Mod}\mathcal{O}$. □

4. The spectral category $\mathbb{K}$

In this section the definition of the V-spectral category $\mathbb{K}$ is given. It is obtained by taking K-theory symmetric spectra $K(\mathcal{A}(U,X))$ of certain additive categories $\mathcal{A}(U,X)$, $U,X \in \mathrm{Sm}/k$. To define these categories we need some preliminaries.

Notation 4.1 Let $U,X \in \mathrm{Sm}/k$. Define $\mathrm{Supp}(U \times X/X)$ as the set of all closed subsets in $U \times X$ of the form $A = \cup_{j \in J} B_j$, where J is a finite set and each B_j is a closed irreducible subset in $U \times X$ which is finite and surjective over U. The empty subset in $U \times X$ is also regarded as an element of $\mathrm{Supp}(U \times X/X)$.

Notation 4.2 Given $U,X \in \mathrm{Sm}/k$ and $A \in \mathrm{Supp}(U \times X/X)$, let $I_A \subset \mathcal{O}_{U \times X}$ be the ideal sheaf of the closed set $A \subset U \times X$. Denote by A_m the closed subscheme in $U \times X$ of the form $(A, \mathcal{O}_{U \times X}/I_A^m)$. If $m = 0$, then A_m is the empty subscheme. Define $\mathrm{SubSch}(U \times X/X)$ as the set of all closed subschemes in $U \times X$ of the form A_m.

For any $Z \in \mathrm{SubSch}(U \times X/X)$ we write $p_U^Z : Z \to U$ to denote $p \circ i$, where $i : Z \hookrightarrow U \times X$ is the closed embedding and $p : U \times X \to U$ is the projection. If there is no likelihood of confusion we shall write p_U instead of p_U^Z, dropping Z from notation.

Clearly, for any $Z \in \mathrm{SubSch}(U \times X/X)$ the reduced scheme Z^{red}, regarded as a closed subset of $U \times X$, belongs to $\mathrm{Supp}(U \times X/X)$.

Notation 4.3 Let $V,U,X \in \mathrm{Sm}/k$. Let $A \in \mathrm{Supp}(V \times U/U)$, $B \in \mathrm{Supp}(U \times X/X)$. Set

$$B \circ A = p_{VX}(V \times B \cap A \times X) \subset V \times X,$$

where $p_{VX} : V \times U \times X \to V \times X$ is the projection. One can check that

$$B \circ A \in \mathrm{Supp}(V \times X/X).$$

Notation 4.4 Let $V,U,X \in \mathrm{Sm}/k$. Let $S \in \mathrm{SubSch}(V \times U/U)$, $Z \in \mathrm{Subsch}(U \times X/X)$. By 4.2 one has $S^{red} \in \mathrm{Supp}(V \times U/U)$, $Z^{red} \in \mathrm{Supp}(U \times X/X)$. By 4.3 one has $Z^{red} \circ S^{red} \in \mathrm{Supp}(V \times X/X)$. One can show that for some integer $k \gg 0$ there exists a scheme morphism

$$\pi_k : T = S \times X \cap V \times Z \to (Z^{red} \circ S^{red})_k$$

such that $i_k \circ \pi_k = p_{VX} \circ i_T : T \to V \times X$. Here $i_k : (Z^{red} \circ S^{red})_k \hookrightarrow V \times X$, $i_T : T \hookrightarrow V \times U \times X$ are closed embeddings and $p_{VX} : V \times U \times X \to V \times X$ is the projection.

If there exists π_k satisfying the condition above then it is unique. Moreover, for any $m > k$ one has $i_k^m \circ \pi_k = \pi_m$, where $i_k^m : (Z^{red} \circ S^{red})_k \hookrightarrow (Z^{red} \circ S^{red})_m$ is the closed embedding.

We shall often write $Z \circ S$ to denote $(Z^{red} \circ S^{red})_k$, provided that there exists the required π_k. In this case we shall also write π to denote $\pi_k : T \to (Z \circ S)$.

Definition 4.5 (of additive categories $\mathcal{A}(U,X)$) For any $U,X \in \mathrm{Sm}/k$ we define objects of $\mathcal{A}(U,X)$ as equivalence classes of the triples

$$(n, Z, \varphi : p_{U,*}(\mathcal{O}_Z) \to M_n(\mathcal{O}_U)),$$

where n is a nonnegative integer, $Z \in \mathrm{SubSch}(U \times X/X)$ and φ is a non-unital homomorphism of sheaves of $\mathcal{O}_U$-algebras. Let $p(\varphi)$ be the idempotent $\varphi(1) \in M_n(\Gamma(U,\mathcal{O}_U))$, then $P(\varphi) = \mathrm{Im}(p(\varphi))$ can be regarded as a $p_{U,*}(\mathcal{O}_Z)$-module by means of φ.

By definition, two triples (n,Z,φ), (n',Z',φ') are equivalent if $n = n'$ and there is a triple (n'',Z'',φ'') such that $n = n' = n''$, $Z,Z' \subset Z''$ are closed subschemes in Z'', and the diagrams

$$\begin{tikzcd} p_{U,*}(\mathcal{O}_Z) \arrow[rr, "\varphi"] & & M_n(\mathcal{O}_U) \\ & p_{U,*}(\mathcal{O}_{Z''}) \arrow[ul, "can"] \arrow[ur, "\varphi''"'] & \end{tikzcd} \qquad \begin{tikzcd} p_{U,*}(\mathcal{O}_{Z'}) \arrow[rr, "\varphi'"] & & M_n(\mathcal{O}_U) \\ & p_{U,*}(\mathcal{O}_{Z''}) \arrow[ul, "can"] \arrow[ur, "\varphi''"'] & \end{tikzcd}$$

are commutative. We shall often denote an equivalence class for the triples by Φ. Though Z is not uniquely defined by Φ, nevertheless we shall also refer to $Z \subset U \times X$ as the *support* of Φ.

Given $\Phi, \Phi' \in \mathcal{A}(U,X)$ we first equalize supports Z, Z' of the objects Φ, Φ' and then set

$$\mathrm{Hom}_{\mathcal{A}(U,X)}(\Phi,\Phi') = \mathrm{Hom}_{p_{U,*}(\mathcal{O}_Z)}(P(\varphi), P(\varphi')),$$

where the right hand side is an Abelian group in the usual way. Given any three objects $\Phi, \Phi', \Phi'' \in \mathcal{A}(U,X)$ a composition law

$$\mathrm{Hom}_{\mathcal{A}(U,X)}(\Phi,\Phi') \circ \mathrm{Hom}_{\mathcal{A}(U,X)}(\Phi',\Phi'') \to \mathrm{Hom}_{\mathcal{A}(U,X)}(\Phi,\Phi'')$$

is defined in the obvious way. This makes therefore $\mathcal{A}(U,X)$ an additive category. The zero object is the equivalence class of the triple $(0,\emptyset,0)$. By definition,

$$\begin{aligned}\Phi_1 \oplus \Phi_2 = (n_1+n_2, Z_1 \cup Z_2, p_{U,*}(\mathcal{O}_{Z_1 \cup Z_2}) &\to p_{U,*}(\mathcal{O}_{Z_1}) \times p_{U,*}(\mathcal{O}_{Z_2}) \\ &\to M_{n_1}(\mathcal{O}_U)) \times M_{n_2}(\mathcal{O}_U)) \hookrightarrow M_{n_1+n_2}(\mathcal{O}_U)).\end{aligned}$$

Clearly, $P(\varphi_1 \oplus \varphi_2) \cong P(\varphi_1) \oplus P(\varphi_2)$. Definition of the additive category $\mathcal{A}(U,X)$ is finished.

We now want to construct a bilinear pairing

$$\mathcal{A}(V,U)\times\mathcal{A}(U,X)\xrightarrow{\circ}\mathcal{A}(V,X),\quad U,V,X\in \mathrm{Sm}/k. \tag{3}$$

First, define it on objects. Namely,

$$((n_1,Z_1,\varphi_1),(n_2,Z_2,\varphi_2))\longmapsto(n_1n_2,Z_2\circ Z_1,\varphi_2\circ\varphi_1),$$

where $Z_2\circ Z_1\in \mathrm{SubSch}(V\times X/X)$ is a closed subscheme of $V\times X$ defined in Notation 4.4. The nonunital homomorphism $\varphi_2\circ\varphi_1 : p_{V,*}^{(Z_2\circ Z_1)}(\mathcal{O}_{Z_2\circ Z_1})\to M_{n_2n_1}(\mathcal{O}_V)$ is given by the composition

$$\begin{tikzcd}
 & M_{n_2}(M_{n_1}(\mathcal{O}_V)) \arrow[r, "L"', "\cong"] & M_{n_2n_1}(\mathcal{O}_V) \\
q_{V,*}(\mathcal{O}_{Z_1\times_U Z_2}) = p_{V,*}(p_{Z_1,*}(\mathcal{O}_{Z_1\times_U Z_2})) \arrow[r, "p_{V,*}(\varphi_{2,Z_1})"] \arrow[d, equal] & M_{n_2}(p_{V,*}(\mathcal{O}_{Z_1})) \arrow[u, "M_{n_2}(\varphi_1)"'] & \\
p_{V,*}^{(Z_2\circ Z_1)}(\pi_*(\mathcal{O}_{Z_1\times_U Z_2})) & & \\
p_{V,*}^{(Z_2\circ Z_1)}(\mathcal{O}_{Z_2\circ Z_1}) \arrow[u, "p_{V,*}^{Z_2\circ Z_1}(\pi^*)=can'"] & &
\end{tikzcd} \tag{4}$$

where L is a canonical isomorphism obtained by inserting (n_1,n_1)-matrices into entries of a (n_2,n_2)-matrix, the diagrams

$$\begin{tikzcd}
Z_1\times_U Z_2 \arrow[r] \arrow[d, "p_{Z_1}"] \arrow[dd, bend right, "q_V"'] & Z_2 \arrow[r] \arrow[d, "p_U"'] & X \\
Z_1 \arrow[r, "r"'] \arrow[d, "p_V"] & U & \\
V & &
\end{tikzcd}
\qquad
\begin{tikzcd}
Z_1\times_U Z_2 \arrow[dr] \arrow[drr, "\pi"] \arrow[dd, "q_V"'] & & \\
 & V\times X \arrow[dl] & Z_2\circ Z_1 \arrow[l, hook'] \arrow[dll, "p_V^{Z_2\circ Z_1}"] \\
V & &
\end{tikzcd}$$

are commutative, and $\pi^* : \mathcal{O}_{Z_2\circ Z_1}\to\pi_*(\mathcal{O}_{Z_1\times_U Z_2})$ is induced by the scheme morphism $\pi : Z_1\times_U Z_2\to Z_2\circ Z_1$ from Notation 4.4. Finally, $\varphi_{2,Z_1} : p_{Z_1,*}(\mathcal{O}_{Z_1\times_U Z_2})\to M_{n_2}(\mathcal{O}_{Z_1})$ is defined as a unique non-unital homomorphism of sheaves of $\mathcal{O}_{Z_1}$-algebras such that for any open affine $U'\subset U$ and any open affine $Z_1'\subset Z_1$ with $r(Z_1')\subset U'$ and $Z_2'=p_U^{-1}(U')$ the value of φ_{2,Z_1} on Z_1' coincides with the non-unital homomorphism of $k[Z_1']$-algebras

$$k[Z_1']\otimes_{k[U']}k[Z_2']\xrightarrow{\mathrm{id}\otimes\varphi_2}k[Z_1']\otimes_{k[U']}M_{n_2}(k[U'])\xrightarrow{a\otimes\beta\mapsto a\cdot r^*(\beta)}M_{n_2}(k[Z_1']).$$

For a future use set $p(\varphi_{2,Z_1}) = \varphi_{2,Z_1}(1) \in M_{n_2}(\Gamma(Z_1,\mathcal{O}_{Z_1}))$ and $P(\varphi_{2,Z_1}) = \operatorname{Im}[p(\varphi_{2,Z_1}) : \mathcal{O}_{Z_1}^{n_2} \to \mathcal{O}_{Z_1}^{n_2}]$.

