GROTHENDIECK CATEGORIES OF ENRICHED FUNCTORS

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ABSTRACT. It is shown that the category of enriched functors $[\mathscr{C}, \mathscr{V}]$ is Grothendieck whenever $\mathscr{V}$ is a closed symmetric monoidal Grothendieck category and $\mathscr{C}$ is a category enriched over $\mathscr{V}$. Localizations in $[\mathscr{C}, \mathscr{V}]$ associated to collections of objects of $\mathscr{C}$ are studied. Also, the category of chain complexes of generalized modules $\mathrm{Ch}(\mathscr{C}_R)$ is shown to be identified with the Grothendieck category of enriched functors $[\mathrm{mod}\,R, \mathrm{Ch}(\mathrm{Mod}\,R)]$ over a commutative ring R, where the category of finitely presented R-modules $\mathrm{mod}\,R$ is enriched over the closed symmetric monoidal Grothendieck category $\mathrm{Ch}(\mathrm{Mod}\,R)$ as complexes concentrated in zeroth degree. As an application, it is proved that $\mathrm{Ch}(\mathscr{C}_R)$ is a closed symmetric monoidal Grothendieck model category with explicit formulas for tensor product and internal Hom-objects. Furthermore, the class of unital algebraic almost stable homotopy categories generalizing unital algebraic stable homotopy categories of Hovey–Palmieri–Strickland [14] is introduced. It is shown that the derived category of generalized modules $\mathscr{D}(\mathscr{C}_R)$ over commutative rings is a unital algebraic almost stable homotopy category which is not an algebraic stable homotopy category.

1. INTRODUCTION

In the present paper we study categories of enriched functors

$$[\mathscr{C}, \mathscr{V}],$$

where $\mathscr{V}$ is a closed symmetric monoidal Grothendieck category and $\mathscr{C}$ is a category enriched over $\mathscr{V}$ (i.e. a $\mathscr{V}$-category). The main result here states that the category $[\mathscr{C}, \mathscr{V}]$ is Grothendieck with an explicit collection of generators. Namely, the following theorem is true.

Theorem. *Let $\mathscr{V}$ be a closed symmetric monoidal Grothendieck category with a set of generators $\{g_i\}_I$. If $\mathscr{C}$ is a small $\mathscr{V}$-category, then the category of enriched functors $[\mathscr{C}, \mathscr{V}]$ is a Grothendieck $\mathscr{V}$-category with the set of generators $\{\mathscr{V}(c,-) \oslash g_i \mid c \in \mathrm{Ob}\,\mathscr{C}, i \in I\}$. Moreover, if $\mathscr{C}$ is a small symmetric monoidal $\mathscr{V}$-category, then $[\mathscr{C}, \mathscr{V}]$ is closed symmetric monoidal with explicit formulas for monoidal product and internal* Hom-*object.*

Taking into account this theorem, we refer to $[\mathscr{C}, \mathscr{V}]$ as a *Grothendieck category of enriched functors*. The usual Grothendieck category of additive functors

$$(\mathscr{B}, \mathrm{Ab})$$

from a pre-additive category $\mathscr{B}$ to abelian groups Ab is recovered from the preceding theorem in the case when $\mathscr{V} = \mathrm{Ab}$ ($\mathscr{B}$ is a $\mathscr{V}$-category). Further examples on how the category $[\mathscr{C}, \mathscr{V}]$ recovers some Grothendieck categories are given in Section 4.

The following result is an extension to enriched categories of similar results of [6, 9].

Theorem. *Suppose $\mathscr{V}$ is a closed symmetric monoidal Grothendieck category. Let $\mathscr{C}$ be a $\mathscr{V}$-category and let $\mathscr{P}$ consist of a collection of objects of $\mathscr{C}$. Let $\mathscr{S}_{\mathscr{P}} = \{G \in [\mathscr{C}, \mathscr{V}] \mid G(p) = 0 \text{ for all } p \in \mathscr{P}\}$. Then $\mathscr{S}_{\mathscr{P}}$ is a localizing subcategory of $[\mathscr{C}, \mathscr{V}]$ and $[\mathscr{P}, \mathscr{V}]$ is equivalent to the quotient category $[\mathscr{C}, \mathscr{V}]/\mathscr{S}_{\mathscr{P}}$.*

We apply Grothendieck categories of enriched functors to study homological algebra for generalized modules. The category of generalized modules

$$\mathscr{C}_R = (\mathrm{mod}\, R, \mathrm{Ab})$$

consists of the additive functors from the category of finitely presented R-modules, $\mathrm{mod}\, R$, to the category of abelian groups, Ab. Its morphisms are the natural transformations of functors. It is called the category of generalized R-modules for the reason that there is a fully faithful, right exact functor

$$M \mapsto -\otimes_R M$$

from the category of all R-modules to $\mathscr{C}_R$.

The category $\mathscr{C}_R$ has a number of remarkable properties which led to powerful applications in ring and module theory and representation theory (see, e.g., the books by Prest [20, 21]). The category $\mathscr{C}_R$ also provides a natural connecting langauge between algebra and model theory of modules [12, 20, 21].

The following theorem states that the category $\mathrm{Ch}(\mathscr{C}_R)$ of chain complexes of $\mathscr{C}_R$ over a commutative ring can be regarded as a Grothendieck category of enriched functors.

Theorem. *Suppose R is a commutative ring. Then the category of chain complexes of generalized R-modules $\mathrm{Ch}(\mathscr{C}_R)$ can naturally be identified with the Grothendieck category of enriched functors $[\mathrm{mod}\, R, \mathrm{Ch}(\mathrm{Mod}\, R)]$, where the category of finitely presented modules $\mathrm{mod}\, R$ is naturally enriched over $\mathrm{Ch}(\mathrm{Mod}\, R)$ as complexes concentrated in zeroth degree.*

A Grothendieck category of enriched functors $[\mathscr{C}, \mathscr{V}]$ can also contain a homotopy information whenever $\mathscr{V}$ is a reasonable model category in the sense of Quillen [22]. As an application of the preceding theorem, we show the following

Theorem. *Let R be a commutative ring, then $\mathrm{Ch}(\mathscr{C}_R)$ is a left and right proper closed symmetric monoidal $\mathscr{V}$-model category, where $\mathscr{V} = \mathrm{Ch}(\mathrm{Mod}\, R)$. The tensor product of two complexes $F_\bullet, G_\bullet \in \mathrm{Ch}(\mathscr{C}_R)$ is given by*

$$F_\bullet \odot G_\bullet = \int^{(M,N) \in \mathrm{mod}\, R \otimes \mathrm{mod}\, R} F_\bullet(M) \otimes_R G_\bullet(N) \otimes_R \mathrm{Hom}_R(M \otimes_R N, -).$$

Here $\mathrm{Hom}_R(M \otimes_R N, -)$ is regarded as a complex concentrated in zeroth degree. The internal Hom-object is defined as

$$\underline{\mathrm{Hom}}(F_\bullet, G_\bullet)(M) = \int_{N \in \mathrm{mod}\, R} \underline{\mathrm{Hom}}_{\mathrm{Ch}(\mathrm{Mod}\, R)}(F_\bullet(N), G_\bullet(M \otimes_R N)).$$

There are various ways to construct closed symmetric monoidal structures on the derived category of a reasonable closed symmetric abelian category (see [13, 25] for some examples). For the case of the derived category $\mathscr{D}(\mathscr{C}_R)$ of generalized modules with R a commutative ring we apply the preceding theorem as well as some facts for compactly generated triangulated categories to establish the following

Theorem. *Let R be a commutative ring. Then the derived category $\mathscr{D}(\mathscr{C}_R)$ of the Grothendieck category $\mathscr{C}_R$ is a compactly generated triangulated closed symmetric monoidal category, where*

the above formulas yield the derived tensor product $F_\bullet \odot^L G_\bullet$ and derived internal Hom-*object* $R\underline{\mathrm{Hom}}(F_\bullet, G_\bullet)$. *The compact objects of $\mathscr{D}(\mathscr{C}_R)$ are the complexes isomorphic to bounded complexes of coherent functors.*

In the classical stable homotopy theory (see, for example, Hovey–Palmieri–Strickland [14]) the category of compact objects of a stable homotopy category possesses a duality, which in some cases is also known as the Spanier–Whitehead Duality. In order to find a duality on the category of compact objects $\mathscr{D}(\mathscr{C}_R)^c$ of $\mathscr{D}(\mathscr{C}_R)$, we use the Auslander–Gruson–Jensen Duality [1, 10, 12] for coherent objects $\mathrm{coh}\,\mathscr{C}_R$. In the model theory of modules, this duality corresponds to elementary duality, introduced by Prest [20, Chapter 8] and developed by Herzog in [11], for positive-primitive formulas. We show that the Auslander–Gruson–Jensen Duality makes sense for compact objects of $\mathscr{D}(\mathscr{C}_R)$. More precisely, the following result is true.

Theorem (Auslander–Gruson–Jensen Duality for compact objects). *Let $\mathscr{D}(\mathscr{C}_R)^c$ be the full triangulated subcategory of $\mathscr{D}(\mathscr{C}_R)$ of compact objects. Then there is a duality*

$$D : (\mathscr{D}(\mathscr{C}_R)^c)^{\mathrm{op}} \to \mathscr{D}(\mathscr{C}_R)^c$$

that takes a compact object $C_\bullet$ to $DC_\bullet := R\underline{\mathrm{Hom}}(C_\bullet, - \otimes_R R)$.

Basing on the above results for $\mathscr{D}(\mathscr{C}_R)$, we introduce the class of unital algebraic almost stable homotopy categories. These essentially the same with unital algebraic stable homotopy categories in the sense of Hovey–Palmieri–Strickland [14] except that the compact objects do not have to be strongly dualizable, but must have a duality. We finish the paper by proving the following

Theorem. *Let R be a commutative ring. Then $\mathscr{D}(\mathscr{C}_R)$ is a unital algebraic almost stable homotopy category, which is not an algebraic stable homotopy category in the sense of Hovey–Palmieri–Strickland.*

2. ENRICHED CATEGORY THEORY

In this section we collect basic facts about enriched categories we shall need later. We refer the reader to [2, 18] for details. Throughout this paper $(\mathscr{V}, \otimes, \underline{\mathrm{Hom}}, e)$ is a closed symmetric monoidal category with monoidal product $\otimes$, internal Hom-object $\underline{\mathrm{Hom}}$ and monoidal unit e. We sometimes write $[a,b]$ to denote $\underline{\mathrm{Hom}}(a,b)$, where $a, b \in \mathrm{Ob}\,\mathscr{V}$. We have structure isomorphisms in $\mathscr{V}$

$$a_{abc} : (a \otimes b) \otimes c \to a \otimes (b \otimes c), \quad l_a : e \otimes a \to a, \quad r_a : a \otimes e \to a$$

with $a, b, c \in \mathrm{Ob}\,\mathscr{V}$.

Definition 2.1. A $\mathscr{V}$-*category* $\mathscr{C}$, or *a category enriched over* $\mathscr{V}$, consists of the following data:
  (1) a class $\mathrm{Ob}\,(\mathscr{C})$ of objects;
  (2) for every pair $a, b \in \mathrm{Ob}\,(\mathscr{C})$ of objects, an object $\mathscr{V}_{\mathscr{C}}(a,b)$ of $\mathscr{V}$;
  (3) for every triple $a, b, c \in \mathrm{Ob}\,(\mathscr{C})$ of objects, a composition morphism in $\mathscr{V}$,

$$c_{abc} : \mathscr{V}_{\mathscr{C}}(a,b) \otimes \mathscr{V}_{\mathscr{C}}(b,c) \to \mathscr{V}_{\mathscr{C}}(a,c);$$

  (4) for every object $a \in \mathscr{C}$, a unit morphism $u_a : e \to \mathscr{V}_{\mathscr{C}}(a,a)$ in $\mathscr{V}$.
     These data must satisfy the following conditions:
     ⋄ given objects $a, b, c, d \in \mathscr{C}$, diagram (1) below is commutative (associativity axiom);

$\diamond$ given objects $a, b \in \mathscr{C}$, diagram (2) below is commutative (unit axiom).

$$(\mathscr{V}_{\mathscr{C}}(a,b) \otimes \mathscr{V}_{\mathscr{C}}(b,c)) \otimes \mathscr{V}_{\mathscr{C}}(c,d) \xrightarrow{\ c_{abc} \otimes 1\ } \mathscr{V}_{\mathscr{C}}(a,c) \otimes \mathscr{V}_{\mathscr{C}}(c,d) \tag{1}$$

$$a_{\mathscr{V}_{\mathscr{C}}(a,b)\mathscr{V}_{\mathscr{C}}(b,c)\mathscr{V}_{\mathscr{C}}(c,d)} \downarrow$$

$$\mathscr{V}_{\mathscr{C}}(a,b) \otimes (\mathscr{V}_{\mathscr{C}}(b,c) \otimes \mathscr{V}_{\mathscr{C}}(c,d))$$

$$1 \otimes c_{bcd} \downarrow \qquad\qquad c_{acd}$$

$$\mathscr{V}_{\mathscr{C}}(a,b) \otimes \mathscr{V}_{\mathscr{C}}(b,d) \xrightarrow{\qquad c_{abd} \qquad} \mathscr{V}_{\mathscr{C}}(a,d)$$

$$e \otimes \mathscr{V}_{\mathscr{C}}(a,b) \xrightarrow{\ l_{\mathscr{V}_{\mathscr{C}}(a,b)}\ } \mathscr{V}_{\mathscr{C}}(a,b) \xleftarrow{\ r_{\mathscr{V}_{\mathscr{C}}(a,b)}\ } \mathscr{V}_{\mathscr{C}}(a,b) \otimes e \tag{2}$$

$$u_a \otimes 1 \downarrow \qquad\qquad 1_{\mathscr{V}_{\mathscr{C}}(a,b)} \downarrow \qquad\qquad 1 \otimes u_b \downarrow$$

$$\mathscr{V}_{\mathscr{C}}(a,a) \otimes \mathscr{V}_{\mathscr{C}}(a,b) \xrightarrow{\ c_{aab}\ } \mathscr{V}_{\mathscr{C}}(a,b) \xleftarrow{\ c_{abb}\ } \mathscr{V}_{\mathscr{C}}(a,b) \otimes \mathscr{V}_{\mathscr{C}}(b,b)$$

When $\mathrm{Ob}\,\mathscr{C}$ is a set, the $\mathscr{V}$-category $\mathscr{C}$ is called a *small $\mathscr{V}$-category*.