In order to define pairing (3) on morphisms, we need some preparations. Let $\Phi_1 \in \mathcal{A}(V,U)$ and $\Phi_2 \in \mathcal{A}(U,X)$. Consider the diagram

$$\begin{tikzcd}
p_{V,*}(\mathcal{O}_{Z_1}^{n_2}) \otimes_{p_{V,*}(\mathcal{O}_{Z_1})} P(\varphi_1) \arrow[r, "can", "\cong"'] & P(\varphi_1)^{n_2} \arrow[r, tail, "i_1"] & (\mathcal{O}_V^{n_1})^{n_2} \arrow[r, "\ell", "\cong"'] & \mathcal{O}_V^{n_2 n_1} \arrow[d, shift left, "p(\varphi_2\circ\varphi_1)"] \\
p_{V,*}(P(\varphi_{2,Z_1})) \otimes_{p_{V,*}(\mathcal{O}_{Z_1})} P(\varphi_1) \arrow[u, "p_{V,*}(i_{2,Z_1})\otimes \mathrm{id}"] \arrow[rrr, "\sigma_{12}"] & & & P(\varphi_2 \circ \varphi_1), \arrow[u, shift left, "i(\varphi_2\circ\varphi_1)"]
\end{tikzcd}$$

where $\sigma_{12} = p(\varphi_2 \circ \varphi_1) \circ \ell \circ i_1 \circ can \circ (i_{2,Z_1} \otimes \mathrm{id})$ (here $\ell(e_{i,j}) = e_{i+(j-1)n_1}$). It is worth to note that the isomorphism ℓ induces an $\mathcal{O}_V$-algebra isomorphism $M_{n_2}(M_{n_1}(\mathcal{O}_V)) \cong M_{n_2 n_1}(\mathcal{O}_V)$ which coincides with the canonical isomorphism L obtained by inserting (n_1,n_1)-matrices into entries of a (n_2,n_2)-matrix.

Definition 4.6 An $p_{V,*}^{(Z_2\circ Z_1)}(\mathcal{O}_{Z_2\circ Z_1})$-module structure on $p_{V,*}(P(\varphi_{2,Z_1}))\otimes_{p_{V,*}(\mathcal{O}_{Z_1})} P(\varphi_1)$ is defined as follows. For any open $V^0 \subset V$, $f \in \Gamma(V^0, p_{V,*}^{(Z_2\circ Z_1)}(\mathcal{O}_{Z_2\circ Z_1}))$, $m_1 \in \Gamma(V^0, P(\varphi_1))$, and $m_2 \in \Gamma(V^0, p_{V,*}(P(\varphi_{2,Z_1})))$ set

$$f(m_2 \otimes m_1) = ((p_{V,*}(\varphi_{2,Z_1}) \circ can')(f))(m_2) \otimes m_1.$$

An $p_{V,*}^{(Z_2\circ Z_1)}(\mathcal{O}_{Z_2\circ Z_1})$-module structure on $P(\varphi_2 \circ \varphi_1)$ is defined as follows. For any open $V^0 \subset V$, $f \in \Gamma(V^0, p_{V,*}^{(Z_2\circ Z_1)}(\mathcal{O}_{Z_2\circ Z_1}))$, and $m \in \Gamma(V^0, P(\varphi_2 \circ \varphi_1))$ set

$$fm = ((\varphi_2 \circ \varphi_1)(f))(m).$$

In particular,

$$1 \cdot m = ((\varphi_2 \circ \varphi_1)(1))(m) = p(\varphi_2 \circ \varphi_1)(m) = m,$$

because $m \in \operatorname{Im}(p(\varphi_2 \circ \varphi_1))$.

Lemma 4.7 *The map σ_{12} is an isomorphism of $\mathcal{O}_V$-modules and, moreover, an isomorphism of the $p_{V,*}^{(Z_2\circ Z_1)}(\mathcal{O}_{Z_2\circ Z_1})$-modules.*

Let $\alpha_1 : \Phi_1 \to \Phi_1'$ and $\alpha_2 : \Phi_2 \to \Phi_2'$ be morphism in $\mathcal{A}(V,U)$ and $\mathcal{A}(U,X)$ respectively. We set

$$\alpha_2 \odot \alpha_1 = \sigma_{12}' \circ (\alpha_2 \otimes \alpha_1) \circ \sigma_{12}^{-1} : P(\varphi_2 \circ \varphi_1) \to P(\varphi_2' \circ \varphi_1'). \tag{5}$$

The definition of pairing (3) is finished. It is defined on objects above and on morphisms by formula (5).

Lemma 4.8 *The functor* $\mathcal{A}(V,U) \times \mathcal{A}(U,X) \xrightarrow{\circ} \mathcal{A}(V,X)$ *is bilinear for all* $U,V,X \in Sm/k$.

For any $X \in \mathrm{Sm}/k$ we define an object $\mathrm{id}_X \in \mathrm{Ob}\mathcal{A}(X,X)$ by

$$\mathrm{id}_X = (1, \Delta_X, \mathrm{id} : \mathcal{O}_X \to \mathcal{O}_X).$$

Lemma 4.9 *For any* $U,X \in Sm/k$ *the functors* $\{\mathrm{id}_U\} \times \mathcal{A}(U,X) \xrightarrow{\circ} \mathcal{A}(U,X)$ *and* $\mathcal{A}(U,X) \times \{\mathrm{id}_X\} \xrightarrow{\circ} \mathcal{A}(U,X)$ *are identities on* $\mathcal{A}(U,X)$.

Lemma 4.10 *For any* $U,V,W,X \in Sm/k$ *and any* $\Phi_1 \in \mathcal{A}(W,V), \Phi_2 \in \mathcal{A}(V,U), \Phi_3 \in \mathcal{A}(U,X)$ *the following statements are true:*

1. $\Phi_3 \circ (\Phi_2 \circ \Phi_1) = (\Phi_3 \circ \Phi_2) \circ \Phi_1 \in \mathrm{Ob}\mathcal{A}(W,X)$;
2. $p(\varphi_3 \circ (\varphi_2 \circ \varphi_1)) = p((\varphi_3 \circ \varphi_2) \circ \varphi_1)$ *and* $P(\varphi_3 \circ (\varphi_2 \circ \varphi_1)) = P((\varphi_3 \circ \varphi_2) \circ \varphi_1)$;
3. *suppose* $\alpha_i : \Phi_i \to \Phi_i'$ *are morphisms* $(i = 1,2,3)$, *then* $\alpha_3 \odot (\alpha_2 \odot \alpha_1) = (\alpha_3 \odot \alpha_2) \odot \alpha_1 \in \mathrm{Hom}_{\mathcal{A}(W,X)}(P(\varphi_3 \circ (\varphi_2 \circ \varphi_1)), P((\varphi_3' \circ \varphi_2') \circ \varphi_1'))$.

Proposition 4.11 *For any* $U,V,W,X \in Sm/k$ *the diagram of functors*

$$\begin{tikzcd}
 & (\mathcal{A}(W,V) \times \mathcal{A}(V,U)) \times \mathcal{A}(U,X) \arrow[r, "\circ\times\mathrm{id}"] & \mathcal{A}(W,U) \times \mathcal{A}(U,X) \arrow[d, "\circ"] \\
\mathcal{A}(W,V) \times (\mathcal{A}(V,U) \times \mathcal{A}(U,X)) \arrow[ur, "\cong"] \arrow[r, "\mathrm{id}\times\circ"'] & \mathcal{A}(W,V) \times \mathcal{A}(V,X) \arrow[r, "\circ"] & \mathcal{A}(W,X)
\end{tikzcd}$$

is strictly commutative.

Lemma 4.12 *Pairings* (3) *together with* $\{\mathrm{id}_X\}_{X \in Sm/k}$ *determine a category* $\mathcal{A}$ *on* Sm/k *which is also enriched over additive categories. Moreover, the rules* $X \mapsto X$ *and* $f \mapsto \Phi_f = (1, \Gamma_f, \mathrm{id} : \mathcal{O}_U \to \mathcal{O}_U)$ *give a functor* $\sigma : Sm/k \to \mathcal{A}$.

The following notation will be useful later.

Notation 4.13 Let $X, X', Y \in \mathrm{Sm}/k$ and $f : X' \to X$ be a morphism in Sm/k. Define a functor $f^* : \mathcal{A}(X,Y) \to \mathcal{A}(X',Y)$ as the additive functor

$$\mathcal{A}(X,Y) \xrightarrow{-\circ\sigma(f)} \mathcal{A}(X',Y).$$

More precisely, $f^*(\Phi) = \Phi \circ \sigma(f)$ and $f^*(\alpha) = \alpha \odot \mathrm{id}_{\sigma(f)}$.

Let $X, Y, Y' \in \mathrm{Sm}/k$ and $g : Y \to Y'$ be a morphism in Sm/k. Define a functor $g_* : \mathcal{A}(X,Y) \to \mathcal{A}(X,Y')$ as the additive functor

$$\mathcal{A}(X,Y) \xrightarrow{\sigma(g)\circ-} \mathcal{A}(X,Y').$$

Namely, $g_*(\Phi) = \sigma(g) \circ \Phi$ and $g_*(\alpha) = \mathrm{id}_{\sigma(g)} \odot \alpha$.