Definition 2.2. Given $\mathscr{V}$-categories $\mathscr{A}, \mathscr{B}$, a *$\mathscr{V}$-functor* or an *enriched functor* $F : \mathscr{A} \to \mathscr{B}$ consists in giving:

(1) for every object $a \in \mathscr{A}$, an object $F(a) \in \mathscr{B}$;
(2) for every pair $a, b \in \mathscr{A}$ of objects, a morphism in $\mathscr{V}$,

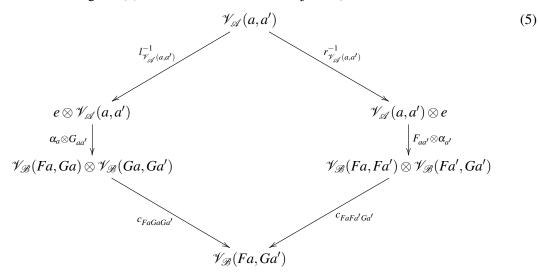$$F_{ab} : \mathscr{V}_{\mathscr{A}}(a,b) \to \mathscr{V}_{\mathscr{B}}(F(a), F(b))$$

in such a way that the following axioms hold:
$\diamond$ for all objects $a, a', a'' \in \mathscr{A}$, diagram (3) below commutes (composition axiom);
$\diamond$ for every object $a \in \mathscr{A}$, diagram (4) below commutes (unit axiom).

$$\mathscr{V}_{\mathscr{A}}(a,a') \otimes \mathscr{V}_{\mathscr{A}}(a',a'') \xrightarrow{\ c_{aa'a''}\ } \mathscr{V}_{\mathscr{A}}(a,a'') \tag{3}$$

$$F_{aa'} \otimes F_{a'a''} \downarrow \qquad\qquad\qquad \downarrow F_{aa''}$$

$$\mathscr{V}_{\mathscr{B}}(Fa, Fa') \otimes \mathscr{V}_{\mathscr{B}}(Fa', Fa'') \xrightarrow{\ c_{Fa,Fa',Fa''}\ } \mathscr{V}_{\mathscr{B}}(Fa, Fa'')$$

$$e \xrightarrow{\ u_a\ } \mathscr{V}_{\mathscr{A}}(a,a) \tag{4}$$

$$u_{Fa} \searrow \qquad \downarrow F_{aa}$$

$$\mathscr{V}_{\mathscr{B}}(Fa, Fa)$$

Definition 2.3. Let $\mathscr{A}, \mathscr{B}$ be two $\mathscr{V}$-categories and $F, G : \mathscr{A} \to \mathscr{B}$ two $\mathscr{V}$-functors. A *$\mathscr{V}$-natural transformation* $\alpha : F \Rightarrow G$ consists in giving, for every object $a \in \mathscr{A}$, a morphism

$$\alpha_a : e \to \mathscr{V}_{\mathscr{B}}(F(a), G(a))$$

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in $\mathscr{V}$ such that diagram (5) below commutes, for all objects $a, a' \in \mathscr{A}$.

$$\mathscr{V}_{\mathscr{A}}(a,a') \tag{5}$$

$$l^{-1}_{\mathscr{V}_{\mathscr{A}}(a,a')} \qquad\qquad r^{-1}_{\mathscr{V}_{\mathscr{A}}(a,a')}$$

$$e \otimes \mathscr{V}_{\mathscr{A}}(a,a') \qquad\qquad \mathscr{V}_{\mathscr{A}}(a,a') \otimes e$$

$$\alpha_a \otimes G_{aa'} \downarrow \qquad\qquad \downarrow F_{aa'} \otimes \alpha_{a'}$$

$$\mathscr{V}_{\mathscr{B}}(Fa,Ga) \otimes \mathscr{V}_{\mathscr{B}}(Ga,Ga') \qquad\qquad \mathscr{V}_{\mathscr{B}}(Fa,Fa') \otimes \mathscr{V}_{\mathscr{B}}(Fa',Ga')$$

$$c_{FaGaGa'} \qquad\qquad c_{FaFa'Ga'}$$

$$\mathscr{V}_{\mathscr{B}}(Fa,Ga')$$

By **Set** we shall mean the closed symmetric monoidal category of sets. Categories in the usual sense are **Set**-categories (categories enriched over **Set**). If $\mathscr{A}$ is a category, let $\mathbf{Set}_{\mathscr{A}}(a,b)$ denote the set of maps in $\mathscr{A}$ from a to b. The closed symmetric monoidal category $\mathscr{V}$ is a $\mathscr{V}$-category due to its internal Hom-objects. Any $\mathscr{V}$-category $\mathscr{C}$ defines a **Set**-category $\mathscr{U}\mathscr{C}$, also called the *underlying category*. Its class of objects is $\mathrm{Ob}\,\mathscr{C}$, the morphism sets are $\mathbf{Set}_{\mathscr{U}\mathscr{C}}(a,b) = \mathbf{Set}_{\mathscr{V}}(e, \mathscr{V}_{\mathscr{C}}(a,b))$ (see [2, p. 316]).

Proposition 2.4. *Let $\mathscr{V}$ be a symmetric monoidal closed category. If $\mathscr{A}$ is a $\mathscr{V}$-category and $F, G : \mathscr{A} \rightrightarrows \mathscr{V}$ are $\mathscr{V}$-functors, giving a $\mathscr{V}$-natural transformation $\alpha : F \Rightarrow G$ is equivalent to giving a family of morphism $\alpha : F(a) \to G(a)$ in $\mathscr{V}$, for $a \in \mathscr{A}$, in such a way that the following diagram commutes for all $a, a' \in \mathscr{A}$*

$$\begin{array}{ccc}
\mathscr{V}_{\mathscr{A}}(a,a') & \xrightarrow{F_{aa'}} & [F(a), F(a')] \\
G_{aa'} \downarrow & & \downarrow [1, \alpha_{a'}] \\
[G(a), G(a')] & \xrightarrow[{[\alpha_a, 1]}]{} & [F(a), G(a')]
\end{array}$$

Proof. See [2, 6.2.8]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 2.5. *Let $\mathscr{V}$ be a symmetric monoidal closed category. If $\mathscr{A}$ is a $\mathscr{V}$-category and $F, G : \mathscr{A} \rightrightarrows \mathscr{V}$ are $\mathscr{V}$-functors, giving a $\mathscr{V}$-natural transformation $\alpha : F \Rightarrow G$ is equivalent to giving a family of morphism $\alpha : F(a) \to G(a)$ in $\mathscr{V}$, for $a \in \mathscr{A}$, in such a way that the following diagram commutes for all $a, a' \in \mathscr{A}$*

$$\begin{array}{ccc}
\mathscr{V}_{\mathscr{A}}(a,a') \otimes F(a) & \xrightarrow{\eta_F} & F(a') \\
1 \otimes \alpha_a \downarrow & & \downarrow \alpha_{a'} \\
\mathscr{V}_{\mathscr{A}}(a,a') \otimes G(a) & \xrightarrow{\eta_G} & G(a'),
\end{array}$$

where η_F, η_G are the morphisms corresponding to the structure morphisms $F_{aa'}$ and $G_{aa'}$ respectively.

Corollary 2.6. *Let $\mathscr{V}$ be a symmetric monoidal closed category. If $\mathscr{A}$ is a small $\mathscr{V}$-category and $F, G : \mathscr{A} \rightrightarrows \mathscr{V}$ are $\mathscr{V}$-functors, suppose $\alpha : F \Rightarrow G$ is a $\mathscr{V}$-natural transformation such that each $\alpha_a : F(a) \to G(a)$, $a \in \mathrm{Ob}\,\mathscr{A}$, is an isomorphism in $\mathscr{V}$. Then α is an isomorphism in $[\mathscr{A}, \mathscr{V}]$.*

Proof. This follows from the preceding corollary if we define $\alpha^{-1} : G \Rightarrow F$ by the collection of arrows α_a^{-1}, $a \in \mathrm{Ob}\,\mathscr{A}$. $\qquad\square$

Let $\mathscr{C}, \mathscr{D}$ be two $\mathscr{V}$-categories. The *monoidal product* $\mathscr{C} \otimes \mathscr{D}$ is the $\mathscr{V}$-category, where

$$\mathrm{Ob}(\mathscr{C} \otimes \mathscr{D}) := \mathrm{Ob}\,\mathscr{C} \times \mathrm{Ob}\,\mathscr{D}$$

and

$$\mathscr{V}_{\mathscr{C} \otimes \mathscr{D}}((a,x),(b,y)) := \mathscr{V}_{\mathscr{C}}(a,b) \otimes \mathscr{V}_{\mathscr{D}}(x,y), \quad a, b \in \mathscr{C}, x, y \in \mathscr{D}.$$

Definition 2.7. A $\mathscr{V}$-category $\mathscr{C}$ is a *right $\mathscr{V}$-module* if there is a $\mathscr{V}$-functor $\mathrm{act} : \mathscr{C} \otimes \mathscr{V} \to \mathscr{C}$, denoted $(c, A) \mapsto c \oslash A$ and a $\mathscr{V}$-natural unit isomorphism $r_c : \mathrm{act}(c, e) \to c$ subject to the following conditions:

(1) there are coherent natural associativity isomorphisms $c \oslash (A \otimes B) \to (c \oslash A) \otimes B$;
(2) the isomorphisms $c \oslash (e \otimes A) \rightrightarrows c \oslash A$ coincide.

A right $\mathscr{V}$-module is *closed* if there is a $\mathscr{V}$-functor

$$\mathrm{coact} : \mathscr{V}^{\mathrm{op}} \otimes \mathscr{C} \to \mathscr{C}$$

such that for all $A \in \mathrm{Ob}\,\mathscr{V}$, and $c \in \mathrm{Ob}\,\mathscr{C}$, the $\mathscr{V}$-functor $\mathrm{act}(-, A) : \mathscr{C} \to \mathscr{C}$ is left $\mathscr{V}$-adjoint to $\mathrm{coact}(A, -)$ and $\mathrm{act}(c, -) : \mathscr{V} \to \mathscr{C}$ is left $\mathscr{V}$-adjoint to $\mathscr{V}_{\mathscr{C}}(c, -)$.

If $\mathscr{C}$ is a small $\mathscr{V}$-category, $\mathscr{V}$-functors from $\mathscr{C}$ to $\mathscr{V}$ and their $\mathscr{V}$-natural transformations form the category $[\mathscr{C}, \mathscr{V}]$ of $\mathscr{V}$-functors from $\mathscr{C}$ to $\mathscr{V}$. If $\mathscr{V}$ is complete, then $[\mathscr{C}, \mathscr{V}]$ is also a $\mathscr{V}$-category. We also denote this $\mathscr{V}$-category by $\mathscr{F}(\mathscr{C})$, or $\mathscr{F}$ if no confusion can arise. The morphism $\mathscr{V}$-object $\mathscr{V}_{\mathscr{F}}(X, Y)$ is the end

$$\int_{\mathrm{Ob}\,\mathscr{C}} \mathscr{V}(X(c), Y(c)). \tag{6}$$

Note that the underlying category $\mathscr{U}\mathscr{F}$ of the $\mathscr{V}$-category $\mathscr{F}$ is $[\mathscr{C}, \mathscr{V}]$.

Given $c \in \mathrm{Ob}\,\mathscr{C}$, $X \mapsto X(c)$ defines the $\mathscr{V}$-functor $\mathrm{Ev}_c : \mathscr{F} \to \mathscr{V}$ called *evaluation at c*. The assignment $c \mapsto \mathscr{V}_{\mathscr{C}}(c, -)$ from $\mathscr{C}$ to $\mathscr{F}$ is again a $\mathscr{V}$-functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{F}$, called the *$\mathscr{V}$-Yoneda embedding*. $\mathscr{V}_{\mathscr{C}}(c, -)$ is a representable functor, represented by c.

Lemma 2.8 (The Enriched Yoneda Lemma). *Let $\mathscr{V}$ be a complete closed symmetric monoidal category and $\mathscr{C}$ a small $\mathscr{V}$-category. For every $\mathscr{V}$-functor $X : \mathscr{C} \to \mathscr{V}$ and every $c \in \mathrm{Ob}\,\mathscr{C}$, there is a $\mathscr{V}$-natural isomorphism $X(c) \cong \mathscr{V}_{\mathscr{F}}(\mathscr{V}_{\mathscr{C}}(c, -), X)$.*

Lemma 2.9. *If $\mathscr{V}$ is a bicomplete closed symmetric monoidal category and $\mathscr{C}$ is a small $\mathscr{V}$-category, then $[\mathscr{C}, \mathscr{V}]$ is bicomplete. (Co)limits are formed pointwise.*

Proof. See [2, 6.6.17]. $\qquad\square$

Corollary 2.10. *Assume $\mathscr{V}$ is bicomplete, and let $\mathscr{C}$ be a small $\mathscr{V}$-category. Then any $\mathscr{V}$-functor $X : \mathscr{C} \to \mathscr{V}$ is $\mathscr{V}$-naturally isomorphic to the coend*

$$X \cong \int^{\mathrm{Ob}\,\mathscr{C}} \mathscr{V}_{\mathscr{C}}(c, -) \oslash X(c).$$

Proof. See [2, 6.6.18]. $\qquad\square$

A *monoidal $\mathscr{V}$-category* is a $\mathscr{V}$-category $\mathscr{C}$ together with a $\mathscr{V}$-functor $\diamond : \mathscr{C} \otimes \mathscr{C} \to \mathscr{C}$, a unit $u \in \mathrm{Ob}\,\mathscr{C}$, a $\mathscr{V}$-natural associativity isomorphism and two $\mathscr{V}$-natural unit isomorphisms. Symmetric monoidal and closed symmetric monoidal $\mathscr{V}$-categories are defined similarly.

Suppose $(\mathscr{C}, \diamond, u)$ is a small symmetric monoidal $\mathscr{V}$-category where $\mathscr{V}$ is bicomplete. In [4], a closed symmetric monoidal product was constructed on the category $[\mathscr{C}, \mathscr{V}]$ of $\mathscr{V}$-functors from $\mathscr{C}$ to $\mathscr{V}$. For $X, Y \in \mathrm{Ob}[\mathscr{C}, \mathscr{V}]$, the monoidal product $X \odot Y \in \mathrm{Ob}[\mathscr{C}, \mathscr{V}]$ is the coend

$$X \odot Y := \int^{\mathrm{Ob}(\mathscr{C} \otimes \mathscr{C})} \mathscr{V}_{\mathscr{C}}(c \diamond d, -) \otimes (X(c) \otimes Y(d)) : \mathscr{C} \to \mathscr{V}. \tag{7}$$

The following theorem is due to Day [4] and plays an important role in our analysis.

Theorem 2.11 (Day [4]). *Let $(\mathscr{V}, \otimes, e)$ be a bicomplete closed symmetric monoidal category and $(\mathscr{C}, \diamond, u)$ a small symmetric monoidal $\mathscr{V}$-category. Then $([\mathscr{C}, \mathscr{V}], \odot, \mathscr{V}_{\mathscr{C}}(u, -))$ is a closed symmetric monoidal category. The internal* Hom-*functor in $[\mathscr{C}, \mathscr{V}]$ is given by the end*

$$\mathscr{F}(X, Y)(c) = \mathscr{V}_{\mathscr{F}}(X, Y(c \diamond -)) = \int_{d \in \mathrm{Ob}\,\mathscr{C}} \mathscr{V}(X(d), Y(c \diamond d)). \tag{8}$$

The next lemma computes the tensor product of representable $\mathscr{V}$-functors.

Lemma 2.12. *The tensor product of representable functors is again representable. Precisely, there is a natural isomorphism*

$$\mathscr{V}_{\mathscr{C}}(c, -) \odot \mathscr{V}_{\mathscr{C}}(d, -) \cong \mathscr{V}_{\mathscr{C}}(c \diamond d, -).$$

3. GROTHENDIECK CATEGORIES

In this section we collect basic facts about Grothendieck categories. We mostly follow Herzog [12] and Stenström [24].

Definition 3.1. A family $\{U_i\}_I$ of objects of an abelian category $\mathscr{A}$ is a *family of generators* for $\mathscr{A}$ if for each non-zero morphism $\alpha : B \to C$ in $\mathscr{A}$ there exist a morphism $\beta : U_i \to B$ for some $i \in I$, such that $\alpha\beta \neq 0$.

Recall that an abelian category is *cocomplete* or an *Ab3-category* if it has arbitrary direct sums. The cocomplete abelian category $\mathscr{C}$ is said to be an *Ab5-category* if for any directed family $\{A_i\}_{i \in I}$ if subobjects of A and for any subobject B of A, the relation

$$\left(\sum_{i \in I} A_i\right) \cap B = \sum_{i \in I} (A_i \cap B)$$

holds.

The condition *Ab*3 is equivalent to the existance of arbitrary direct limits. Also *Ab*5 is equivalent to the fact that there exist inductive limits and the inductive limits over directed families of indices are exact, i.e. if I is a directed set and

$$0 \longrightarrow A_i \longrightarrow B_i \longrightarrow C_i \longrightarrow 0$$

is an exact sequence for any $i \in I$, then

$$0 \longrightarrow \varinjlim A_i \longrightarrow \varinjlim B_i \longrightarrow \varinjlim C_i \longrightarrow 0$$

is an exact sequence.

An abelian category which satisfies the condition *Ab*5 and which possesses a family of generators is called a *Grothendieck category*.

Example 3.2. Given any associative ring R with identity, the category $\operatorname{Mod} R$ of R-modules is a Grothendieck category, where R is a generator.

Example 3.3. By [12], another example of a Grothendieck category is the category of additive functors $(\mathscr{B}, \operatorname{Ab})$ from a small preadditive category $\mathscr{B}$ to the category of abelian groups Ab, in which limits and colimits of functors are defined objectwise. A family of projective generators for $(\mathscr{B}, \operatorname{Ab})$ is given by the collection of representable functors $\{h^B\}_{B \in \operatorname{Ob} \mathscr{B}}$. In what follows we shall also write $(B, -)$ to denote the representable functor h^B, $B \in \operatorname{Ob} \mathscr{B}$.

Below we shall need the following useful fact:

Proposition 3.4. *The category of unbounded chain complexes* $\operatorname{Ch}(\mathscr{A})$ *of a Grothendieck category $\mathscr{A}$ is again a Grothendieck category.*

Proof. Colimits and limits are taken dimensionwise, filtered colimits are obviously exact. Following notation of Hovey [13], denote by $D^n X$, $n \in \mathbb{Z}$ and $X \in \mathscr{A}$, the complex which is X in degree n and $n-1$ and 0 elsewhere, with interesting differential being the identity map. If U is a generator of $\mathscr{A}$, then $\{D^n U\}_{n \in \mathbb{Z}}$ are generators of $\operatorname{Ch}(\mathscr{A})$. To see that these generate $\operatorname{Ch}(\mathscr{A})$, use the adjunction relation $\operatorname{Hom}_{\operatorname{Ch}(\mathscr{A})}(D^n U, X) \cong \operatorname{Hom}_{\mathscr{A}}(U, X_n)$. $\qquad\square$

Remark 3.5. This remark is to warn the reader that one should not confuse generators in abelian and triangulated categories. Precisely, we shall also work with the derived category $\mathscr{D}(\mathscr{A})$ of unbounded complexes of a Grothendieck category $\mathscr{A}$. Then generators for $\mathscr{D}(\mathscr{A})$ cannot be generators for $\operatorname{Ch}(\mathscr{A})$ and vice versa in general. Indeed, the generators $\{D^n U\}_{n \in \mathbb{Z}}$ of $\operatorname{Ch}(\mathscr{A})$ are contractible complexes, and hence zero in $\mathscr{D}(\mathscr{A})$.

On the other hand, suppose U is a generator for $\mathscr{A}$. Denote by $S^n U$, $n \in \mathbb{Z}$, the complex which is U in degree n and 0 elsewhere. Then $\{S^n U\}_{n \in \mathbb{Z}}$ is a family of generators for the derived category $\mathscr{D}(\mathscr{A})$ in the sense that for every non-zero object $X \in \mathscr{D}(\mathscr{A})$ there is a non-zero morphism in $\mathscr{D}(\mathscr{A})$ from some $S^n U$ to X. But these cannot generate $\operatorname{Ch}(A)$ as the following example shows.

Suppose K is a field, and

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & 0 & \longrightarrow & K & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \\
& \downarrow & & {\scriptstyle \alpha}\downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & 0 & \longrightarrow & K \oplus K & \overset{d_0}{\longrightarrow} & K & \longrightarrow & 0 & \longrightarrow \cdots \\
& \downarrow & & {\scriptstyle f}\downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & 0 & \longrightarrow & K & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots
\end{array}
$$

is a commutative diagram in $\operatorname{Ch}(\operatorname{Mod} K)$, where $d_0(x,y) = f(x,y) = y$. We suppose the middle complex is concentrated in degrees 0 and -1. Clearly $\alpha(1) \in \operatorname{Ker} d_0$ implies $\alpha(1) = (x', 0)$ for some $x' \in K$. But $f\alpha(1) = f(x', 0) = 0$, so $f\alpha = 0$. Thus there is no non-zero map from $S^0 K$ to the middle complex such that the composite $f\alpha \neq 0$. Since there is no non-zero morphism from $S^n K$ to the middle complex such that $f\alpha \neq 0$ for any $n \neq 0$, we see that $\{S^n K\}_{n \in \mathbb{Z}}$ are not generators for $\operatorname{Ch}(\operatorname{Mod} K)$.

Definition 3.6. Recall that an object $A \in \mathscr{C}$ is *finitely generated* if whenever there are subobjects $A_i \subseteq A$ for $i \in I$ satisfying $A = \sum_{i \in I} A_i$, then there is a finite subset $J \subset I$ such that $A = \sum_{i \in J} A_i$. The category of finitely generated subobjects of $\mathscr{C}$ is denoted by $\operatorname{fg}\mathscr{C}$. The category is *locally finitely generated* provided that every object $X \in \mathscr{C}$ is a directed sum $X = \sum_{i \in I} X_i$ of finitely generated subobjects X_i, or equivalently, $\mathscr{C}$ possesses a family of finitely generated generators.

Definition 3.7. A finitely generated object $B \in \mathscr{C}$ is *finitely presented* provided that every epimorphism $\eta : A \to B$ with A finitely generated has a finitely generated kernel $\mathrm{Ker}\,\eta$. The subcategory of finitely presented objects of $\mathscr{C}$ is denoted by $\mathrm{fp}\,\mathscr{C}$. The corresponding categories of finitely presented left and right R-modules over the ring R are denoted by $R\,\mathrm{mod} = \mathrm{fp}(R\,\mathrm{Mod})$ and $\mathrm{mod}\,R = \mathrm{fp}(\mathrm{Mod}\,R)$, respectively. Note that the subcategory $\mathrm{fp}\,\mathscr{C}$ of $\mathscr{C}$ is closed under extensions. Moreover, if

$$0 \to A \to B \to C \to 0$$

is a short exact sequence in $\mathscr{C}$ with B finitely presented, then C is finitely presented if and only if A is finitely generated. The category is *locally finitely presented* provided that it is locally finitely generated and every object $X \in \mathscr{C}$ is a direct limit $X = \varinjlim_{i \in I} X_i$ of finitely presented objects X_i, or equivalently, $\mathscr{C}$ possesses a family of finitely presented generators.

Definition 3.8. A finitely presented object C of a locally finitely presented Grothendieck category $\mathscr{C}$ is *coherent* if every finitely generated subobject B of C is finitely presented. Equivalently, every epimorphism $h : C \to A$ with A finitely presented has a finitely presented kernel. Evidently, a finitely generated subobject of a coherent object is also coherent. The subcategory of coherent objects of $\mathscr{C}$ is denoted by $\mathrm{coh}\,\mathscr{C}$. The category $\mathscr{C}$ is *locally coherent* provided that it is locally finitely presented and every object $X \in \mathscr{C}$ is a direct limit $X = \varinjlim_{i \in I} X_i$ of coherent objects X_i, or equivalently, $\mathscr{C}$ possesses a family of coherent generators.

The subcategories consisting of finitely generated, finitely presented and coherent objects are ordered by inclusion as follows:

$$\mathscr{C} \supseteq \mathrm{fg}\,\mathscr{C} \supseteq \mathrm{fp}\,\mathscr{C} \supseteq \mathrm{coh}\,\mathscr{C}.$$

Examples 3.9. (1) The categories $\mathrm{Mod}\,R$ and $(\mathscr{B}, \mathrm{Ab})$ (see Examples 3.2-3.3) are locally finitely presented. The representable functors $\{h^B\}_{B \in \mathscr{B}}$ are projective finitely presented generators of $(\mathscr{B}, \mathrm{Ab})$ (see [12] for details). The category $\mathrm{Ch}(\mathscr{A})$ with $\mathscr{A}$ a Grothendieck category (see Proposition 3.4) is locally finitely presented whenever $\mathscr{A}$ is. If $\{U_i\}_{i \in I}$ are finitely presented generators of $\mathscr{A}$, then $\{D^n U_i\}_{n \in \mathbb{Z}, i \in I}$ are finitely presented generators of $\mathrm{Ch}(\mathscr{A})$.

(2) $\mathrm{Mod}\,R$ is locally coherent if and only if the ring R is right coherent. $(\mathscr{B}, \mathrm{Ab})$ is locally coherent whenever $\mathscr{B}$ is closed under cokernels (see [12] for details). In turn, $\mathrm{Ch}(\mathscr{A})$ with $\mathscr{A}$ a Grothendieck category is locally coherent whenever $\mathscr{A}$ is.

4. GROTHENDIECK CATEGORIES OF ENRICHED FUNCTORS

In this section we prove that the category of enriched functors $[\mathscr{C}, \mathscr{V}]$ is a Grothendieck category whenever $\mathscr{V}$ is a closed symmetric monoidal Grothendieck category, giving us new Grothendieck categories in practice. Also, the localization theory of Grothendieck categories becomes available for $[\mathscr{C}, \mathscr{V}]$. Moreover, $[\mathscr{C}, \mathscr{V}]$ is closed symmetric monoidal whenever $\mathscr{C}$ is a symmetric monoidal $\mathscr{V}$-category.

Here are some examples of closed symmetric monoidal Grothendieck categories.

Examples 4.1. (1) Given any commutative ring R, the triple $(\mathrm{Mod}\,R, \otimes_R, R)$ is a closed symmetric monoidal Grothendieck category.

(2) More generally, let X be a quasi-compact quasi-separated scheme. Consider the category $\mathrm{Qcoh}(\mathscr{O}_X)$ of quasi-coherent $\mathscr{O}_X$-modules. By [15, 3.1] $\mathrm{Qcoh}(\mathscr{O}_X)$ is a locally finitely presented Grothendieck category, where quasi-coherent $\mathscr{O}_X$-modules of finite type form a family

of finitely presented generators. The tensor product on $\mathscr{O}_X$-modules preserves quasi-coherence, and induces a closed symmetric monoidal structure on $\mathrm{Qcoh}(\mathscr{O}_X)$.