Using this notation and Proposition 4.11, one has the following

Corollary 4.14 *Let $f : X' \to X$ and $g : Y \to Y'$ be morphisms in Sm/k. Then $f^* \circ g_* = g_* \circ f^* : \mathcal{A}(X,Y) \to \mathcal{A}(X',Y')$. If $f' : X'' \to X'$ is a map in Sm/k then $(f \circ f')^* = (f')^* \circ f^* : \mathcal{A}(X,Y) \to \mathcal{A}(X'',Y)$. Also, for any map $g' : Y' \to Y''$ in Sm/k one has $(g' \circ g)_* = (g')_* \circ g_* : \mathcal{A}(X,Y) \to \mathcal{A}(X,Y'')$.*

By [4, §6.1] for an additive category $\mathcal{C}$, one can define the structure of a symmetric spectrum on the Waldhausen K-theory spectrum $K(\mathcal{C})$. By definition,

$$K(\mathcal{C})_n = |\mathrm{Ob}\, S.^{Q}\mathcal{C}|, \quad Q = \{1,\dots,n\}.$$

Moreover, strictly associative bilinear pairings of additive categories induce strictly associative pairings of their K-theory symmetric spectra (see [4, §6.1]). The spectrum $K(\mathcal{C})$ is connective as any Waldhausen K-theory spectrum.

Notation 4.15 For any $U, X \in \mathrm{Sm}/k$, we denote by $\mathbb{K}(U,X)$ the Waldhausen K-theory symmetric spectrum $K(\mathcal{A}(U,X))$, where $\mathcal{A}(U,X)$ is the additive category in the sense of Definition 4.5.

Pairing (3) yields a pairing of symmetric spectra

$$\mathbb{K}(V,U) \wedge \mathbb{K}(U,X) \to \mathbb{K}(V,X). \tag{6}$$

Proposition 4.11 implies that (6) is a strictly associative pairing. Moreover, for any $X \in \mathrm{Sm}/k$ there is a map $\mathbf{1} : S \to \mathbb{K}(X,X)$ which is subject to the unit coherence law (see [4, section 6.1]). Note that $\mathbf{1}_0 : S^0 \to \mathbb{K}(X,X)_0$ is the map which sends the basepoint to the null object and the non-basepoint to the unit object id_X.

Thus we get the following

Theorem 4.16 *The triple $(\mathbb{K}, \wedge, \mathbf{1})$ determines a spectral category. Moreover, the functor $\sigma : Sm/k \to \mathcal{A}$ of Lemma 4.12 gives a spectral functor*

$$\sigma : \mathcal{O}_{naive} \to \mathbb{K}$$

between spectral categories such that the pair $(\mathbb{K}, \sigma)$ is a spectral category over Sm/k in the sense of Definition 2.1(6).

We now want to define a spectral functor

$$\boxdot : \mathbb{K} \wedge \mathcal{O}_{naive} \to \mathbb{K}.$$

It is in fact determined by additive functors

$$f^\star : \mathcal{A}(X,X') \to \mathcal{A}(X \times U, X' \times U'), \quad f : U \to U' \in \mathrm{Mor}(\mathrm{Sm}/k),$$

satisfying certain reasonable properties mentioned below. If

$$(n, Z, \varphi : p_{X,*}^{Z}(\mathcal{O}_Z) \to M_n(\mathcal{O}_X))$$

is a representative for $\Phi \in \mathcal{A}(X, X')$, then $f^\star(\Phi)$ is represented by the triple

$$(n, Z \times \Gamma_f, \varphi \boxtimes \mathrm{id}_U : (p_X^Z \times \mathrm{id})_*(\mathcal{O}_{Z \times \Gamma_f}) \to M_n(\mathcal{O}_{X \times U})).$$

Here $\varphi \boxtimes \mathrm{id}_U$ is a unique non-unital homomorphism of sheaves of $\mathcal{O}_{X\times U}$-algebras such that for any affine open subsets $X_0 \subset X$, $U_0 \subset U$ and for $Z_0 = (p_X^Z)^{-1}(X_0) \subset Z$ the value of $\varphi \boxtimes \mathrm{id}_U$ on $X_0 \times U_0$ is the following non-unital homomorphism of $k[Z_0 \times U_0]$-algebras:

$$(\varphi \boxtimes \mathrm{id}_U)(a \boxtimes b) := (q_{X,0})^*(\varphi(a)) \cdot (q_{U,0})^*(b) \in M_n(k[X_0 \times U_0]),$$

where $q_{X,0} : X^0 \times U^0 \to X^0$ and $q_{U,0} : X^0 \times U^0 \to U^0$ are the projections.

To define $f^\star$ on morphisms, we note that the canonical morphism

$$adj : q_X^*(P(\varphi)) \xrightarrow{q_X^*(i_{P(\varphi)})} q_X^*(\mathcal{O}_X^n) \xrightarrow{can} \mathcal{O}_{X\times U}^n \xrightarrow{p(f^\star(\Phi))} P(f^\star(\Phi))$$

is an isomorphism. Given a morphism $\alpha : \Phi \to \Phi'$ in $\mathcal{A}(X, X')$, we set

$$f^\star(\alpha) = adj' \circ q_X^*(\alpha) \circ adj^{-1} : P(f^\star(\Phi)) \to P(f^\star(\Phi')).$$

Clearly, $f^\star$ is an additive functor.

Proposition 4.17 *Let $f_1 : U \to U'$, $f_2 : U' \to U''$ be two maps in Sm/k, $\Phi_1, \Phi_1' \in \mathrm{Ob}\mathcal{A}(X, X')$, $\Phi_2, \Phi_2' \in \mathrm{Ob}\mathcal{A}(X', X'')$, let $\alpha_1 : \Phi_1 \to \Phi_1'$ be a morphism in $\mathcal{A}(X, X')$ and let $\alpha_2 : \Phi_2 \to \Phi_2'$ be a morphism in $\mathcal{A}(X', X'')$. Then*

1. $(f_2 \circ f_1)^\star(\Phi_2 \circ \Phi_1) = f_2^\star(\Phi_2) \circ f_1^\star(\Phi_1)$;
2. $(f_2 \circ f_1)^\star(\alpha_2 \odot \alpha_1) = f_2^\star(\alpha_2) \odot f_1^\star(\alpha_1)$.

Corollary 4.18 *Under the assumptions of Proposition 4.17 the diagram of functors*

$$\begin{tikzcd}
\mathcal{A}(X,X') \times \mathcal{A}(X',X'') \arrow[r, "\circ"] \arrow[d, "f_1^\star \times f_2^\star"'] & \mathcal{A}(X,X'') \arrow[d, "(f_2\circ f_1)^\star"] \\
\mathcal{A}(X\times U, X'\times U') \times \mathcal{A}(X'\times U', X''\times U'') \arrow[r, "\circ"] & \mathcal{A}(X\times U, X''\times U'')
\end{tikzcd}$$

is commutative.

Corollary 4.19 *We have a spectral functor*

$$\boxdot : \mathbb{K} \wedge \mathcal{O}_{naive} \to \mathbb{K}$$

such that $(X,U) \in Sm/k \times Sm/k$ is mapped to $X \times U \in Sm/k$. Moreover, for any morphism $h : X \to X'$ in Sm/k, regarded as the object $\sigma(h)$ of $\mathcal{A}(X,X')$, one has

$$f^\star(\sigma(h)) = \sigma(f \times h) \in \operatorname{Ob}\mathcal{A}(X \times U, X' \times U')$$

for every morphism of k-smooth schemes $f : U \to U'$.

In what follows we shall denote by $\mathbb{K}_0$ the ringoid $\pi_0(\mathbb{K})$.

Theorem 4.20 (Knizel [8]) *For any $\mathbb{K}_0$-presheaf of abelian groups $\mathcal{F}$, i.e. $\mathcal{F}$ is a contravariant functor from $\mathbb{K}_0$ to abelian groups, the associated Nisnevich sheaf $\mathcal{F}_{\mathrm{nis}}$ has a unique structure of a $\mathbb{K}_0$-presheaf for which the canonical homomorphism $\mathcal{F} \to \mathcal{F}_{\mathrm{nis}}$ is a homomorphism of $\mathbb{K}_0$-presheaves. If $\mathcal{F}$ is homotopy invariant then so is $\mathcal{F}_{\mathrm{nis}}$. Moreover, if the field k is perfect then every $\mathbb{A}^1$-invariant Nisnevich $\mathbb{K}_0$-sheaf is strictly $\mathbb{A}^1$-invariant.*

Remark 4.21 Although the category $\mathcal{A}(X,Y)$ is different from the category of bimodules $\mathcal{P}(X,Y)$ (see Appendix for the definition of $\mathcal{P}(X,Y)$), the proof of the preceding theorem is in spirit similar to the proof of the same fact for K_0-presheaves obtained by Walker [18].

Proposition 4.22 *$\mathbb{K}_0$ is a V-ringoid. If the field k is perfect then it is also a strict $\mathbb{A}^1$-invariant V-ringoid.*

Proof: The proof of [2, 5.9] shows that for any elementary distinguished square the sequence of Nisnevich sheaves associated to representable presheaves

$$0 \to \mathbb{K}_{0,\mathrm{nis}}(-,U') \to \mathbb{K}_{0,\mathrm{nis}}(-,U) \oplus \mathbb{K}_{0,\mathrm{nis}}(-,X') \to \mathbb{K}_{0,\mathrm{nis}}(-,X) \to 0$$

is exact.

Let $\rho : \mathrm{Sm}/k \to \mathbb{K}_0$ be the composite functor

$$\mathrm{Sm}/k \to \pi_0\mathcal{O}_{naive} \xrightarrow{\pi_0(\sigma)} \pi_0(\mathbb{K}) = \mathbb{K}_0, \tag{7}$$

where $\sigma : \mathcal{O}_{naive} \to \mathbb{K}$ is the spectral functor constructed in Theorem 4.16. Also, let a functor $\boxtimes : \mathbb{K}_0 \times \mathrm{Sm}/k \to \mathbb{K}_0$ be the composite functor

$$\mathbb{K}_0 \times \mathrm{Sm}/k \to \mathbb{K}_0 \times \pi_0\mathcal{O}_{naive} \to \pi_0(\mathbb{K} \wedge \mathcal{O}_{naive}) \xrightarrow{\pi_0(\boxdot)} \mathbb{K}_0, \tag{8}$$

where $\boxdot : \mathbb{K} \wedge \mathcal{O}_{naive} \to \mathbb{K}$ is the spectral functor constructed in Corollary 4.19. Then we have that $\mathrm{id}_X \boxtimes f = \rho(\mathrm{id}_X \times f)$, $(u + v) \boxtimes f = u \boxtimes f + v \boxtimes f$ for all $u,v \in \mathrm{Mor}(\mathbb{K}_0)$ and $f \in \mathrm{Mor}(\mathrm{Sm}/k)$.

Theorem 4.20 now implies $\mathbb{K}_0$ is a V-ringoid. It also follows from Theorem 4.20 that it is a strict $\mathbb{A}^1$-invariant V-ringoid over perfect fields. □

We are now in a position to prove the main result of the section.

Theorem 4.23 *The spectral category* $\mathbb{K}$ *together with the spectral functor* $\sigma : \mathcal{O}_{naive} \to \mathbb{K}$ *of Theorem 4.16 is a V-spectral category in the sense of Definition 2.5. If the field k is perfect then it is also a strict V-spectral category.*

Proof: $\mathbb{K}$ is connective by construction. It is proved similar to [2, 5.9] that $\mathbb{K}$ is Nisnevich excisive. The structure spectral functor

$$\sigma : \mathcal{O}_{naive} \to \mathbb{K}$$

is constructed in Theorem 4.16.

It follows from Corollary 4.19 that there is a spectral functor

$$\boxdot : \mathbb{K} \wedge \mathcal{O}_{naive} \to \mathbb{K}$$

sending $(X,U) \in \mathrm{Sm}/k \times \mathrm{Sm}/k$ to $X \times U \in \mathrm{Sm}/k$ and such that $\mathrm{id}_X \boxdot f = \sigma(\mathrm{id}_X \times f)$ for all $f \in \mathrm{Mor}(\mathrm{Sm}/k)$. Proposition 4.22 implies the ringoid $\mathbb{K}_0$ together with structure functors (7) and (8) is a V-ringoid which is strict $\mathbb{A}^1$-invariant whenever the base field k is perfect. □

We are now able to introduce the triangulated category of K-motives.

Definition 4.24 Suppose k is a perfect field. The *triangulated category of K-motives* $DK_{-}^{\mathrm{eff}}(k)$ is the triangulated category $D\mathcal{O}_{-}^{\mathrm{eff}}(k)$ constructed in Section 3 associated to the strict V-spectral category $\mathcal{O} = \mathbb{K}$ of Theorem 4.23.

To conclude the section, we discuss further useful properties of categories $\mathcal{A}(U,X)$-s.

Proposition 4.25 *Under Notation 4.13 and the notation of Lemma 4.12 and the notation which are just above Proposition 4.17 for any* $X,Y \in Sm/k$ *and any morphism* $f : U \to V$ *in* Sm/k *the following square of additive functors is strictly commutative*

$$\begin{array}{ccc}
\mathcal{A}(X \times V, Y \times V) & \xrightarrow{(1_X \times f)^*} & \mathcal{A}(X \times U, Y \times V) \\
\uparrow{\scriptstyle \mathrm{id}_V^\star} & & \uparrow{\scriptstyle (1_Y \times f)_*} \\
\mathcal{A}(X,Y) & \xrightarrow{\mathrm{id}_U^\star} & \mathcal{A}(X \times U, Y \times U).
\end{array}$$

Notation 4.26 For every $X \in \mathrm{Sm}/k, Y \in \mathrm{Sm}/k$ and $n > 0$, denote by $\mathcal{A}(X,Y)(\mathbb{G}_m^{\times n})$ the category whose objects are the tuples $(\Phi,\theta_1,\ldots,\theta_n)$, where $\Phi \in \mathcal{A}(X,Y)$ and $(\theta_1,\ldots,\theta_n)$ are commuting automorphisms of Φ. Morphisms from $(\Phi,\theta_1,\ldots,\theta_n)$

to $(\Phi',\theta_1',\dots,\theta_n')$ are given by morphisms $\alpha : \Phi \to \Phi'$ in $\mathcal{A}(X,Y)$ such that $\alpha \circ \theta_i = \theta_i' \circ \alpha$ for every $i = 1,\dots,n$.

Using Notation 4.13 for a morphism $f : X' \to X$ in Sm/k, define an additive functor

$$f_n^* : \mathcal{A}(X,Y)(\mathbb{G}_m^{\times n}) \to \mathcal{A}(X',Y)(\mathbb{G}_m^{\times n})$$

as follows: $f_n^*(\Phi,\theta_1,\dots,\theta_n) = (f^*(\Phi), f^*(\theta_1),\dots,f^*(\theta_n))$ on objects and $f_n^*(\alpha) = f^*(\alpha)$ on morphisms.

Using Notation 4.13 for a morphism $g : Y \to Y'$ in Sm/k, define an additive functor

$$g_{*,n} : \mathcal{A}(X,Y)(\mathbb{G}_m^{\times n}) \to \mathcal{A}(X,Y')(\mathbb{G}_m^{\times n})$$

as follows: $g_{*,n}(\Phi,\theta_1,\dots,\theta_n) = (g_*(\Phi), g_*(\theta_1)\dots,g_*(\theta_n))$ on objects and $g_{n,*}(\alpha) = g_*(\alpha)$ on morphisms.

Definition 4.27 Given $X \in \mathrm{Sm}/k, Y \in \mathrm{Sm}/k$ and $n > 0$, we define an additive functor

$$\rho_{X,Y,n} : \mathcal{A}(X,Y \times \mathbb{G}_m^{\times n}) \to \mathcal{A}(X,Y)(\mathbb{G}_m^{\times n})$$

by using the functor $(pr_Y)_* : \mathcal{A}(X,Y \times \mathbb{G}_m^{\times n}) \to A(X,Y)$ from Notation 4.13 as follows. Given an object $\Phi \in \mathcal{A}(X,Y \times \mathbb{G}_m^{\times n})$ and its representative

$$(n, Z, \varphi : p_{X,*}(\mathcal{O}_Z) \to M_n(\mathcal{O}_X)),$$

we have n automorphisms $[t_i]$'s of Φ of the form $m \mapsto \varphi(t_i|_Z)m$, where each $t_i = p_i^*(t) \in \Gamma(X \times Y \times \mathbb{G}_m^{\times n})$ and $p_i : X \times Y \times \mathbb{G}_m^{\times n} \to \mathbb{G}_m$ is the projection. One sets

$$\rho_{X,Y,n}(\Phi) = ((pr_Y)_*(\Phi), (pr_Y)_*([t_1]),\dots,(pr_Y)_*([t_n]))$$

on objects and $\rho_{X,Y,n}(\Phi)(\alpha) = (pr_Y)_*(\alpha)$ on morphisms.

The following lemma is a straightforward consequence of Corollary 4.14.

Lemma 4.28 *The bivariant additive category*

$$\mathcal{A} : (Sm/k)^{\mathrm{op}} \times Sm/k \to AddCats, \quad (X,Y) \mapsto \mathcal{A}(X,Y),$$

satisfies the following property:

($\mathfrak{Aut}$) *for every $X \in Sm/k, Y \in Sm/k$ and $n > 0$, the functors $\rho_{X,Y,n}$ meet the following two conditions:*
(a) for any $f : X' \to X$ in Sm/k and $n > 0$ one has $f_n^* \circ \rho_{X,Y,n} = \rho_{X',Y,n} \circ f^*$, where $f^* : \mathcal{A}(X,Y \times \mathbb{G}_m^{\times n}) \to \mathcal{A}(X',Y \times \mathbb{G}_m^{\times n})$ is defined in Notation 4.13;
(b) using Notation 4.13, for any $g : Y \to Y'$ in Sm/k and $n > 0$ one has

$$g_{*,n} \circ \rho_{X,Y,n} = \rho_{X,Y',n} \circ (g \times id_n)_*,$$

where id_n is the identity morphism of $\mathbb{G}_m^{\times n}$.

The following proposition is true as well.

Proposition 4.29 *For every* $X \in Sm/k, Y \in Sm/k$ *and* $n > 0$ *the additive functor*

$$\rho_{X,Y,n} : \mathcal{A}(X, Y \times \mathbb{G}_m^{\times n}) \to \mathcal{A}(X,Y)(\mathbb{G}_m^{\times n})$$

is a category isomorphism (it is not just an equivalence of categories).

5. Comparing $\mathcal{A}(X,Y)$ with $\tilde{\mathcal{P}}(X,Y)$

Let X, Y be two k-schemes of finite type over the base field k. We denote by $\mathcal{P}(X,Y)$ the category of coherent $\mathcal{O}_{X\times Y}$-modules $P_{X,Y}$ such that $\mathrm{Supp}(P_{X,Y})$ is finite over X and the coherent $\mathcal{O}_X$-module $(p_X)_*(P_{X,Y})$ is locally free. A disadvantage of the category $\mathcal{P}(X,Y)$ is that whenever we have two maps $f : X \to X'$ and $g : X' \to X''$ then the functor $(g \circ f)^*$ agrees with $f^* \circ g^*$ only up to a canonical isomorphism. To fix the problem, we replace $\mathcal{P}(X,Y)$ by the equivalent additive category of big bimodules $\tilde{\mathcal{P}}(X,Y)$ which is functorial in both arguments. This is done in Appendix.

In this section for any $X \in \mathrm{Sm}/k$ and $Y \in \mathrm{AffSm}/k$ a canonical functor

$$F_{X,Y} : \mathcal{A}(X,Y) \to \tilde{\mathcal{P}}(X,Y)$$

is constructed. Logically, one should now read Appendix about big bimodules, and then return to this section.

As an application, we obtain canonical isomorphisms over a perfect field k

$$K_i(X) \cong DK_{-}^{\mathrm{eff}}(k)(M_{\mathbb{K}}(X)[i], M_{\mathbb{K}}(pt)), \quad X \in \mathrm{Sm}/k, i \in \mathbb{Z}, pt = \mathrm{Spec}\, k,$$

where $K(X)$ is an algebraic K-theory spectrum defined as the Waldhausen symmetric K-theory spectrum $K(\tilde{\mathcal{P}}(X, pt))$ and $DK_{-}^{\mathrm{eff}}(k)$ is the triangulated category of K-motives (see Definition 4.24).