(3) Let R be any commutative ring. Let $C' = \{C'_n, \partial'_n\}$ and $C'' = \{C''_n, \partial''_n\}$ be two chain complexes of R-modules. Their tensor product $C' \otimes_R C'' = \{(C' \otimes_R C'')_n, \partial_n\}$ is the chain complex defined by

$$(C' \otimes_R C'')_n = \bigoplus_{i+j=n} (C'_i \otimes_R C''_j),$$

and

$$\partial_n(t'_i \otimes s''_j) = \partial'_i(t'_i) \otimes s''_j + (-1)^i t'_i \otimes \partial''_j(s''_j), \quad \text{for all } t'_i \in C'_i, \ s''_j \in C''_j, \ (i+j=n),$$

where $C'_i \otimes_R C''_j$ denotes the tensor product of R-modules C'_i and C''_j. Then the triple $(\mathrm{Ch}(\mathrm{Mod}\,R), \otimes_R, R)$ is a closed symmetric monoidal Grothendieck category. Here R is regarded as a complex concentrated in the zeroth degree.

(4) $(\mathrm{Mod}\,kG, \otimes_k, k)$ is closed symmetric monoidal Grothendieck category, where k is a field and G is a finite group.

The main result of this section is as follows.

Theorem 4.2. *Let $\mathscr{V}$ be a closed symmetric monoidal Grothendieck category with a set of generators $\{g_i\}_I$. If $\mathscr{C}$ is a small $\mathscr{V}$-category, then the category of enriched functors $[\mathscr{C}, \mathscr{V}]$ is a Grothendieck $\mathscr{V}$-category with the set of generators $\{\mathscr{V}(c,-) \oslash g_i \mid c \in \mathrm{Ob}\,\mathscr{C}, i \in I\}$. Moreover, if $\mathscr{C}$ is a small symmetric monoidal $\mathscr{V}$-category, then $[\mathscr{C}, \mathscr{V}]$ is closed symmetric monoidal with monoidal product and internal* Hom-*object computed by the formulas* (7) *and* (8) *of Day.*

Proof. If $\mathscr{C}$ is a small $\mathscr{V}$-category, then $[\mathscr{C}, \mathscr{V}]$ is a $\mathscr{V}$-category (see p. 6). The internal Hom-object is given by (6). Let show that $[\mathscr{C}, \mathscr{V}]$ is a preadditive category. Given $\mathscr{V}$-functors $X, Y \in [\mathscr{C}, \mathscr{V}]$, we have that

$$\mathrm{Hom}_{[\mathscr{C},\mathscr{V}]}(X,Y) = \mathrm{Hom}_{\mathscr{V}}\left(e, \int_{c \in \mathrm{Ob}\,\mathscr{C}} \mathscr{V}(X(c), Y(c))\right)$$

is an abelian group, because $\mathscr{V}$ is preadditive. We can also describe explicitly the abelian group structure as follows. The morphisms of $[\mathscr{C}, \mathscr{V}]$ are, by definition, the $\mathscr{V}$-functors from $\mathscr{C}$ to $\mathscr{V}$. Using Corollary 2.5, for any $\mathscr{V}$-natural transformations

$$\alpha, \alpha' : X \to Y$$

its sum $\alpha + \alpha'$ is determined by the arrows

$$\alpha_c + \alpha'_c : e \to \mathscr{V}(X(c), Y(c)).$$

Recall that $\mathrm{Hom}_{\mathscr{V}}(X(c), Y(c)) = \mathrm{Hom}_{\mathscr{V}}(e, \mathscr{V}(X(c), Y(c)))$ and $\alpha_c + \alpha'_c$ is addition of α_c and α'_c in the abelian group $\mathrm{Hom}_{\mathscr{V}}(X(c), Y(c))$.

To show that the addition is bilinear, let

$$\beta \in \mathrm{Hom}_{\mathscr{V}}\left(e, \int_c \mathscr{V}(Y(c), Z(c))\right) = \int_c \mathrm{Hom}_{\mathscr{V}}(e, \mathscr{V}(Y(c), Z(c)))$$

$$= \int_c \mathrm{Hom}_{\mathscr{V}}(Y(c), Z(c)).$$

Using Corollary 2.5, we set

$$(\beta\alpha)_c := \beta_c \circ \alpha_c.$$

Using the fact that $\mathscr{V}$ is preadditive, we have

$$(\beta(\alpha+\alpha'))_c = \beta_c(\alpha_c+\alpha'_c)$$
$$= \beta_c \circ \alpha_c + \beta_c \circ \alpha'_c$$
$$= (\beta\alpha+\beta\alpha')_c.$$

Similarly, $(\alpha+\alpha')\gamma = \alpha\gamma+\alpha'\gamma$. We see that $[\mathscr{C},\mathscr{V}]$ is preadditive.

Since $\mathscr{V}$ is a bicomplete closed symmetric monoidal category and $\mathscr{C}$ is a small $\mathscr{V}$-category, then by Lemma 2.9 the category $[\mathscr{C},\mathscr{V}]$ is bicomplete. Moreover, limits and colimits are formed objectwise. In particular, $[\mathscr{C},\mathscr{V}]$ has finite products. It follows from [18, VIII.2.2] that $[\mathscr{C},\mathscr{V}]$ is an additive category. Furthermore, $[\mathscr{C},\mathscr{V}]$ has kernels and cokernels which are defined objectwise.

Given a morphism α in $[\mathscr{C},\mathscr{V}]$, the canonical map

$$\bar{\alpha} : \mathrm{Coker}(\ker\alpha) \to \mathrm{Ker}(\mathrm{coker}\,\alpha)$$

is an isomorphism objectwise. Corollary 2.6 implies $\bar{\alpha}$ is an isomorphism. It follows that $[\mathscr{C},\mathscr{V}]$ is an abelian category.

Next, direct limits exist in $[\mathscr{C},\mathscr{V}]$ and are defined objectwise. They are exact in $[\mathscr{C},\mathscr{V}]$, because direct limits in $\mathscr{V}$ are exact (by assumption, $\mathscr{V}$ is a Grothendieck category). So, $[\mathscr{C},\mathscr{V}]$ is an Ab5-category.

It remains to find generators for $[\mathscr{C},\mathscr{V}]$. By [5, 2.4] $[\mathscr{C},\mathscr{V}]$ is a closed $\mathscr{V}$-module, and hence there is an action

$$\oslash : [\mathscr{C},\mathscr{V}]\otimes\mathscr{V} \to [\mathscr{C},\mathscr{V}].$$

Now for any non-zero functor $X \in [\mathscr{C},\mathscr{V}]$ we have natural isomorphisms

$$\mathrm{Hom}_{[\mathscr{C},\mathscr{V}]}(\mathscr{V}(c,-)\oslash g_i, X)) \cong \mathrm{Hom}_{\mathscr{V}}(g_i, \mathscr{V}_{[\mathscr{C},\mathscr{V}]}(\mathscr{V}(c,-),X))$$
$$\cong \mathrm{Hom}_{\mathscr{V}}(g_i, X(c)).$$

Let $\alpha : X \to Y$ be a non-zero map in $[\mathscr{C},\mathscr{V}]$. We want to show that there are $i \in I$, a map $\beta : \mathscr{V}(c,-)\oslash g_i \to X$ such that $\alpha\beta \neq 0$.

Since α is non-zero, then $\alpha_c : X(c) \to Y(c)$ is a non-zero map in $\mathscr{V}$ for some $c \in \mathrm{Ob}\,\mathscr{C}$. By assumption $\{g_i\}_I$ are generators of $\mathscr{V}$, and so there is a map $\bar{\beta} : g_i \to X(c)$ such that $\alpha_c\bar{\beta} \neq 0$. By the above isomorphism we can find a unique map $\beta : \mathscr{V}(c,-)\oslash g_i \to X$ corresponding to $\bar{\beta}$. Now $\alpha_c\bar{\beta} \neq 0$ implies $\alpha\beta \neq 0$ as required.

If $\mathscr{C}$ is a small symmetric monoidal $\mathscr{V}$-category, then $[\mathscr{C},\mathscr{V}]$ is closed symmetric monoidal by Day's Theorem 2.11. The monoidal product and internal Hom-object are computed by formulas of Day (7) and (8). $\square$

Below we give a couple of examples illustrating the preceding theorem.

Example 4.3. Let R be a commutative ring with unit. Consider the closed symmetric monoidal category $\mathscr{V} = (\mathrm{Mod}\,R, \otimes_R, R)$. Consider a $\mathscr{V}$-category $\mathscr{C}$ defined as follows. Its objects are integers $\mathrm{Ob}\,\mathscr{C} = \mathbb{Z}$. Given two integers $m,n \in \mathrm{Ob}\,\mathscr{C}$, we define the Hom-object as

$$\mathscr{V}_{\mathscr{C}}(m,n) = \begin{cases} 0 & \text{if } m \neq n; \\ R & \text{if } m = n. \end{cases}$$

Clearly, $\mathscr{C}$ is a $\mathscr{V}$-category and the category $[\mathscr{C},\mathscr{V}]$ of $\mathscr{V}$-functors from $\mathscr{C}$ to $\mathscr{V}$ is the product category $\prod_{\mathbb{Z}} \mathrm{Mod}\,R$. By definition, $\mathrm{Ob}(\prod_{\mathbb{Z}} \mathrm{Mod}\,R)$ are tuples $(M_i)_{i\in\mathbb{Z}}$ and morphisms are tuples of R-homomorphisms $(f_i : M_i \to N_i)_{i\in\mathbb{Z}}$.

Let $\mathrm{Gr}\,R$ be the category of $\mathbb{Z}$-graded R-modules and graded homomorphisms. It is easy to see that the functor

$$\prod_{\mathbb{Z}}\mathrm{Mod}\,R \to \mathrm{Gr}\,R, \quad (M_i)_{i\in\mathbb{Z}} \mapsto \oplus_{\mathbb{Z}}M_i,$$

is an isomorphism of categories. It is well known that $\mathrm{Gr}\,R$ is a closed symmetric monoidal Grothendieck category with tensor product

$$(M \otimes_R N)_k := \bigoplus_{i+j=k} M_i \otimes_R N_j \text{ for all } M,N \in \mathrm{Gr}\,R. \tag{9}$$

We want to show that this tensor product is recovered from Day's theorem for $[\mathscr{C},\mathscr{V}]$. Indeed, we define a symmetric monoidal product on $\mathscr{C}$ as follows. For every $m,n \in \mathrm{Ob}\,\mathscr{C}$

$$m \boxtimes n := m+n.$$

Given $a \in \mathscr{V}_{\mathscr{C}}(m,m) = R$ and $b \in \mathscr{V}_{\mathscr{C}}(n,n) = R$, we set

$$a \boxtimes b = a \cdot b \in \mathscr{V}_{\mathscr{C}}(m+n,m+n) = R.$$

Clearly, $m \boxtimes n = n \boxtimes m$. Since R is commutative, it follows that $\boxtimes$ defines a strictly symmetric monoidal tensor product on $\mathscr{C}$.

Now for every $M,N \in [\mathscr{C},\mathscr{V}] \cong \mathrm{Gr}\,R$, Day's theorem implies

$$M \odot N = \int^{(m,n)\in\mathbb{Z}\boxtimes\mathbb{Z}} M(m) \otimes_R N(n) \otimes_R \mathscr{V}_{\mathscr{C}}(m \boxtimes n, -)$$
$$= \int^{(m,n)\in\mathbb{Z}\boxtimes\mathbb{Z}} M_m \otimes_R N_n \otimes_R \mathscr{V}_{\mathscr{C}}(m+n, -).$$

Thus,

$$(M \odot N)_k = \bigoplus_{m+n=k} M_m \otimes_R N_n$$

and

$$\underline{\mathrm{Hom}}(M,N)(n) = \int_{m\in\mathrm{Ob}\,\mathscr{C}=\mathbb{Z}} \mathrm{Hom}_R(M(m),N(m \boxtimes n))$$
$$= \int_m \mathrm{Hom}_R(M_m, N_{m+n}) = \mathrm{Hom}_{\mathrm{Gr}\,R}(M,N(n)).$$

So the tensor product (9) as well as internal Hom-functor for graded modules are recovered from Day's theorem.

By Theorem 4.2 $[\mathscr{C},\mathscr{V}]$ is Grothendieck with $\{R(n)\}_{n\in\mathbb{Z}}$ generators, where

$$R(n)_m = \begin{cases} 0, & \text{if } m \neq n; \\ R, & \text{if } m = n. \end{cases}$$

It follows that $\mathrm{Gr}\,R$ is Grothendieck. Moreover, $[\mathscr{C},\mathscr{V}] \cong \mathrm{Gr}\,R$ is closed symmetric monoidal.

Example 4.4. Let R be a commutative ring. Recall that an R-linear DG-category $\mathscr{C}$ is just a category enriched over $\mathscr{V} = \mathrm{Ch}(\mathrm{Mod}\,R)$. Right DG-modules over the DG-category $\mathscr{C}$ are just contravariant $\mathscr{V}$-functors from $\mathscr{C}$ to $\mathscr{V}$. In turn, DG-morphisms of DG-modules are nothing but $\mathscr{V}$-natural transformations. The category of right DG-modules and DG-morphisms is denoted by $\mathrm{Mod}\,\mathscr{C}$. Using our notation, one has, by definition, $\mathrm{Mod}\,\mathscr{C} = [\mathscr{C}^{\mathrm{op}},\mathscr{V}]$.

Theorem 4.2 and Proposition 3.4 imply $\mathrm{Mod}\,\mathscr{C}$ is a Grothendieck category with the family of generators

$$D^n(\mathscr{V}_{\mathscr{C}}(-,c)) := \mathscr{V}_{\mathscr{C}}(-,c) \oslash D^n R, \quad c \in \mathrm{Ob}\,\mathscr{C}, n \in \mathbb{Z}.$$

Here $D^n R$ stands for the complex which is R in degrees n and $n-1$ and zero elsewhere, with interesting differential being the identity.