Let $X, Y \in \mathrm{Sm}/k$ and assume that Y is affine. Let $\mathcal{A}(X,Y)$ be the additive category in the sense of Definition 4.5 and let $\tilde{\mathcal{P}}(X,Y)$ be the additive category of big bimodules defined in Appendix. If $f : X' \to X$ is a morphism in Sm/k, then there is an additive functor $f^* : \mathcal{A}(X,Y) \to \mathcal{A}(X',Y)$ defined in Notation 4.13. By Corollary 4.14 the assignments $X \mapsto \mathcal{A}(X,Y)$ and $f \mapsto f^*$ yield a presheaf of small additive categories on Sm/k. By Lemma A.1 the assignments $X \mapsto \tilde{\mathcal{P}}(X,Y)$ and $f \mapsto (f^* : \tilde{\mathcal{P}}(X,Y) \to \tilde{\mathcal{P}}(X',Y))$ yield another presheaf of small additive categories on Sm/k.

The main goal of this section is to prove the following

Theorem 5.1 *Let Y be an affine k-smooth variety. Then there is a morphism*

$$F : \mathcal{A}(-,Y) \to \tilde{\mathcal{P}}(-,Y)$$

of presheaves of additive categories on Sm/k such that for any k-smooth affine X the functor $F_{X,Y} : \mathcal{A}(X,Y) \to \tilde{\mathcal{P}}(X,Y)$ is an equivalence of categories.

We postpone the proof but first construct a functor

$$F_{X,Y} : \mathcal{A}(X,Y) \to \tilde{\mathcal{P}}(X,Y)$$

which is an equivalence of categories whenever X is affine. We shall do this in several steps. Let $\Phi \in \mathcal{A}(X,Y)$ be an object. It is represented by a triple

$$(n, Z, \varphi : p_{X,*}(\mathcal{O}_Z) \to M_n(\mathcal{O}_X)),$$

where n is a nonnegative integer, $Z \in \mathrm{SubSch}(X \times Y/Y)$ (see Notation 4.2) and φ is a non-unital homomorphism of sheaves of $\mathcal{O}_X$-algebras. Thus one can consider the composite of non-unital k-algebra homomorphisms

$$\Phi_X : k[Y] \to k[X \times Y] \to k[Z] \xrightarrow{\varphi} M_n(k[X]).$$

Clearly, it does not depend on the choice of a triple representing the object Φ.

Let Sch/X be the category of X-schemes of finite type. For an X-scheme $f : U \to X$ in Sch/X set

$$\Phi_U := M_n(f^*) \circ \Phi_X : k[Y] \to M_n(k[U]).$$

Note that Φ_U depends not only on U itself but rather on the X-scheme U. The assignment $U/X \mapsto \Phi_U$ defines a morphism of presheaves of non-unital k-algebras $(U/X \mapsto k[Y]) \to (U/X \mapsto M_n(k[U]))$.

One has a compatible family of projectors given by $U/X \mapsto p_U^\Phi = \Phi_U(1) \in M_n(k[U])$. Set $P_U^\Phi = \mathrm{Im}(p_U^\Phi) \subset k[U]^n$. Then the assignment

$$U \mapsto P_U^\Phi \tag{9}$$

is a presheaf of $(U/X \mapsto k[U])$-modules. Given $U/X \in Sch/X$ and a point $u \in U$, we set

$$p_{U,u}^\Phi := \mathrm{colim}_{u \in V \subset U}\, p_V^\Phi \in M_n(\mathcal{O}_{U,u}), \quad P_{U,u}^\Phi := \mathrm{Im}(p_{U,u}^\Phi) \subset \mathcal{O}_{U,u}^n.$$

The stalk of the presheaf $(U \mapsto P_U^\Phi)$ of $(U/X \mapsto k[U])$-modules at the point $u \in U$ is the $\mathcal{O}_{U,u}$-module $P_{U,u}^\Phi$.

The presheaf of $(U/X \mapsto k[U])$-modules $U/X \mapsto P_U^{\Phi}$ has, moreover, a $k[Y]$-module structure. Namely, for each $U/X \in Sch/X$ the k-algebra $k[Y]$ acts on the $k[U]$-module P_U^{Φ} by means of the non-unital k-algebra homomorphism $\Phi_U : k[Y] \to M_n(k[U])$. Therefore the k-algebra $k[Y]$ acts on the $\mathcal{O}_{U,u}$-module $P_{U,u}^{\Phi}$ by means of a non-unital k-algebra homomorphism

$$\Phi_{U,u} : k[Y] \xrightarrow{\Phi_U} M_n(k[U]) \xrightarrow{M_n(can)} M_n(\mathcal{O}_{U,u}),$$

where can is the localization homomorphism $k[U] \to \mathcal{O}_{U,u}$. *In what follows we will regard the $\mathcal{O}_{U,u}$-module $P_{U,u}^{\Phi}$ as an $\mathcal{O}_{U,u} \otimes_k k[Y]$-module via the non-unital k-algebra homomorphism $\Phi_{U,u}$.*

Definition 5.2 Let $U/X \in Sch/X$ and $q \in U \times Y$ be a point. Let $u = pr_U(q) \in U$ be its image in U. Set

$$\mathcal{P}_{U,q}^{\Phi} := \left\{ \frac{m}{g} \mid m \in P_{U,u}^{\Phi}, g \in \mathcal{O}_{U,u} \otimes_k k[Y] \text{ such that } g(q) \neq 0 \right\} / \sim,$$

where "$\sim$" is the standard equivalence relation for fractions. Clearly, $\mathcal{P}_{U,q}^{\Phi}$ is an $\mathcal{O}_{U\times Y,q}$-module.

Now define a Zariski sheaf $\mathcal{P}_U^{\Phi}$ of $\mathcal{O}_{U\times Y}$-modules on $U \times Y$ as follows. Its sections on an open set $W \subset U \times Y$ are a compatible family of elements $\{n_q \in \mathcal{P}_{U,q}^{\Phi}\}_{q\in W}$. More precisely, we give the following

Definition 5.3 Set $\mathcal{P}_U^{\Phi}(W)$ to consist of the tuples $(n_q) \in \prod_{q\in W} \mathcal{P}_{U,q}^{\Phi}$ such that there is an affine cover $U = \cup U_i$ and for any i there is an affine cover of the form $(W \cap U_i \times Y) = \cup (U_i \times Y)_{g_{i_j}}$ with $g_{i_j} \in k[U_i \times Y]$ and there are elements $n_{i_j} \in (P_{U_i}^{\Phi})_{g_{i_j}}$ such that for any i and any i_j and any point $q \in (U_i \times Y)_{g_{i_j}}$ one has $n_{i_j} = n_q \in \mathcal{P}_{U,q}^{\Phi}$. Here $(U_i \times Y)_{g_{i_j}}$ stands for the principal open set associated with g_{i_j}.

Clearly, the assignment $W \mapsto \mathcal{P}_U^{\Phi}(W)$ is a Zariski sheaf of $\mathcal{O}_{U\times Y}$-modules on $U \times Y$. The $\mathcal{O}_{U\times Y}$-module structure on this sheaf is given as follows: for $f \in k[W]$ and $(n_q) \in \mathcal{P}_U^{\Phi}(W)$ set $f \cdot (n_q) = (f \cdot n_q)$.

Next, for any morphism $f : V \to U$ of objects in Sch/X construct a sheaf morphism

$$\sigma_f : \mathcal{P}_U^{\Phi} \to F_*(\mathcal{P}_V^{\Phi}),$$

where $F = f \times \mathrm{id} : V \times Y \to U \times Y$. Given a point $v \in V$ and its image $u \in U$, set $F_v^* = p_{V,v}^{\Phi} \circ f^* \circ i_{U,u}^{\Phi} : P_{U,u}^{\Phi} \to P_{V,v}^{\Phi}$, where $i_{U,u}^{\Phi} : P_{U,u}^{\Phi} \hookrightarrow \mathcal{O}_{U,u}^n$ is the inclusion.

For any point $r \in V \times Y$ and its image $s = F(r) \in U \times Y$ set $v = pr_V(r)$ and $u = pr_U(s)$. Clearly, $f(v) = u$. The k-algebra homomorphism $\mathcal{O}_{U\times Y,s} \to \mathcal{O}_{V\times Y,r}$

makes $\mathcal{P}^{\Phi}_{V,r}$ an $\mathcal{O}_{U\times Y,s}$-module. There is a unique homomorphism $F_r^* : \mathcal{P}^{\Phi}_{U,s} \to \mathcal{P}^{\Phi}_{V,r}$ of $\mathcal{O}_{U\times Y,s}$-modules making the diagram commutative

$$\begin{tikzcd} P^{\Phi}_{U,u} \arrow[r] \arrow[d, "F_v^*"'] & \mathcal{P}^{\Phi}_{U,s} \arrow[d, "F_r^*"] \\ P^{\Phi}_{V,v} \arrow[r] & \mathcal{P}^{\Phi}_{V,r} \end{tikzcd}$$

Let $W \subset U \times Y$ be an open subset. By definition,

$$\mathcal{P}^{\Phi}_{U}(W) = \{(n_s) \in \prod_{s\in W} \mathcal{P}^{\Phi}_{U,s} \mid n_s \text{ are locally compatible}\}$$

and

$$F_*(\mathcal{P}^{\Phi}_V)(W) = \mathcal{P}^{\Phi}_V(F^{-1}(W)) = \{(n_r) \in \prod_{r\in F^{-1}(W)} \mathcal{P}^{\Phi}_{V,r} \mid n_r \text{ are locally compatible}\}.$$

Define $F_W^* : \mathcal{P}^{\Phi}_U(W) \to F_*(\mathcal{P}^{\Phi}_V)(W)$ as follows. Given a section $(n_s \in \mathcal{P}^{\Phi}_{U,s})_{s\in W}$ of $\mathcal{P}^{\Phi}_U$ over W, set

$$F_W^*((n_s \in \mathcal{P}^{\Phi}_{U,s})_{s\in W}) := ((F_r^*(n_s)_{f(r)=s}))_{s\in W}.$$

It is straightforward to check that the assignment $W \mapsto F_W^*$ defines an $\mathcal{O}_{U\times Y}$-sheaf morphism

$$\sigma_f : \mathcal{P}^{\Phi}_U \to F_*(\mathcal{P}^{\Phi}_V).$$

Moreover, for a pair of morphisms $g : U_3 \to U_2$ and $f : U_2 \to U_1$ in Sch/X one has

$$\sigma_{f\circ g} = (f \times \mathrm{id}_Y)_*(\sigma_g) \circ \sigma_f : \mathcal{P}^{\Phi}_{U_1} \to (F \circ G)_*(\mathcal{P}^{\Phi}_{U_3}) = F_*(G_*(\mathcal{P}^{\Phi}_{U_3})).$$