Example 4.5. Any preadditive category $\mathscr{B}$ is nothing but a category enriched over abelian groups $\mathscr{V} = \mathrm{Ab}$. $\mathscr{V}$-functors from $\mathscr{B}$ to $\mathscr{V}$ are the same as additive functors. Theorem 4.2 says that the category of additive functors $(\mathscr{B}, \mathrm{Ab})$ is Grothendieck with representable functors $\{h^B = \mathscr{V}_{\mathscr{B}}(B, -) \oslash \mathbb{Z}\}_{B \in \mathscr{B}}$ being a family of generators. Thus the fact that $(\mathscr{B}, \mathrm{Ab})$ is Grothendieck (see Example 3.3) follows from Theorem 4.2.

5. LOCALIZATION WITH RESPECT TO ENRICHED SUBCATEGORIES

We say that a full subcategory $\mathscr{S}$ of an abelian category $\mathscr{C}$ is a *Serre subcategory* if for any short exact sequence
$$0 \to X \to Y \to Z \to 0$$
in $\mathscr{C}$ an object $Y \in \mathscr{S}$ if and only if $X, Z \in \mathscr{S}$. A Serre subcategory $\mathscr{S}$ of a Grothendieck category $\mathscr{C}$ is *localizing* if it is closed under taking direct limits. Equivalently, the inclusion functor $i : \mathscr{S} \to \mathscr{C}$ admits the right adjoint $t = t_{\mathscr{S}} : \mathscr{C} \to \mathscr{S}$ which takes every object $X \in \mathscr{C}$ to the maximal subobject $t(X)$ of X belonging to $\mathscr{S}$. The functor t we call the *torsion functor*. An object C of $\mathscr{C}$ is said to be $\mathscr{S}$-*torsionfree* if $t(C) = 0$. Given a localizing subcategory $\mathscr{S}$ of $\mathscr{C}$ the *quotient category* $\mathscr{C}/\mathscr{S}$ consists of $C \in \mathscr{C}$ such that $t(C) = t^1(C) = 0$, where t^1 stands for the first derived functor associated with t. The objects from $\mathscr{C}/\mathscr{S}$ we call $\mathscr{S}$-*closed objects*. Given $C \in \mathscr{C}$ there exists a canonical exact sequence
$$0 \to A' \to C \xrightarrow{\lambda_C} C_{\mathscr{S}} \to A'' \to 0$$
with $A' = t(C)$, $A'' \in \mathscr{S}$, and where $C_{\mathscr{S}} \in \mathscr{C}/\mathscr{S}$ is the maximal essential extension of $\widetilde{C} = C/t(C)$ such that $C_{\mathscr{S}}/\widetilde{C} \in \mathscr{S}$. The object $C_{\mathscr{S}}$ is uniquely defined up to a canonical isomorphism and is called the $\mathscr{S}$-*envelope* of C. Moreover, the inclusion functor $\iota : \mathscr{C}/\mathscr{S} \to \mathscr{C}$ has the left adjoint *localizing functor* $(-)_{\mathscr{S}} : \mathscr{C} \to \mathscr{C}/\mathscr{S}$, which is also exact. It takes each $C \in \mathscr{C}$ to $C_{\mathscr{S}} \in \mathscr{C}/\mathscr{S}$. Then,
$$\mathrm{Hom}_{\mathscr{C}}(X, Y) \cong \mathrm{Hom}_{\mathscr{C}/\mathscr{S}}(X_{\mathscr{S}}, Y)$$
for all $X \in \mathscr{C}$ and $Y \in \mathscr{C}/\mathscr{S}$.

If $\mathscr{C}$ and $\mathscr{D}$ are Grothendieck categories, $q : \mathscr{C} \to \mathscr{D}$ is an exact functor, and a functor $s : \mathscr{D} \to \mathscr{C}$ is fully faithful and right adjoint to q, then $\mathscr{S} := \mathrm{Ker}\, q$ is a localizing subcategory and there exists an equivalence $\mathscr{C}/\mathscr{S} \overset{H}{\cong} \mathscr{D}$ such that $H \circ (-)_{\mathscr{S}} = q$. We shall refer to the pair (q, s) as the *localization pair*.

Example 5.1. Let $\mathscr{A}$ be a small preadditive category. Consider the category $(\mathscr{A}, \mathrm{Ab})$ of additive functors from $\mathscr{A}$ to Ab. Let $p \in \mathrm{Ob}\,\mathscr{A}$, then we have
$$\mathrm{Hom}_{(\mathscr{A}, \mathrm{Ab})}((p, -), (p, -)) = \mathrm{End}_{\mathscr{A}}\, p.$$
Let
$$\mathscr{S}_p = \{F \in (\mathscr{A}, \mathrm{Ab}) \mid F(p) = 0\}.$$
Then $\mathscr{S}_p$ is a localizing subcategory of $(\mathscr{A}, \mathrm{Ab})$. By [6] and [9] there is an equivalence of categories
$$(\mathscr{A}, \mathrm{Ab})/\mathscr{S}_p \cong \mathrm{Mod}(\mathrm{End}_{\mathscr{A}}\, p)^{\mathrm{op}}.$$
This result has some applications in ring and module theory (see [7, 8]).

More generally, given a collection of objects $\mathscr{P}$ in $\mathscr{A}$, we can consider the localizing subcategory

$$\mathscr{S}_{\mathscr{P}} = \{F \in (\mathscr{A}, \mathrm{Ab}) \mid F(p) = 0 \text{ for all } p \in \mathscr{P}\}$$

of $(\mathscr{A}, \mathrm{Ab})$ and then

$$(\mathscr{A}, \mathrm{Ab})/\mathscr{S}_{\mathscr{P}} \cong (\mathscr{P}, \mathrm{Ab}),$$

where $\mathscr{P}$ on the right hand side is regarded as a full subcategory in $\mathscr{A}$ (see [6] for details).

Our next goal is to obtain an enriched analog of this result.

Suppose $\mathscr{V}$ is a closed symmetric monoidal Grothendieck category. Let $\mathscr{C}$ be a $\mathscr{V}$-category. By Theorem 4.2 $[\mathscr{C}, \mathscr{V}]$ is a Grothendieck category. Suppose $\mathscr{P}$ is a collection of objects in $\mathscr{C}$. We shall also regard $\mathscr{P}$ as a natural $\mathscr{V}$-subcategory. Then

$$\mathscr{S}_{\mathscr{P}} = \{F \in [\mathscr{C}, \mathscr{V}] \mid F(p) = 0 \text{ for all } p \in \mathscr{P}\}$$

is localizing in $[\mathscr{C}, \mathscr{V}]$.

We shall prove below that there is a natural equivalence of Grothendieck categories

$$[\mathscr{C}, \mathscr{V}]/\mathscr{S}_{\mathscr{P}} \cong [\mathscr{P}, \mathscr{V}].$$

Thus the same result of [6, 9] for the category of additive functors $(\mathscr{A}, \mathrm{Ab})$ with $\mathscr{A}$ preaddtive (see Example 5.1 above) is recovered from the case when $\mathscr{V} = \mathrm{Ab}$. But first we prove the following

Proposition 5.2. *Suppose $\mathscr{V}$ is a closed symmetric monoidal Grothendieck category. Let $\mathscr{C}$ be a $\mathscr{V}$-category and let $\mathscr{P}$ consist of a collection of objects of $\mathscr{C}$. Then the inclusion map $i : \mathscr{P} \to \mathscr{C}$ is a $\mathscr{V}$-functor. It induces two adjoint functors*

$$i_* : [\mathscr{P}, \mathscr{V}] \rightleftarrows [\mathscr{C}, \mathscr{V}] : i^*$$

where i_* is the enriched left Kan extension and i^* is just restriction to $\mathscr{P}$.

Proof. Although this fact is a consequence of [2, 6.7.7], we give a proof here for the convenience of the reader.

If $F \in [\mathscr{P}, \mathscr{V}]$ then by Corollary 2.10 we have

$$F \cong \int^{\mathrm{Ob}\,\mathscr{P}} \mathscr{V}(p, -) \oslash F(p).$$

By definition of the left Kan extension, we have

$$i_* F \cong \int^{\mathrm{Ob}\,\mathscr{P}} \mathscr{V}(i(p), -) \oslash F(p).$$

We want to show that

$$\mathrm{Hom}_{[\mathscr{C}, \mathscr{V}]}(i_* F, G) \cong \mathrm{Hom}_{[\mathscr{P}, \mathscr{V}]}(F, i^* G).$$

Using (6), one has isomorphisms of $\mathscr{V}$-objects

$$\mathscr{V}_{\mathscr{F}}(F, i^* G) = \mathscr{V}_{\mathscr{F}}(F, G \circ i) = \int_{\mathrm{Ob}\,\mathscr{P}} \mathscr{V}(F(p), G(i(p))). \tag{10}$$

On the other hand,

$$\mathscr{V}_{\mathscr{F}}(i_*F,G) = \mathscr{V}_{\mathscr{F}}\left(\int^{\mathrm{Ob}\,\mathscr{P}} \mathscr{V}(i(p),-) \oslash F(p), G\right)$$

$$= \int_{\mathrm{Ob}\,\mathscr{P}} \mathscr{V}_{\mathscr{F}}(F(p), \mathscr{V}_{[\mathscr{C},\mathscr{V}]}(\mathscr{V}((i(p),-),G)))$$

$$= \int_{\mathrm{Ob}\,\mathscr{P}} \mathscr{V}(F(p), G(i(p))). \tag{11}$$

We have used here the fact that the functor $\mathscr{V}_{\mathscr{F}}(-,G)$ takes $\mathscr{V}$-coends to $\mathscr{V}$-ends [2, 6.6.11] as well as the fact that $[\mathscr{C},\mathscr{V}]$ is a closed $\mathscr{V}$-module. Now (10) and (11) imply i_* and i^* are adjoint functors. $\qquad\square$

Let $\mathscr{S}_{\mathscr{P}} = \{G \in [\mathscr{C},\mathscr{V}] \mid G(p) = 0 \text{ for all } p \in \mathscr{P}\}$. Obviously, $\mathscr{S}_{\mathscr{P}}$ is localizing. Denote by $\varkappa : [\mathscr{P},\mathscr{V}] \to [\mathscr{C},\mathscr{V}]/\mathscr{S}_{\mathscr{P}}$ the composition of the left Kan extension functor $i_* : [\mathscr{P},\mathscr{V}] \to [\mathscr{C},\mathscr{V}]$ and the localization functor $(-)_{\mathscr{S}_{\mathscr{P}}} : [\mathscr{C},\mathscr{V}] \to [\mathscr{C},\mathscr{V}]/\mathscr{S}_{\mathscr{P}}$.

Theorem 5.3. *The functor* $\varkappa : [\mathscr{P},\mathscr{V}] \to [\mathscr{C},\mathscr{V}]/\mathscr{S}_{\mathscr{P}}$ *is an equivalence of categories.*

Proof. First observe that

$$i^* i_* F \cong F$$

for all $F \in [\mathscr{P},\mathscr{V}]$. Second, given $G \in [\mathscr{C},\mathscr{V}]$ the adjunction map $\beta : i_* i^* G \to G$ is such that $\mathrm{Ker}\,\beta, \mathrm{Coker}\,\beta \in \mathscr{S}_{\mathscr{P}}$. Indeed, applying the exact functor i^* to the exact sequence

$$\mathrm{Ker}\,\beta \rightarrowtail i_* i^* G \xrightarrow{\;\beta\;} G \twoheadrightarrow \mathrm{Coker}\,\beta,$$

we get an exact sequence in $[\mathscr{P},\mathscr{V}]$

$$i^*(\mathrm{Ker}\,\beta) \rightarrowtail i^* i_* i^* G \xrightarrow{\;i^*(\beta)\;} i^* G \twoheadrightarrow i^*(\mathrm{Coker}\,\beta).$$

Since the composite map

$$i^* G \to i^* i_* i^* G \xrightarrow{\;i^*(\beta)\;} i^* G$$

is the identity map and the left arrow is an isomorphism, then so is the right arrow. Thus

$$i^*(\mathrm{Ker}\,\beta) = i^*(\mathrm{Coker}\,\beta) = 0,$$

and hence $\mathrm{Ker}\,\beta, \mathrm{Coker}\,\beta \in \mathscr{S}_{\mathscr{P}}$. It also follows that

$$(i_* i^* G)_{\mathscr{S}_{\mathscr{P}}} \cong G_{\mathscr{S}_{\mathscr{P}}}. \tag{12}$$

We have for all $F, F' \in [\mathscr{P},\mathscr{V}]$

$$\mathrm{Hom}_{[\mathscr{C},\mathscr{V}]/\mathscr{S}_{\mathscr{P}}}(\varkappa(F), \varkappa(F')) \cong \mathrm{Hom}_{[\mathscr{C},\mathscr{V}]/\mathscr{S}_{\mathscr{P}}}((i_*(F)_{\mathscr{S}_{\mathscr{P}}}, i_*(F')_{\mathscr{S}_{\mathscr{P}}})$$

$$\cong \mathrm{Hom}_{[\mathscr{C},\mathscr{V}]}(i_*F, (i_*F')_{\mathscr{S}_{\mathscr{P}}})$$

$$\cong \mathrm{Hom}_{[\mathscr{P},\mathscr{V}]}(F, i^*((i_*F')_{\mathscr{S}_{\mathscr{P}}}))$$

$$\cong \mathrm{Hom}_{[\mathscr{P},\mathscr{V}]}(F, i^* i_* F')$$

$$\cong \mathrm{Hom}_{[\mathscr{P},\mathscr{V}]}(F, F').$$

We use here an isomorphism $i^* G \cong i^*(G_{\mathscr{S}_{\mathscr{P}}})$ for any $G \in [\mathscr{C},\mathscr{V}]$. The isomorphism is obtained by applying the exact functor i^* to the exact sequence

$$S \rightarrowtail G \xrightarrow{\;\lambda_G\;} G_{\mathscr{S}_{\mathscr{P}}} \twoheadrightarrow S', \qquad S, S' \in \mathscr{S}_{\mathscr{P}},$$

where λ_G is the $\mathscr{S}_{\mathscr{P}}$-envelope of G.