Lemma 5.4 *The data $U/X \mapsto \mathcal{P}^{\Phi}_U$ and $(f : V \to U) \mapsto (\sigma_f : \mathcal{P}^{\Phi}_U \to F_*(\mathcal{P}^{\Phi}_V))$ defined above determine an object of the category $\tilde{\mathcal{P}}(X,Y)$. We shall denote this object by $F_{X,Y}(\Phi)$.*

Now define the functor $F_{X,Y} : \mathcal{A}(X,Y) \to \tilde{\mathcal{P}}(X,Y)$ on morphisms. Let $\alpha : \Phi \to \Psi$ be a morphism in $\mathcal{A}(X,Y)$. The morphism α is a Zariski sheaf morphism

$$(U/X \mapsto P^{\Phi}_U) \to (U/X \mapsto P^{\Psi}_U)$$

on small Zariski site X_{Zar} respecting the $k[Y]$-module structure on both sides. We write $\alpha_U : P^{\Phi}_U \to P^{\Psi}_U$ for the value of α at U. For any point $x \in X$ the Zariski sheaf morphism α induces a morphism of stalks

$$\alpha_x : P^{\Phi}_{X,x} \to P^{\Psi}_{X,x}.$$

Finally, for any point $q \in X \times Y$ and its image $x = p_X(q) \in X$ one has a homomorphism

$$\alpha_q : \mathcal{P}^{\Phi}_{X,q} \to \mathcal{P}^{\Psi}_{X,q}$$

given by $\alpha_q\big(\frac{m}{g}\big) = \frac{\alpha_x(m)}{g}$ for any $m \in P^{\Phi}_{X,x}$ and any $g \in \mathcal{O}_{X,x} \otimes_k k[Y]$ with $g(q) \neq 0$.

Definition 5.5 Define a morphism $F_{X,Y}(\alpha) : F_{X,Y}(\Phi) \to F_{X,Y}(\Psi)$ as follows. Given a Zariski open subset $W \subset X \times Y$ and a section $s = (n_q) \in \mathcal{P}^{\Phi}_X(W)$, set $\alpha_W(s) = (\alpha_q(n_q))$. Clearly, the family $(\alpha_q(n_q))$ is an element of $\mathcal{P}^{\Psi}_X(W)$. Moreover, α_W is a homomorphism and the assignment $W \mapsto \alpha_W$ is a morphism in $\tilde{\mathcal{P}}(X,Y)$. We shall write $F_{X,Y}(\alpha)$ for this morphism in $\tilde{\mathcal{P}}(X,Y)$.

Lemma 5.6 *The assignments $\Phi \mapsto F_{X,Y}(\Phi)$ and $\alpha \mapsto F_{X,Y}(\alpha)$ determine an additive functor $F_{X,Y} : \mathcal{A}(X,Y) \to \tilde{\mathcal{P}}(X,Y)$. Moreover, for a given affine k-smooth variety Y the assignment $X \mapsto F_{X,Y}$ determines a morphism of presheaves of additive categories.*

Lemma 5.6 implies that in order to prove Theorem 5.1, it remains to check that for affine $X,Y \in \mathrm{AffSm}/k$ the functor $F_{X,Y}$ is an equivalence of categories. Firstly describe a plan of the proof. Given $X,Y \in \mathrm{AffSm}/k$ we shall construct a square of additive categories and additive functors

$$\begin{array}{ccc} \mathcal{A}(X,Y) & \xrightarrow{F_{X,Y}} & \tilde{\mathcal{P}}(X,Y) \\ \Gamma \big\downarrow & & \big\downarrow R \\ A(X,Y) & \xrightarrow[a_{X,Y}]{} & \mathcal{P}(X,Y) \end{array} \tag{10}$$

which commutes up to an isomorphism of additive functors. We shall prove that the functors Γ, $a_{X,Y}$ and R are equivalences of categories. As a consequence, the functor $F_{X,Y}$ will be an equivalence of categories.

Definition 5.7 For affine schemes $X,Y \in \mathrm{AffSm}/k$ define a category $A(X,Y)$ as follows. Objects of $A(X,Y)$ are the pairs (n,φ), where $n \geqslant 0$ and $\varphi : k[Y] \to M_n(k[X])$ is a non-unital k-algebra homomorphism. The homomorphism φ defines a projector $\varphi(1) \in M_n(k[X])$. The projector $\varphi(1)$ defines a projective $k[X]$-module $\mathrm{Im}\big(\varphi(1) : k[X]^n \to k[X]^n\big)$. This $k[X]$-module has also a $k[Y]$-module structure which is given by the non-unital homomorphism φ. Namely, $mf := \varphi(f)(m)$. Thus $\mathrm{Im}(\varphi(1))$ is a $k[X \times Y]$-module. Set

$$\mathrm{Mor}_{A(X,Y)}((n_1,\varphi_1),(n_1,\varphi_1)) = \mathrm{Hom}_{k[X\times Y]}(\mathrm{Im}(\varphi_1(1)),\mathrm{Im}(\varphi_2(1))).$$

Definition 5.8 Given affine schemes $X,Y \in \mathrm{AffSm}/k$, define a functor

$$\Gamma : \mathcal{A}(X,Y) \to A(X,Y)$$

as follows. Given an object $\Psi \in \mathcal{A}(X,Y)$, choose its representative $(n, Z, \psi : p_{X,*}(\mathcal{O}_Z) \to M_n(\mathcal{O}_X))$. This representative gives rise to a pair

$$\Gamma(\Psi) := (n, \varphi : k[Y] \to k[X \times Y] \to \Gamma(Z, \mathcal{O}_Z) \xrightarrow{\psi} M_n(k[X])),$$

which is an object of $A(X,Y)$. Clearly, this pair does not depend on the choice of a representative. One has an equality $\Gamma(X, P(\psi)) = P_X^{\Psi}$, where $P(\psi)$ is defined in Definition 4.5 and P_X^{Ψ} is given by (9). If $\alpha : \Psi_1 \to \Psi_2$ is a morphism in $\mathcal{A}(X,Y)$, then equalizing the supports of Ψ_1 and Ψ_2 and taking the global sections on X, we get an isomorphism

$$\mathrm{Mor}_{\mathcal{A}(X,Y)}(\Psi_1, \Psi_2) = \mathrm{Hom}_{p_{X,*}(\mathcal{O}_Z)}(P(\psi_1), P(\psi_2)) \xrightarrow{\Gamma(\alpha)}$$
$$\xrightarrow{\Gamma(\alpha)} \mathrm{Hom}_{k[X \times Y]}(P_X^{\Psi_1}, P_X^{\Psi_1}) = \mathrm{Hom}_{A(X,Y)}(\Gamma(\Psi_1), \Gamma(\Psi_2)).$$

This completes the definition of the functor Γ.

Lemma 5.9 *The functor $\Gamma : \mathcal{A}(X,Y) \to A(X,Y)$ is an equivalence of additive categories.*

Proof: Define a functor $a : A(X,Y) \to \mathcal{A}(X,Y)$ on objects as follows. An object (n, φ) in $A(X,Y)$ defines a projector $\varphi(1) \in M_n(k[X])$. The image $\mathrm{Im}(\varphi(1))$ in $k[X]^n$ has a $k[Y]$-module structure given by the non-unital homomorphism φ. In this way $\mathrm{Im}(\varphi(1))$ is a $k[X \times Y]$-module. Let $A \subset X \times Y$ be the support of $\mathrm{Im}(\varphi(1))$. Using Notation 4.1, it is easy to see that $A \in \mathrm{Supp}(X \times Y/Y)$. Thus there exists an integer $m \geqslant 0$ such that $I_A^m \cdot \mathrm{Im}(\varphi(1)) = (0)$. The latter means that $\mathrm{Im}(\varphi(1))$ is a $k[X \times Y]/I_A^m$-module, and therefore the non-unital k-algebra homomorphism φ can be presented in the form

$$k[X \times Y] \xrightarrow{can_{A,m}} k[X \times Y]/I_A^m \xrightarrow{\bar{\varphi}_{A,m}} M_n(k[X])$$

for a unique $\bar{\varphi}_{A,m}$. Let $Z = \mathrm{Spec}(k[X \times Y]/I_A^m)$ and let $(\bar{\varphi}_{A,m})^{\sim} : p_{X,*}(\mathcal{O}_Z) \to M_n(\mathcal{O}_X)$ be the sheaf homomorphism associated to $\bar{\varphi}_{A,m}$. Set

$$a(n, \varphi) := \text{ the equivalence class of the triple } (n, Z, (\bar{\varphi}_{A,m})^{\sim}).$$

This equivalence class remains unchanged when enlarging A in $\mathrm{Supp}(X \times Y/Y)$ and the integer m. In fact, if $A' \in \mathrm{Supp}(X \times Y/Y)$ is such that $A \subset A'$ and $m' \geqslant m$, then $I_{A'}^{m'} \subset I_A^m$. Thus $\varphi = \bar{\varphi}_{A',m'} \circ can_{A',m'}$ for a unique $\bar{\varphi}_{A',m'}$. Let $Z' = \mathrm{Spec}(k[X \times Y]/I_{A'}^{m'})$ and let $(\bar{\varphi}_{A',m'})^{\sim} : p_{X,*}(\mathcal{O}_{Z'}) \to M_n(\mathcal{O}_X)$ be the sheaf homomorphism associated to $\bar{\varphi}_{A',m'}$. Clearly, the equivalence class of the triple $(n, Z, (\bar{\varphi}_{A,m})^{\sim})$ coincides with the equivalence class of the triple $(n, Z', (\bar{\varphi}_{A',m'})^{\sim})$.

Define the functor $a : A(X,Y) \to \mathcal{A}(X,Y)$ on morphisms as follows. Let $\alpha : (n_1,\varphi_1) \to (n_2,\varphi_2)$ be a morphism in $A(X,Y)$. Let A_i be the support of the $k[X\times Y]$-module $\mathrm{Im}(\varphi_i(1))$ and let m_i be an integer such that $I_A^{m_i} \cdot \mathrm{Im}(\varphi_i(1)) = (0)$. Enlarging A_1 and A_2 in $\mathrm{Supp}(X \times Y/Y)$ if necessary, we may assume that $A_1 = A = A_2$. Enlarging m_1 and m_2, we may as well assume that $m_1 = m = m_2$. Therefore we may assume that $Z_1 = Z = Z_2$. Now applying the functor from $k[X]$-modules to $\mathcal{O}_X$-modules, we get a homomorphism

$$\begin{aligned}\mathrm{Hom}_{A(X,Y)}((n_1,\varphi_1),(n_1,\varphi_1)) &= \mathrm{Hom}_{k[X\times Y]}(\mathrm{Im}(\varphi_1(1)),\mathrm{Im}(\varphi_2(1))) = \\ &= \mathrm{Hom}_{k[X\times Y]/I_A^m}(\mathrm{Im}(\bar{\varphi}_{1,A,m}(1)),\mathrm{Im}(\bar{\varphi}_{2,A,m})) \\ &\to \mathrm{Hom}_{p_{X,*}(\mathcal{O}_Z)}(\mathrm{Im}(\bar{\varphi}_1(1))^{\sim},\mathrm{Im}(\bar{\varphi}_2(1))^{\sim}).\end{aligned}$$

Set $a(\alpha)$ to be the image of α under this homomorphism. The definition of the functor a is completed.