We see that $\varkappa$ is fully faithful. Now let $G \in [\mathscr{C}, \mathscr{V}]/\mathscr{S}_{\mathscr{P}}$ be any $\mathscr{S}_{\mathscr{P}}$-closed object, then the isomorphism (12) implies

$$\varkappa(i^*G) = (i_* i^* G)_{\mathscr{S}_{\mathscr{P}}} \cong G_{\mathscr{S}_{\mathscr{P}}} \cong G.$$

If we set $F := i^*G$, then $\varkappa(F) \cong G$. This shows that $\varkappa$ is an equivalence of categories. $\qquad\square$

Corollary 5.4 ([6]). *Let $\mathscr{C}$ be a Grothendieck category with finitely generated projective generators $\mathscr{B} = \{p_i\}_{i \in I}$. Let $\mathscr{P} = \{p_j\}_{j \in J}$ be a subfamily in $\mathscr{B}$, where $J \subset I$. Then*

$$\mathscr{S}_{\mathscr{P}} = \{x \in \mathscr{C} \,|\, \mathrm{Hom}_{\mathscr{C}}(p_j, x) = 0 \text{ for all } p_j \in \mathscr{P}\}$$

is localizing and $\mathscr{C}/\mathscr{S}_{\mathscr{P}}$ is a Grothendieck category with $\{(p_j)_{\mathscr{S}_{\mathscr{P}}}\}$ a family of finitely generated projective generators.

Proof. By Mitchell's Theorem $\mathscr{C}$ is equivalent to the category $(\mathscr{B}^{\mathrm{op}}, \mathrm{Ab})$ by means of the functor T sending $x \in \mathscr{C}$ to $(-, x)$. Now the latter category is the same as the category of enriched $\mathscr{V}$-functors from $\mathscr{B}^{\mathrm{op}}$ to $\mathscr{V}$ where $\mathscr{V} = \mathrm{Ab}$. We use as well the fact that a category is preadditive if and only if it is enriched in Ab.

It follows that T induces an equivalence of categories $\mathscr{C}/\mathscr{S}_{\mathscr{P}}$ and $(\mathscr{B}^{\mathrm{op}}, \mathrm{Ab})/\widetilde{\mathscr{S}_{\mathscr{P}}}$, where

$$\widetilde{\mathscr{S}_{\mathscr{P}}} = \{F \in (\mathscr{B}^{\mathrm{op}}, \mathrm{Ab}) \,|\, F(p_j) = 0 \text{ for all } p_j \in \mathscr{P}\}.$$

Theorem 5.3 implies the latter quotient category is equivalent to $(\mathscr{P}^{\mathrm{op}}, \mathrm{Ab})$. The proof of Theorem 5.3 shows that the functor

$$(\mathscr{P}^{\mathrm{op}}, \mathrm{Ab}) \to \mathscr{C}/\mathscr{S}_{\mathscr{P}}$$

which sends $F \in (\mathscr{P}^{\mathrm{op}}, \mathrm{Ab})$ to $(T^{-1}(i_* F))_{\mathscr{S}_{\mathscr{P}}}$ is an equivalence of categories. It follows that $\{(p_j)_{\mathscr{S}_{\mathscr{P}}}\}_{j \in J}$ is a family of finitely generated projective generators of $\mathscr{C}/\mathscr{S}_{\mathscr{P}}$. $\qquad\square$

Example 5.5. Let $\mathscr{C}$ and $\mathscr{V}$ be as in Example 4.3 and let $n \in \mathrm{Ob}\,\mathscr{C} = \mathbb{Z}$. We set

$$\mathscr{S}_n = \{F \in [\mathscr{C}, \mathscr{V}] \cong \mathrm{Gr}\,R \,|\, F(n) = 0\}.$$

Then Theorem 5.3 implies

$$[\mathscr{C}, \mathscr{V}]/\mathscr{S}_n \cong \mathrm{Gr}\,R/\mathscr{S}_n \cong [\mathscr{P}, \mathrm{Mod}\,R],$$

where $\mathscr{P}$ has one object $n \in \mathbb{Z}$ and $\mathscr{V}_{\mathscr{P}}(n, n) = R$. But $[\mathscr{P}, \mathrm{Mod}\,R] = \mathrm{Mod}\,R$, hence $\mathrm{Gr}\,R/\mathscr{S}_n \cong \mathrm{Mod}\,R$.

Let $\mathscr{R}$ be a ring object of a closed symmetric monoidal Grothendieck category $\mathscr{V}$ and let $[\mathscr{C}, \mathscr{V}]$ be the Grothendieck category with $\mathrm{Ob}\,\mathscr{C} = \{*\}$ and $\mathscr{V}_{\mathscr{C}}(*, *) = \mathscr{R}$. There is a natural functor

$$\varphi : \mathscr{R}\,\mathrm{Mod} \to [\mathscr{C}, \mathscr{V}]$$

defined as follows. Given a left $\mathscr{R}$-module M, we set $\varphi(M) : \mathscr{C} \to \mathscr{V}$ to be a $\mathscr{V}$-functor such that $(\varphi(M))(*) = M$ and the structure map

$$\varphi(M) : \mathscr{V}(*, *) \to \mathscr{V}((\varphi(M))(*), (\varphi(M))(*))$$

is the map adjoint to the structure map $\mathscr{R} \otimes M \to M$.

The following lemma is straightforward.

Lemma 5.6. *The functor $\varphi : \mathscr{R}\,\mathrm{Mod} \to [\mathscr{C}, \mathscr{V}]$ is an isomorphism of categories.*

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Example 5.7. To illustrate the previous lemma, let R be a commutative unital ring and let $A = A_0 \oplus A_1 \oplus \cdots$ be a graded R-algebra. Then A is a ring object in $[\mathscr{C}, \mathscr{V}] = \operatorname{Gr} R$ with respect to the tensor product (9). We can regard A as a one object $\mathscr{V}$-category, where $\mathscr{V} = \operatorname{Gr} R$. In our notation $A = \mathscr{V}_{\operatorname{Gr} R}(*,*)$. Lemma 5.6 implies $[\mathscr{C}, \mathscr{V}] \cong A\operatorname{Mod}$ is a Grothendieck category. Moreover, $\{A(n) = A \oslash R(n)\}_{n \in \mathbb{Z}}$ are generators by Theorem 4.2 (see as well Example 4.3).

Corollary 5.8. *Let $\mathscr{C}$ be a small $\mathscr{V}$-category, let c be any object of $\mathscr{C}$, $\mathscr{S}_c = \{G \in [\mathscr{C}, \mathscr{V}] \mid G(c) = 0\}$ and $R = \mathscr{V}(c,c)$. Under the notation of Theorem 5.3 and Lemma 5.6 the composite functor*

$$\varkappa \circ \varphi : R\operatorname{Mod} \to [\mathscr{C}, \mathscr{V}]/\mathscr{S}_c$$

is an equivalence of Grothendieck categories.

Proof. This follows from Theorem 5.3 and Lemma 5.6. $\qquad\square$

6. COMPARING $\operatorname{Ch} \mathscr{C}_R$ AND $[\operatorname{mod} R, \operatorname{Ch}(\operatorname{Mod} R)]$

In the remaining sections we prove that the category of chain complexes of generalized modules $\operatorname{Ch} \mathscr{C}_R$ over a commutative ring R can be identified with the category of enriched functors $[\operatorname{mod} R, \operatorname{Ch}(\operatorname{Mod} R)]$, where the category of finitely presented modules $\operatorname{mod} R$ is regarded as a full subcategory of $\operatorname{Ch}(\operatorname{Mod} R)$ of complexes concentrated in zeroth degree and this single entry is finitely presented. As an application of this, we show that $\operatorname{Ch} \mathscr{C}_R$ is a closed symmetric monoidal category. We shall also establish that $\operatorname{Ch} \mathscr{C}_R$ is a closed symmetric monoidal model category with nice finiteness conditions. As another application, the derived category $D(\mathscr{C}_R)$ will be shown to be closed symmetric monoidal compactly generated triangulated category with duality on compact objects. However, compact objects are not strongly dualisable as it will be shown below. Thus $\mathscr{D}(\mathscr{C}_R)$ is an example of a category which satisfies all the axioms of a unital algebraic stable homotopy theory in the sense of Hovey–Palmieri–Strickland [14] except the property that compact objects are strongly dualisable. This kind of category is new for the authors. We refer to these as *unital algebraic almost stable homotopy theories*, a basic example of which is the category $\mathscr{D}(\mathscr{C}_R)$.

Let $\operatorname{Ch}(\operatorname{Mod} R)$ be the category of chain complexes of modules over a commutative ring R. It is a closed symmetric monoidal cofibrantly generated and weakly finitely generated model category by [13, sections II.3 and IV.2]. Weak equivalences are the quasi-isomorphisms and fibrations are the surjective chain maps. The tensor product and internal Hom-object are defined as above. The monoidal unit is the chain complex with the ring R concentrated in zeroth degree and other degrees are zero.

The category of finitely presented modules $\operatorname{mod} R$ is a small symmetric monoidal category naturally enriched over $\operatorname{Mod} R$. Moreover, $\operatorname{mod} R$ can also be enriched over $\operatorname{Ch} \operatorname{Mod} R$ if we regard a module $M \in \operatorname{mod} R$ as the chain complex with M concentrated in zeroth degree and other degrees are zero. Given, $M, N \in \operatorname{mod} R$ the internal Hom-object is the chain complex

$$\cdots \to 0 \to \operatorname{Hom}_R(M,N) \to 0 \to \cdots$$

where $\operatorname{Hom}_R(M,N)$ is concentrated in zeroth degree. Observe that this complex equals $\underline{\operatorname{Hom}}(M,N)$, where M, N are regarded as complexes defined above.

If there is no likelihood of confusion, we shall also write the internal Hom-chain complex as $\mathscr{V}(A_\bullet, B_\bullet)$, where $A_\bullet, B_\bullet$ are two chain complexes. In other words,

$$\mathscr{V}(A_\bullet, B_\bullet) := \underline{\operatorname{Hom}}(A_\bullet, B_\bullet).$$

Following the notation of section 2 we denoted by $[\mathrm{mod}\,R, \mathrm{Ch}(\mathrm{Mod}\,R)]$ the category of enriched functors from the small symmetric monoidal category $\mathrm{mod}\,R$ to $\mathrm{Ch}(\mathrm{Mod}\,R)$ (see Definition 2.2).

Following Herzog [12], we define the category of generalized modules $\mathscr{C}_R$ as

$$\mathscr{C}_R := (\mathrm{mod}\,R, \mathrm{Ab}),$$

whose objects are the additive functors $F : \mathrm{mod}\,R \to \mathrm{Ab}$ from the category of right finitely presented R-modules $\mathrm{mod}\,R$ to the category of abelian groups Ab. Its morphisms are the natural transformations of functors. Similarly, the category $_R\mathscr{C}$ consists of the additive functors from the category of left finitely presented R-modules to Ab.

In Theorems 6.1-6.2 below we shall construct important identifications of categories. By these we mean explicit isomorphisms of categories. The description of the isomorphisms is lengthy and the reader can easily recover them from the proof of each of these theorems.

Theorem 6.1. *Suppose R is a commutative ring. Then the category of generalized R-modules $\mathscr{C}_R$ can naturally be identified with the category $[\mathrm{mod}\,R, \mathrm{Mod}\,R]$.*

Proof. We briefly recall the proof from [3] for the convenience of the reader.

I. We first associate to any object in $[\mathrm{mod}\,R, \mathrm{Mod}\,R]$ an object in $\mathscr{C}_R$.

Let $F \in \mathrm{Ob}[\mathrm{mod}\,R, \mathrm{Mod}\,R]$. By Definition 2.2 F takes $\mathrm{Ob}(\mathrm{mod}\,R)$ to $\mathrm{Ob}(\mathrm{Mod}\,R)$ and for all $M, M' \in \mathrm{mod}\,R$ there is a R-module homomorphism

$$F_{MM'} : \mathrm{Hom}_R(M, M') \to \mathrm{Hom}_R(F(M), F(M')). \tag{13}$$

Let $f : M \to M'$ be a homomorphism in $\mathrm{mod}\,R$, then we set $F(f)$ to be the image of f in $\mathrm{Hom}_R(M, M')$ under (13). Observe that an R-module structure on $F(M)$ is given by

$$x \cdot r = F_{MM}(r)(x)$$

for all $x \in F(M)$ and $r \in R$. Here $F_{MM}(r)$ stands for the image of the right multiplication endomorphism $r : M \to M$.

II. Next, we want to show that the morphisms of $[\mathrm{mod}\,R, \mathrm{Mod}\,R]$ can naturally be regarded as morphisms of $\mathscr{C}_R$.

We have to verify that $\mathscr{V}$-natural transformations in $[\mathrm{mod}\,R, \mathrm{Mod}\,R]$ are morphisms in $\mathscr{C}_R$. Given $F, G \in [\mathrm{mod}\,R, \mathrm{Mod}\,R]$, the first step shows that $F, G \in \mathscr{C}_R$. So given a $\mathscr{V}$-natural transformation t in $[\mathrm{mod}\,R, \mathrm{Mod}\,R]$, we want to prove that t yields a morphism in $\mathscr{C}_R$ in a natural way. For t we have structure homomorphisms in $\mathrm{Mod}\,R$ (see Definition 2.3)

$$t_M : R \to \mathrm{Hom}_R(F(M), G(M)).$$

for all $M, M' \in \mathrm{mod}\,R$. For any $M \in \mathrm{Mod}\,R$, set

$$\tau_M := t_M(1) : F(M) \to G(M).$$

Therefore t yields a natural transformation in $\mathscr{C}_R$

$$\tau : F \to G.$$

We see that morphisms in $[\mathrm{mod}\,R, \mathrm{Mod}\,R]$ can naturally be regarded as morphisms of $\mathscr{C}_R$. So $F(M) \in \mathrm{Mod}\,R$.

III. In this step we shall show that any object F of $\mathscr{C}_R$ can be regarded as an enriched functor in $[\mathrm{mod}\,R, \mathrm{Mod}\,R]$.

For any $M \in \mathrm{mod}\,R$ we have that $F(M) \in \mathrm{Ab}$. Let us show that $F(M)$ is an R-module. For all $x \in F(M)$, we have to define $x \cdot r$, where $r \in R$. The element r defines an R-module

homomorphism $r : M \to M$ sending $m \in M$ to $m \cdot r$. We have a morphism $F(r) : F(M) \to F(M)$ and we set

$$x \cdot r := F(r)(x).$$

So $F(M) \in \mathrm{Mod}\, R$. Now we define an enriched functor associated with F. We therefore define a morphism in $\mathrm{Mod}\, R$

$$F_{MM'} : \mathrm{Hom}_R(M, M') \to \mathrm{Hom}_R(F(M), F(M')), \quad F_{MM'}(f) := F(f),$$
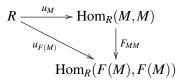
for all M, M' in $\mathrm{mod}\, R$.

Next we construct diagram (4). We have that

$$u_M : R \to \mathrm{Hom}_R(M, M)$$

is given by the right multiplication homomorphism. One has

$$F_{MM}(u_M(r)) = F(r)$$

for all $r \in R$, and hence the diagram

$$
\begin{array}{ccc}
R & \xrightarrow{\;u_M\;} & \mathrm{Hom}_R(M, M) \\
& u_{F(M)} \searrow & \downarrow F_{MM} \\
& & \mathrm{Hom}_R(F(M), F(M))
\end{array}
$$

is commutative. The structure of an enriched functor for F, denoted by the same letter, is completed.

So every object in $\mathscr{C}_R$ can naturally be regarded as an enriched functor in $[\mathrm{mod}\, R, \mathrm{Mod}\, R]$.

IV. In this step we shall show that morphisms in $\mathscr{C}_R$ (recall that these are natural transformations of additive functors) can naturally be regarded as $\mathscr{V}$-natural transformations in $[\mathrm{mod}\, R, \mathrm{Mod}\, R]$, i.e. as morphisms of $[\mathrm{mod}\, R, \mathrm{Mod}\, R]$.

Let $\tau : F \to G$ be any natural transformation in $\mathscr{C}_R$. Then for each object $M \in \mathrm{mod}\, R$, there exists a homomorphism $\tau_M : F(M) \to G(M)$ in Ab and for each homomorphism $f : M \to M'$ in $\mathrm{mod}\, R$ the diagram

$$
\begin{array}{ccc}
F(M) & \xrightarrow{\;\tau_M\;} & G(M) \\
F(f) \downarrow & & \downarrow G(f) \\
F(M') & \xrightarrow[\;\tau_M\;]{} & G(M')
\end{array}
$$

is commutative.

By above we can regard F, G as enriched functors. We want to show that τ yields a $\mathscr{V}$-natural transformation t between the $\mathscr{V}$-functors F and G. For any $M \in \mathrm{mod}\, R$ define a map

$$t_M : R \to \mathrm{Hom}_R(F(M), G(M))$$

as $t_M(r) := \tau_M \circ F(r) = G(r) \circ \tau_M$. By definition, $t_M(1) = \tau_M$. Then the maps t_M yield $\mathscr{V}$-natural transformations between the $\mathscr{V}$-functors F and G. $\qquad\square$

More generally, we have the following

Theorem 6.2. *Suppose R is a commutative ring. Then the category of chain complexes of generalized R-modules $\mathrm{Ch}(\mathscr{C}_R)$ can naturally be identified with the category $[\mathrm{mod}\, R, \mathrm{Ch}(\mathrm{Mod}\, R)]$.*

Proof. Given $M \in \operatorname{mod} R$ we want to describe chain morphisms of the form:

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \xrightarrow{\ 0\ } & M & \xrightarrow{\ 0\ } & 0 & \longrightarrow & \cdots \\
& & \downarrow & & {\scriptstyle\alpha}\downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathscr{V}(A_\bullet,B_\bullet)_1 & \xrightarrow{\ \partial_1\ } & \mathscr{V}(A_\bullet,B_\bullet)_0 & \xrightarrow{\ \partial_0\ } & \mathscr{V}(A_\bullet,B_\bullet)_{-1} & \longrightarrow & \cdots
\end{array}
$$

From the diagram we have $\partial_0 \alpha = 0$. This means $\partial_0 \alpha(m) = 0$, for all $m \in M$.

$$
\alpha(m) \in \mathscr{V}(A_\bullet,B_\bullet)_0 = \prod_{p\in\mathbb{Z}} \operatorname{Hom}_R(A_p,B_p)
$$
$$
\alpha(m) = (\alpha_p(m) : A_p \to B_p)_{p\in\mathbb{Z}}
$$
$$
\partial \alpha_p(m) = 0 = \partial \alpha_p(m) - \alpha(m)_{p-1}\partial.
$$

We get a commutative diagram

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & A_{p+1} & \xrightarrow{\ \partial_{p+1}\ } & A_p & \xrightarrow{\ \partial_p\ } & A_{p-1} & \longrightarrow & \cdots \\
& & {\scriptstyle\alpha_{p+1}(m)}\downarrow & & {\scriptstyle\alpha_p(m)}\downarrow & & \downarrow{\scriptstyle\alpha_{p-1}(m)} & & \\
\cdots & \longrightarrow & B_{p+1} & \xrightarrow[\ \partial_{p+1}\ ]{} & B_p & \xrightarrow[\ \partial_p\ ]{} & B_{p-1} & \longrightarrow & \cdots
\end{array}
$$

This shows $\alpha(m) : A_\bullet \to B_\bullet$ is a chain map.

I. Consider a $\mathscr{V}$-functor $F \in [\operatorname{mod} R, \operatorname{Ch}(\operatorname{Mod} R)]$. By definition, we have $F(M) \in \operatorname{Ch}(\operatorname{Mod} R)$ for any M in $\operatorname{mod} R$. We also have a map

$$
F_{MM'} : \mathscr{V}(M,M') \to \mathscr{V}(F(M),F(M')) \in \operatorname{Ch}(\operatorname{Mod} R).
$$

This is the same as a chain map

$$
F_{MM'} : (\cdots \to 0 \to \operatorname{Hom}_R(M,M') \to 0 \to \cdots) \longrightarrow \underline{\operatorname{Hom}}(F(M),F(M')).
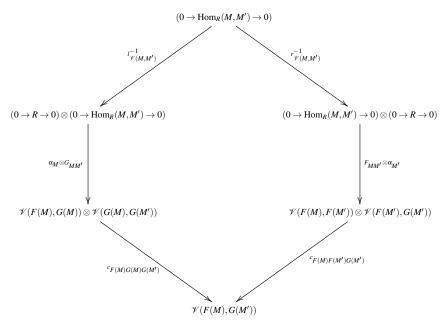$$

By above we have a chain map

$$
F_{MM'}(f) : F(M) \to F(M')
$$

for all $f \in \operatorname{Hom}_R(M,M')$. It is directly verified that $F_{MM}(\operatorname{id}_M) = \operatorname{id}_{F(M)}$ and $F_{MM''}(gf) = F_{M'M''}(g)F_{MM'}(f)$ for any $g \in \operatorname{Hom}_R(M',M'')$ (see Definition 2.2).

If we observe that $\operatorname{Ch}\mathscr{C}_R$ is the same as the category $(\operatorname{mod} R, \operatorname{Ch}(\operatorname{Ab}))$ of additive functors from $\operatorname{mod} R$ to $\operatorname{Ch}(\operatorname{Ab})$, it follows that the enriched functor F gives rise to an object in $\operatorname{Ch}\mathscr{C}_R$. We denote this object by the same letter.

II. Now let $\alpha : F \Rightarrow G$ be a $\mathscr{V}$-map in $[\operatorname{mod} R, \operatorname{Ch}(\operatorname{Mod} R)]$. It consists of giving a chain map

$$
\alpha_M : (0 \to R \to 0) \longrightarrow \mathscr{V}(F(M),G(M)) = \underline{\operatorname{Hom}}(F(M),G(M)),
$$

which is equivalent to giving a chain map $\alpha_M(r) : F(M) \to G(M)$ for all $r \in R$, such that the following diagram commutes

$$(0 \to \mathrm{Hom}_R(M,M') \to 0)$$

$l^{-1}_{\mathscr{V}(M,M')}$
$r^{-1}_{\mathscr{V}(M,M')}$

$$(0 \to R \to 0) \otimes (0 \to \mathrm{Hom}_R(M,M') \to 0) \qquad (0 \to \mathrm{Hom}_R(M,M') \to 0) \otimes (0 \to R \to 0)$$

$\alpha_M \otimes G_{MM'}$
$F_{MM'} \otimes \alpha_{M'}$

$$\mathscr{V}(F(M),G(M)) \otimes \mathscr{V}(G(M),G(M')) \qquad \mathscr{V}(F(M),F(M')) \otimes \mathscr{V}(F(M'),G(M'))$$

$c_{F(M)G(M)G(M')}$
$c_{F(M)F(M')G(M')}$

$$\mathscr{V}(F(M),G(M'))$$

The diagram implies the following:

(a) For all $r \in R$ and $f \in \mathrm{Hom}_R(M,M')$ the chain map $c(\alpha_M(r) \otimes G_{MM'}(f)) : F(M) \to G(M')$ equals the chain map $G_{MM'}(f) \circ \alpha_M(r)$.

(b) For all $r \in R$ and $f \in \mathrm{Hom}_R(M,M')$ the chain map $c(F_{MM'}(f) \otimes \alpha_M(r)) : F(M) \to G(M')$ equals $\alpha_{M'}(r) \circ F_{MM'}(f)$.

So we get a commutative diagram

$$
\begin{array}{ccc}
F(M) & \xrightarrow{\alpha_M(r)} & G(M) \\
{\scriptstyle F_{MM'}(f)}\downarrow & & \downarrow{\scriptstyle G_{MM'}(f)} \\
F(M') & \xrightarrow{\alpha_{M'}(r)} & G(M')
\end{array}
$$

Thus $\alpha_M(r) : F(M) \to G(M)$ gives rise to a morphism in $(\mathrm{mod}\,R, \mathrm{Ch}(\mathrm{Ab})) = \mathrm{Ch}\,\mathscr{C}_R$.

We shall associate to the enriched map $\alpha : F \Rightarrow G$ in $[\mathrm{mod}\,R, \mathrm{Ch}(\mathrm{Mod}\,R)]$ the map $F \to G$ in $\mathrm{Ch}\,\mathscr{C}_R$ given by the chain maps $\alpha_M(1) : F(M) \to G(M)$, $M \in \mathrm{mod}\,R$. We denote the associated chain map by the same letter α.

III. By Theorem 6.1 the category $\mathrm{Ch}[\mathrm{mod}\,R, \mathrm{Mod}\,R]$ is identified with $\mathrm{Ch}\,\mathscr{C}_R$. Let $F_\bullet \in \mathrm{Ch}[\mathrm{mod}\,R, \mathrm{Mod}\,R]$, so we have a chain complex

$$F_\bullet = \cdots \longrightarrow F_{n+1} \xrightarrow{\partial_{n+1}} F_n \xrightarrow{\partial_n} F_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

with each $F_n \in [\mathrm{mod}\,R, \mathrm{Mod}\,R]$ and each ∂_n being a $\mathscr{V}$-natural transformation from $\mathrm{mod}\,R$ to $\mathrm{Mod}\,R$. We have to associate a $\mathscr{V}$-functor in $[\mathrm{mod}\,R, \mathrm{Ch}(\mathrm{Mod}\,R)]$ to $F_\bullet$.

If $M \in \mathrm{mod}\,R$ then

$$F_\bullet(M) = \cdots \longrightarrow F(M)_{n+1} \xrightarrow{\partial_{n+1,M}(1)} F(M)_n \xrightarrow{\partial_{n,M}(1)} F(M)_{n-1} \xrightarrow{\partial_{n-1,M}(1)} \cdots \in \mathrm{Ch}(\mathrm{Mod}\,R)$$

Also, for any map $f : M \to M' \in \text{mod} R$ we have that $F_\bullet(f) : F_\bullet(M) \to F_\bullet(M')$ is a chain map, because each square of the following diagram is commutative

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & F_{n+1}(M) & \xrightarrow{\partial_{n+1}} & F_n(M) & \xrightarrow{\partial_n} & F_{n-1}(M) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle F_{MM'}(f)} & & \downarrow{\scriptstyle F_{MM'}(f)} & & \downarrow{\scriptstyle F_{MM'}(f)} & \\
\cdots \longrightarrow & F_{n+1}(M') & \xrightarrow[\partial_{n+1}]{} & F_n(M')_n & \xrightarrow[\partial_n]{} & F(M')_{n-1}, & \longrightarrow \cdots
\end{array}
$$

Thus $F_{MM'}(f)$ is a chain map, and hence one gets a chain map

$$F_{MM'} : (\cdots \to 0 \to \text{Hom}_R(M, M') \to 0 \to \cdots) \longrightarrow \underline{\text{Hom}}(F_\bullet(M), F_\bullet(M')).$$

Since $F_{MM}(\text{id}_M) = \text{id}_{F_\bullet(M)}$ and $F_{MM''}(gf) = F_{M'M''}(g)F_{MM'}(f)$ for all $g \in \text{Hom}_R(M', M'')$, $F_\bullet$ yields a $\mathscr{V}$-functor in $[\text{mod} R, \text{Ch}(\text{Mod} R)]$ denoted by the same letter.