It is straightforward to check that the functors Γ and a are mutually inverse equivalences of additive categories. For instance, the composite $a \circ \Gamma$ is the identity functor from $A(X,Y)$ to itself. □

Definition 5.10 Define a functor $a_{X,Y} : A(X,Y) \to \mathcal{P}(X,Y)$ as follows. It takes an object (n,φ) to the $\mathcal{O}_{X\times Y}$-module sheaf $\mathrm{Im}(\varphi(1))^{\sim}$ associated with the $k[X \times Y]$-module $\mathrm{Im}(\varphi(1))$ described in Definition 5.7. On morphisms it is defined by the isomorphism

$$\begin{aligned}\mathrm{Hom}_{A(X,Y)}((n_1,\varphi_1),(n_2,\varphi_2)) &= \mathrm{Hom}_{k[X\times Y]}(\mathrm{Im}(\varphi_1(1)),\mathrm{Im}(\varphi_2(1))) \cong \\ \cong \mathrm{Hom}_{\mathcal{O}_{X\times Y}}(\mathrm{Im}(\varphi_1(1))^{\sim},\mathrm{Im}(\varphi_2(1))^{\sim}) &= \mathrm{Hom}_{\mathcal{P}(X,Y)}(a_{X,Y}(n_1,\varphi_1),a_{X,Y}(n_2,\varphi_2)).\end{aligned}$$

Proof of Theorem 5.1: Consider the following square of functors

$$\begin{array}{ccc}\mathcal{A}(X,Y) & \xrightarrow{F_{X,Y}} & \tilde{\mathcal{P}}(X,Y) \\ {\scriptstyle\Gamma}\downarrow & & \downarrow{\scriptstyle R} \\ A(X,Y) & \xrightarrow[a_{X,Y}]{} & \mathcal{P}(X,Y)\end{array}$$

where R takes a big bimodule $P \in \tilde{P}(X,Y)$ to the $\mathcal{O}_{X\times Y}$-module $P_{X,Y} \in \mathcal{P}(X,Y)$ and a morphism $\alpha : P \to Q$ of big bimodules to the morphism $\alpha_{X,Y} : P_{X,Y} \to Q_{X,Y}$ of $\mathcal{O}_{X\times Y}$-modules. We claim that this diagram commutes up to an isomorphism of functors. Since the functors Γ, $a_{X,Y}$, R are equivalences of categories, the functor $F_{X,Y}$ is a category equivalence, too. To complete the proof, it remains to construct a functor isomorphism $a_{X,Y} \circ \Gamma \to R \circ F_{X,Y}$.

Let $\Psi \in \mathcal{A}(X,Y)$ be an object and let $(n,Z,\psi : p_{X,*}(\mathcal{O}_Z) \to M_n(\mathcal{O}_X))$ be a triple representing Ψ (see Definition 4.5). Then $\Gamma(\Psi) = (n,\varphi : k[Y] \to k[X\times Y] \to$

$\Gamma(Z,\mathcal{O}_Z) \xrightarrow{\psi} M_n(k[X]))$ as described in Definition 5.8. Let $\mathrm{Im}(\varphi(1))$ be the $k[X \times Y]$-module described in Definition 5.7. Then $a_{X,Y}(\Gamma(\Psi))$ is the $\mathcal{O}_{X\times Y}$-module sheaf $\mathrm{Im}(\varphi(1))^{\sim}$ associated with the $k[X\times Y]$-module $\mathrm{Im}(\varphi(1))$. On the other hand, following Definition 5.3 and the description of R, one has $R(F_{X,Y}(\Psi)) = \mathcal{P}_X^{\Psi}$. We need to construct an isomorphism $\theta_\Psi : \mathrm{Im}(\varphi(1))^{\sim} \xrightarrow{\cong} \mathcal{P}_X^{\Psi}$, natural in Ψ, of $\mathcal{O}_{X\times Y}$-modules. Giving such a morphism θ_Ψ is the same as giving a $k[X \times Y]$-homomorphism

$$\Theta_\Psi : \mathrm{Im}(\varphi(1)) \to \Gamma(X \times Y, \mathcal{P}_X^{\Psi}).$$

Moreover, θ_Ψ is an isomorphism whenever so is Θ_Ψ. A section of $\mathcal{P}_X^{\Psi}$ on $X\times Y$ is a compatible family of elements $(n_q \in \mathcal{P}_{X,q}^{\Phi})_{q\in X\times Y}$ (see Definitions 5.3 and 5.2). For $s \in \mathrm{Im}(\varphi(1))$, set

$$\Theta_\Psi(s) = \left(\frac{s_{x(q)}}{1}\right) \in \prod_{q\in X\times Y} \mathcal{P}_{X,q}^{\Psi},$$

where $x(q) = p_X(q) \in X$ and $s_{x(q)} \in P_{X,x(q)}^{\Psi}$ is the image of s in $P_{X,x(q)}^{\Psi}$ under the canonical map $P_X^{\Psi} = \mathrm{Im}(p_X^{\Psi}) \to \mathrm{Im}(p_{X,x(q)}^{\Psi}) = P_{X,x(q)}^{\Psi}$ (see the discussion above Definition 5.2). Clearly, $\Theta_\Psi(s)$ belongs to $\Gamma(X \times Y, \mathcal{P}_X^{\Psi})$. We claim that Θ_Ψ is an isomorphism. In fact, if $\frac{s_{x(q)}}{1} = 0$ for all $q \in X \times Y$ then $s = 0$. It follows that Θ_Ψ is injective. If $(n_q) \in \Gamma(X\times Y, \mathcal{P}_X^{\Psi})$, then $(n_q) \in \prod_{q\in X\times Y} \mathcal{P}_{X,q}^{\Psi}$ is a compatible family of elements. It follows from Definition 5.3 that there is a global section s of the sheaf $\mathrm{Im}(\varphi(1))^{\sim}$ such that for each $q \in X \times Y$ one has $s_q = n_q$. Since $\Gamma(X \times Y, \mathrm{Im}(\varphi(1))^{\sim}) = \mathrm{Im}(\varphi(1))$ the map Θ_Ψ is surjective. The fact that the assignment $\Psi \mapsto \Theta_\Psi$ is a functor transformation $a_{X,Y} \circ \Gamma \to R \circ F_{X,Y}$ is obvious. Our theorem now follows. □

Let $DK_{-}^{\mathrm{eff}}(k)$ be the triangulated category of K-motives in the sense of Definition 4.24. Recall that the $\mathbb{K}$-motive $M_{\mathbb{K}}(X)$ of a k-smooth scheme X is the $\mathbb{K}$-module $C_*(\mathbb{K}(-,X))$ (see Definition 3.3 and Notation 4.15). The $\mathbb{K}$-motive $M_{\mathbb{K}}(X)$ belongs to the category $DK_{-}^{\mathrm{eff}}(k)$ as observed just below the proof of Corollary 3.4. To conclude the section, we give the following application of Theorem 5.1.

Theorem 5.11 *Let k be a perfect field and let X be any scheme in Sm/k. Then for every integer $i \in \mathbb{Z}$ there is a natural isomorphism of abelian groups*

$$K_i(X) \cong DK_{-}^{\mathrm{eff}}(k)(M_{\mathbb{K}}(X)[i], M_{\mathbb{K}}(pt)),$$

where $K(X)$ is Quillen's K-theory of X.

A priori, there is no reason for the right hand side to be zero for $i < 0$. However, Theorem 5.1 and the fact that K-theory of X is connective imply this is the case.

Proof: By Theorem 4.23 $\mathbb{K}$ is a strict V-spectral category. By (1) one has a canonical isomorphism for every integer i

$$SH^{\mathrm{nis}}_{S^1}(k)(X[i],\mathbb{K}(-,pt)) \cong SH^{\mathrm{nis}}_{S^1}\mathbb{K}(\mathbb{K}(-,X)[i],\mathbb{K}(-,pt)).$$

Let

$$K : \mathrm{Sm}/k \to Sp^{\Sigma}, \quad X \mapsto K(X) = K(\tilde{\mathcal{P}}(X,pt))$$

be the algebraic K-theory presheaf of symmetric spectra. It follows from Theorem 5.1 that the natural map in $Pre^{\Sigma}(\mathrm{Sm}/k)$

$$F : \mathbb{K}(-,pt) \to K,$$

induced by the additive functors $F_{X,pt} : \mathcal{A}(X,pt) \to \tilde{\mathcal{P}}(X,pt)$, $X \in \mathrm{Sm}/k$, is a Nisnevich local weak equivalence.

Using Thomason's theorem [14] stating that algebraic K-theory satisfies Nisnevich descent, we obtain isomorphisms

$$K_i(X) \cong SH^{\mathrm{nis}}_{S^1}(k)(X[i],K) \cong SH^{\mathrm{nis}}_{S^1}\mathbb{K}(\mathbb{K}(-,X)[i],\mathbb{K}(-,pt)), \quad i \in \mathbb{Z}.$$

Consider a commutative diagram in $Pre^{\Sigma}(\mathrm{Sm}/k)$

$$\begin{tikzcd}
K(\mathcal{A}(-,pt)) \arrow[r] \arrow[d, "F"'] & K(\mathcal{A}(-,pt))_f \arrow[r] \arrow[d, "\delta"'] & |\underline{\mathrm{Hom}}(\Delta^{\cdot},K(\mathcal{A}(-,pt))_f)| \arrow[d, "\gamma"] \\
K \arrow[r, "\alpha"] & K_f \arrow[r, "\beta"] & |\underline{\mathrm{Hom}}(\Delta^{\cdot},K_f)|.
\end{tikzcd}$$

Here the lower f-symbol refers to a fibrant replacement functor in the Nisnevich local model structure on $Pre^{\Sigma}(\mathrm{Sm}/k)$. Theorem 5.1 implies F is a Nisnevich local weak equivalence. By [14] $K(-)$ is Nisnevich excisive, and hence α is a stable weak equivalence. Since $K(-)$ is homotopy invariant, then β is a stable weak equivalence. It follows that δ,γ are stable weak equivalences. Therefore the composition of the upper horizontal maps is a Nisnevich local weak equivalence. Thus the canonical map

$$\mathbb{K}(-,pt) \to M_{\mathbb{K}}(pt)$$

is a Nisnevich local weak equivalence. One has an isomorphism

$$K_i(X) \cong SH^{\mathrm{nis}}_{S^1}\mathbb{K}(\mathbb{K}(-,X)[i],M_{\mathbb{K}}(pt)), \quad i \in \mathbb{Z}.$$

Since $\mathbb{K}(-,X)[i], M_{\mathbb{K}}(pt)$ are bounded below $\mathbb{K}$-modules, then our theorem follows from Theorem 3.5(2). $\square$

A. The category of big bimodules $\tilde{\mathcal{P}}(X,Y)$

Let X,Y be two schemes of finite type over the base field k. We denote by $\mathcal{P}(X,Y)$ the category of coherent $\mathcal{O}_{X\times Y}$-modules $P_{X,Y}$ such that $\mathrm{Supp}(P_{X,Y})$ is finite over X and the coherent $\mathcal{O}_X$-module $(p_X)_*(P_{X,Y})$ is locally free. A disadvantage of the category $\mathcal{P}(X,Y)$ is that whenever we have two maps $f : X \to X'$ and $g : X' \to X''$ then the functor $(g \circ f)^*$ agrees with $f^* \circ g^*$ only up to a canonical isomorphism. We want to replace $\mathcal{P}(X,Y)$ by an equivalent additive category $\tilde{\mathcal{P}}(X,Y)$ which is functorial in both arguments.

To this end, we use the construction which is in spirit like that of Grayson for finitely generated projective modules [5] and Friedlander–Suslin for big vector bundles [1]. Let X be a Noetherian scheme. Consider the big Zariski site Sch/X of all schemes of finite type over X. We define the *category of big bimodules* $\tilde{\mathcal{P}}(X,Y)$ as follows.

An object of $\tilde{\mathcal{P}}(X,Y)$ consists of the following data:

1. For any $U \in Sch/X$ one has a bimodule $P_{U,Y} \in \mathcal{P}(U,Y)$.
2. For any morphism $f : U' \to U$ in Sch/X one has a morphism $\sigma_f : P_{U,Y} \to (f \times 1_Y)_*(P_{U',Y})$ in $\mathcal{P}(U,Y)$ satisfying:
   (a) $\sigma_1 = 1$.
   (b) The morphism $\tau_f : (f \times 1_Y)^*(P_{U,Y}) \to P_{U',Y}$ which is adjoint to σ_f must be an isomorphism in $\mathcal{P}(U',Y)$.
   (c) Given a chain of maps $U'' \xrightarrow{f_1} U' \xrightarrow{f} U$ in Sch/X, the following relation is satisfied
   $$\sigma_{f\circ f_1} = (f \times 1_Y)_*(\sigma_{f_1}) \circ \sigma_f.$$

A morphism of two big bimodules $\alpha : P \to Q$ is a morphism $\alpha_{X,Y} : P_{X,Y} \to Q_{X,Y}$ in $\mathcal{P}(X,Y)$. Clearly, $\tilde{\mathcal{P}}(X,Y)$ is an additive category.

Given a map $g : X' \to X$ of two Noetherian schemes, we define an additive functor
$$g^* : \tilde{\mathcal{P}}(X,Y) \to \tilde{\mathcal{P}}(X',Y)$$
as follows. For any $U \in Sch/X'$ and $P \in \mathcal{P}(X,Y)$ one sets $g^*(P)_{U,Y} = P_{U,Y}$, where U is regarded as an object of Sch/X by means of composition with g. In a similar way, if $h : U' \to U$ is a map in Sch/X' then $\sigma_h : g^*(P)_{U,Y} \to g^*(P)_{U',Y}$ equals σ_h. So we have defined g^* on objects. Let $\alpha : P \to Q$ be a morphism in

$\tilde{\mathcal{P}}(X,Y)$. By definition, it is a morphism $\alpha_{X,Y} : P_{X,Y} \to Q_{X,Y}$ in $\mathcal{P}(X,Y)$. There is a commutative diagram

$$\begin{tikzcd}
(g\times 1_Y)^*(P_{X,Y}) \arrow[r, "\tau_g"] \arrow[d, "(g\times 1_Y)^*(\alpha_{X,Y})"'] & P_{X',Y} \arrow[d, "\alpha_{X',Y}"] \\
(g\times 1_Y)^*(Q_{X,Y}) \arrow[r, "\tau_g"] & Q_{X',Y}
\end{tikzcd}$$

where the horizontal maps are isomorphisms. Then $g^*(\alpha) := \alpha_{X',Y}$. The functor g^* is constructed.

Lemma A.1 *Let $g_1 : X'' \to X'$ and $g : X' \to X$ be two maps of schemes. Then $(g\circ g_1)^* = g_1^* \circ g^*$.*

Proof: This is straightforward. □

Now let us discuss functoriality in Y. For this consider a map $h : Y \to Y'$. We construct an additive functor

$$h_* : \tilde{\mathcal{P}}(X,Y) \to \tilde{\mathcal{P}}(X,Y')$$

in the following way. We set $h_*(P)_{U,Y'} = (1_U \times h)_*(P_{U,Y})$ for any $P \in \tilde{\mathcal{P}}(X,Y)$. If $f : U' \to U$ is a map in Sch/X then

$$(1_U \times h)_*(f\times 1_Y)_* = (f\times 1_{Y'})_*(1_{U'}\times h)_*.$$

We define σ_f for $h_*(P)$ as

$$\begin{aligned}(1_U\times h)_*(\sigma_f) : (1_U\times h)_*(P_{U,Y}) &\to (1_U\times h)_*(f\times 1_Y)_*(P_{U',Y})\\ &= (f\times 1_{Y'})_*(1_{U'}\times h)_*(P_{U',Y}).\end{aligned}$$

By definition, h_* takes a morphism $\alpha_{X,Y}$ in $\mathcal{P}(X,Y)$ to $(1_X\times h)_*(\alpha_{X,Y})$. The construction of the functor h_* is completed.

Lemma A.2 *Let $h_1 : Y' \to Y''$ and $h : Y \to Y'$ be two maps of schemes. Then $(h_1\circ h)_* = (h_1)_* \circ h_*$.*

Proof: This is straightforward. □

We leave the reader to verify the following

Proposition A.3 *The natural functor*

$$R : \tilde{\mathcal{P}}(X,Y) \to \mathcal{P}(X,Y), \quad P \mapsto P_{X,Y},$$

is an equivalence of additive categories.

By Lemmas A.1-A.2 $\tilde{\mathcal{P}}(X,Y)$ has the desired functoriality properties in both arguments.

REFERENCES

1. E. M. Friedlander, A. Suslin, *The spectral sequence relating algebraic K-theory to motivic cohomology*, Ann. Sci. École Norm. Sup. (4) **35** (2002), 773–875.
2. G. Garkusha, I. Panin, *K-motives of algebraic varieties*, Homology Homotopy Appl. **14**(2) (2012), 211–264.
3. G. Garkusha, I. Panin, *On the motivic spectral sequence*, preprint arXiv:1210.2242.
4. T. Geisser, L. Hesselholt, *Topological cyclic homology of schemes*, in Algebraic K-theory (Seattle, WA, 1997), Proc. Sympos. Pure Math. **67**, Amer. Math. Soc., Providence, RI, 1999, pp. 41–87.
5. D. Grayson, *Weight filtrations via commuting automorphisms*, K-Theory **9** (1995), 139–172.
6. M. Hovey, B. Shipley, J. Smith, *Symmetric spectra*, J. Amer. Math. Soc. **13** (2000), 149–208.
7. B. Keller, *Derived categories and their uses,* In Handbook of Algebra, vol. 1, North-Holland, Amsterdam, 1996, pp. 671–701.
8. A. Knizel, *Homotopy invariant presheaves with Kor-transfers*, MSc Diploma, St. Petersburg State University, 2012.
9. F. Morel, *The stable* $\mathbb{A}^1$*-connectivity theorems*, K-Theory **35** (2006), 1–68.
10. F. Morel, V. Voevodsky, $\mathbb{A}^1$*-homotopy theory of schemes*, Publ. Math. IHES **90** (1999), 45–143.
11. S. Schwede, *An untitled book project about symmetric spectra*, available at www.math.uni-bonn.de/~schwede (version v3.0/April 2012).
12. S. Schwede, B. Shipley, *Stable model categories are categories of modules*, Topology **42**(1) (2003), 103–153.
13. A. Suslin, V. Voevodsky, *Bloch–Kato conjecture and motivic cohomology with finite coefficients*, The Arithmetic and Geometry of Algebraic Cycles (Banff, AB, 1998), NATO Sci. Ser. C Math. Phys. Sci. **548**, Kluwer Acad. Publ., Dordrecht (2000), pp. 117–189.
14. R. W. Thomason, T. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift III, Progress in Mathematics **88**, Birkhäuser, 1990, pp. 247–435.
15. V. Voevodsky, *Cohomological theory of presheaves with transfers, in Cycles, Transfers and Motivic Homology Theories* (V. Voevodsky, A. Suslin and E. Friedlander, eds.), Annals of Math. Studies, Princeton University Press, 2000.
16. V. Voevodsky, *Triangulated category of motives over a field, in Cycles, Transfers and Motivic Homology Theories* (V. Voevodsky, A. Suslin and E. Friedlander, eds.), Annals of Math. Studies, Princeton University Press, 2000.

17. F. Waldhausen, *Algebraic K-theory of spaces*, In Algebraic and geometric topology, Proc. Conf., New Brunswick/USA 1983, Lecture Notes in Mathematics **1126**, Springer-Verlag, 1985, pp. 318–419.

18. M. E. Walker, *Motivic cohomology and the K-theory of automorphisms*, PhD Thesis, University of Illinois at Urbana-Champaign, 1996.

GRIGORY GARKUSHA g.garkusha@swansea.ac.uk

Department of Mathematics
Swansea University
Singleton Park, Swansea SA2 8PP
United Kingdom

IVAN PANIN paniniv@gmail.com

St. Petersburg Branch of V. A. Steklov Mathematical Institute
Fontanka 27
191023 St. Petersburg
Russia
and
St. Petersburg State University
Department of Mathematics and Mechanics
Universitetsky prospekt, 28
198504, Peterhof, St. Petersburg
Russia

Received: June 27, 2013