IV. Next let $\beta : F_\bullet \to G_\bullet$ be a chain map in $\text{Ch}[\text{mod} R, \text{Mod} R]$, then we have the following commutative diagram

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & F_{n+1} & \xrightarrow{\partial_{n+1}} & F_n & \xrightarrow{\partial_n} & F_{n-1} & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \beta_{n+1}} & & \downarrow{\scriptstyle \beta_n} & & \downarrow{\scriptstyle \beta_{n-1}} & \\
\cdots \longrightarrow & G_{n+1} & \xrightarrow[\partial_{n+1}]{} & G_n & \xrightarrow[\partial_n]{} & G_{n-1} & \longrightarrow \cdots
\end{array}
$$

Note that each $\beta_{n,M} : R \to \text{Hom}_R(F_n(M), G_n(M))$ is such that diagram (5) is commutative for it. So we are given maps $\beta_{n,M}(r) : F_n(M) \to G_n(M)$ for all $r \in R$.

One has a commutative diagram for all $r \in R$

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & F_{n+1}(M) & \xrightarrow{\partial_{n+1,M}(1)} & F_n(M) & \xrightarrow{\partial_{n,M}(1)} & F_{n-1}(M) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \beta_{n+1,M}(r)} & & \downarrow{\scriptstyle \beta_{n,M}(r)} & & \downarrow{\scriptstyle \beta n,M(r)} & \\
\cdots \longrightarrow & G_{n+1}(M) & \xrightarrow[\partial_{n+1,M}(1)]{} & G_n(M) & \xrightarrow[\partial_{n,M}(1)]{} & G_{n-1}(M)) & \longrightarrow \cdots
\end{array}
$$

In particular, $\beta_M(r) : F_\bullet(M) \to G_\bullet(M)$ is a chain map.

Now we want to show that the diagram

$$
\begin{array}{ccc}
F_\bullet(M) & \xrightarrow{\beta_M(r)} & G_\bullet(M) \\
\downarrow{\scriptstyle F_\bullet(f)} & & \downarrow{\scriptstyle G_\bullet(f)} \\
F_\bullet(M') & \xrightarrow[\beta_{M'}(r)]{} & G_\bullet(M')
\end{array}
\qquad (14)
$$

commutes. Commutativity of diagram (5) implies commutativity of the following diagram for all $n \in \mathbb{Z}$:

$$
\begin{CD}
F_n(M) @>{\beta_{n,M}(r)}>> G_n(M) \\
@V{F_n(f)}VV @VV{G_n(f)}V \\
F_n(M') @>>{\beta_{n,M'}(r)}> G_n(M')
\end{CD}
$$

Commutativity of the latter together with the facts that $\beta_M(r), G_\bullet(f), F_\bullet(f)$ are all chain maps is enough to check commutativity of (14) as shown in the diagram below

Thus we have constructed a $\mathscr{V}$-natural map $\beta : F_\bullet \to G_\bullet$.

It is now easily verified that associations given in steps I–IV yield the desired isomorphisms of categories $[\operatorname{mod} R, \operatorname{Ch}(\operatorname{Mod} R)]$ and $\operatorname{Ch} \mathscr{C}_R$. $\qquad\square$

We are now in a position to prove the main result of this section.

Theorem 6.3. *Let R be a commutative ring, then $\operatorname{Ch}(\mathscr{C}_R)$ is a left and right proper closed symmetric monoidal $\mathscr{V}$-model category, where $\mathscr{V} = \operatorname{Ch}(\operatorname{Mod} R)$, and the monoid axiom in the sense of Schwede–Shipley [23] is satisfied. The tensor product of two complexes $F_\bullet, G_\bullet \in \operatorname{Ch}(\mathscr{C}_R)$ is given by*

$$
F_\bullet \odot G_\bullet = \int^{(M,N) \in \operatorname{mod} R \otimes \operatorname{mod} R} F_\bullet(M) \otimes_R G_\bullet(N) \otimes_R \operatorname{Hom}_R(M \otimes_R N, -). \tag{15}
$$

Here $\operatorname{Hom}_R(M \otimes_R N, -)$ is regarded as a complex concentrated in zeroth degree. The internal Hom-object is defined as

$$
\underline{\operatorname{Hom}}(F_\bullet, G_\bullet)(M) = \int_{N \in \operatorname{mod} R} \underline{\operatorname{Hom}}_{\operatorname{Ch}(\operatorname{Mod} R)}(F_\bullet(N), G_\bullet(M \otimes_R N)). \tag{16}
$$

Proof. By Theorem 6.2 we know that $\operatorname{Ch} \mathscr{C}_R$ can be identified with the category $[\operatorname{mod} R, \operatorname{Ch}(\operatorname{Mod} R)]$. Note that $\operatorname{mod} R$ is symmetric monoidal category enriched over $\operatorname{Ch}(\operatorname{Mod} R)$. Now formulas (15)-(16) as well as the fact that $\operatorname{Ch} \mathscr{C}_R$ is closed symmetric monoidal follow from Theorem 2.11.

It remains to show that $\operatorname{Ch} \mathscr{C}_R$ is a left and right proper closed symmetric monoidal $\mathscr{V}$-model category, and the monoid axiom is satisfied. Since $\operatorname{Ch} \mathscr{C}_R$ can be identified with $[\operatorname{mod} R, \operatorname{Ch}(\operatorname{Mod} R)]$ by Theorem 6.2, it is enough to verify this for $[\operatorname{mod} R, \operatorname{Ch}(\operatorname{Mod} R)]$. The category $\operatorname{Ch}(\operatorname{Mod} R)$ is a closed symmetric monoidal cofibrantly generated model category, where the weak equivalences

are quasi-isomorphisms and the fibrations are surjective chain morphisms (see [13, 2.3.11] and [13, 4.2.13]). Moreover, $\mathrm{Ch}(\mathrm{Mod}\,R)$ is weakly finitely generated in the sense of [5, Section 3.1]. Also, $\mathrm{Ch}(\mathrm{Mod}\,R)$ satisfies the monoid axiom in the sense of [23]. It follows from [5, 4.2] that $[\mathrm{mod}\,R, \mathrm{Ch}(\mathrm{Mod}\,R)]$ is a weakly finitely generated model category, where fibrations and weak equivalences are defined objectwise. Furthermore, $[\mathrm{mod}\,R, \mathrm{Ch}(\mathrm{Mod}\,R)]$ is a monoidal $\mathscr{V}$-enriched model category satisfying the monoid axiom by [5, 4.4], because $\mathrm{mod}\,R$ is a symmetric monoidal category enriched over $\mathrm{Ch}(\mathrm{Mod}\,R)$. Finally, [5, 4.8] implies $[\mathrm{mod}\,R, \mathrm{Ch}(\mathrm{Mod}\,R)]$ is both left and right proper. $\qquad\square$

Corollary 6.4. *Let $\mathscr{R}$ be a ring object in $\mathrm{Ch}(\mathscr{C}_R)$. Then the category of left $\mathscr{R}$-modules is a cofibrantly generated model Grothendieck category. If $\mathscr{R}$ is a commutative ring object, then the category of $\mathscr{R}$-modules is a cofibrantly generated, monoidal model category satisfying the monoid axiom, and the category of $\mathscr{R}$-algebras is a cofibrantly generated model category.*

Proof. This is a consequence of Lemma 5.6, Theorem 6.3 and [23, 4.1]. $\qquad\square$

Example 6.5. A typical example of a ring object in $\mathrm{Ch}(\mathscr{C}_R)$ is constructed as follows. Let C be a finitely presented R-module. We regard the representable functor $(C, -) \in \mathscr{C}_R$ as a complex in zeroth degree. Lemma 2.12 implies a natural isomorphism of representable functors

$$(C, -) \odot (C, -) \cong (C \otimes_R C, -).$$

It follows that $(C, -)$ is a ring object of $\mathrm{Ch}(\mathscr{C}_R)$ if and only if C is an R-coalgebra. It is a commutative ring object if and only if C is a cocommutative R-coalgebra. By the previous corollary we have a Grothendieck model category of $(C, -)$-modules inside $\mathrm{Ch}(\mathscr{C}_R)$.

7. Almost stable homotopy category structure for $\mathscr{D}(\mathscr{C}_R)$

In this section we give an application of Theorem 6.3. Namely, we prove that the derived category $\mathscr{D}(\mathscr{C}_R)$ of the Grothendieck category of generalized modules is a closed symmetric monoidal compactly generated triangulated category with duality on compact objects. However, compact objects are not strongly dualizable as it will be shown below. Thus $\mathscr{D}(\mathscr{C}_R)$ is an example of a category which satisfies all the axioms of a unital algebraic stable homotopy theory in the sense of Hovey–Palmieri–Strickland [14] except the property that compact objects are strongly dualizable. We refer to these as *unital algebraic almost stable homotopy categories*. A basic example of this class of categories is $\mathscr{D}(\mathscr{C}_R)$.

Definition 7.1. Let $\mathscr{C}$ be a triangulated category in which all set indexed direct sums exist. An object A of $\mathscr{C}$ is called *compact* if the canonial map

$$\bigoplus_{i \in I} \mathrm{Hom}(A, E_i) \to \mathrm{Hom}(A, \bigoplus_{i \in I} E_i)$$

is an isomorphism for any set of objects E_i in $\mathscr{C}$ and $i \in I$. A triangulated category $\mathscr{C}$ is *compactly generated* if there is a set S of compact objects with the following property, if for every object $E \in \mathscr{C}$ we have

$$\mathrm{Hom}(A, E) = 0 \text{ for all } A \in S \text{ implies } E = 0.$$

Recall that the derived tensor product $F_\bullet \odot^L G_\bullet$ of two complexes in $\mathrm{Ch}\,\mathscr{C}_R$ is defined as $F_\bullet^c \odot G_\bullet^c$, where $F_\bullet^c, G_\bullet^c$ are cofibrant replacements of $F_\bullet$ and $G_\bullet$ respectively in the monoidal model category $\mathrm{Ch}\,\mathscr{C}_R$. We start with the following

Theorem 7.2. *Let R be a commutative ring. Then the derived category $\mathscr{D}(\mathscr{C}_R)$ of the Grothendieck category $\mathscr{C}_R$ is a compactly generated triangulated closed symmetric monoidal category, where formulas (15) and (16) yield the derived tensor product $F_\bullet \odot^L G_\bullet$ and derived internal Hom-object $R\underline{\mathrm{Hom}}(F_\bullet, G_\bullet)$. The compact objects of $\mathscr{D}(\mathscr{C}_R)$ are the complexes isomorphic to bounded complexes of coherent functors in $\mathrm{coh}\,\mathscr{C}_R$.*

Proof. By Theorem 6.3 $\mathrm{Ch}(\mathscr{C}_R)$ is a left and right proper closed symmetric monoidal $\mathscr{V}$-model category with $\mathscr{V} = \mathrm{Ch}(\mathrm{Mod}\,R)$ and the weak equivalences quasi-isomorphisms. Hence the derived category $\mathscr{D}(\mathscr{C}_R)$ is identified with the homotopy category $\mathrm{Ho}(\mathrm{Ch}(\mathscr{C}_R))$, because both categories are obtained from $\mathrm{Ch}(\mathscr{C}_R)$ by inverting the quiasi-isomorphisms (see [13, §1.2] and [25, §10.4]). But the latter category is a closed symmetric monoidal category by [13, 4.3.2] with derived tensor product and internal interal Hom-functors induced by (15) and (16) from Theorem 6.3.

Since $\mathscr{C}_R$ is a Grothendieck category with finitely generated projective generators $\{(M, -)\}_{M \in \mathrm{mod}\,R}$, then its derived category $\mathscr{D}(\mathscr{C}_R)$ is a compactly generated triangulated category. The compact objects are those complexes which are isomorphic in $\mathscr{D}(\mathscr{C}_R)$ to bounded complexes of representable functors. They are also called perfect complexes. Since every coherent functor $C \in \mathrm{coh}\,\mathscr{C}_R$ has a resolution (see, e.g., [12, 2.1])

$$0 \longrightarrow (L, -) \longrightarrow (K, -) \longrightarrow (M, -) \longrightarrow C \longrightarrow 0,$$

where $K, L, M \in \mathrm{mod}\,R$, then every bounded complex of coherent functors is quasi-isomorphic to a bounded complex of representable functors. It follows that every bounded complex of coherent functors is quasi-isomorphic to a perfect complex. This finishes the proof. $\qquad\square$

Remark 7.3. A monoidal unit in $\mathrm{Ch}\,\mathscr{C}_R$ and $\mathscr{D}(\mathscr{C}_R)$ is $(R, -) \cong - \otimes_R R$ regarded as a complex concentrated in zeroth degree.

Definition 7.4. Let $\mathscr{V}$ be a closed symmetric monoidal additive category, with monoidal product $x \otimes y$, unit e, and internal function objects $\mathscr{V}(x, y)$. An object $x \in \mathscr{V}$ is *strongly dualizable* if the natural map $\mathscr{V}(x, e) \otimes y \to \mathscr{V}(x, y)$ is an isomorphism for all y. We shall also write $x^\vee$ to denote $\mathscr{V}(x, e)$.

It follows from [17, Theorem 7.1.6] that the functor

$$x \in \mathscr{V} \mapsto x^\vee \in \mathscr{V}$$

puts the full subcategory of strongly dualizable objects of $\mathscr{V}$ in duality.

We want to show below that $\mathscr{D}(\mathscr{C}_R)$ has a duality on compact objects but these are not strongly dualizable in general. To this end we shall need the categorical duality

$$D : (\mathrm{coh}\,\mathscr{C}_R)^{\mathrm{op}} \to \mathrm{coh}\,_R\mathscr{C}$$

of Auslander [1] and Gruson–Jensen [10] (see [12] as well) defined over any non-commutative ring R as follows. Given $_RN \in R\mathrm{mod}$, we have

$$(DC)(_RN) := \mathrm{Hom}_{\mathscr{C}_R}(C, - \otimes_R N).$$

If $\eta : B \to C$ is a morphism in $\mathrm{coh}\,\mathscr{C}_R$, then

$$D(\eta)_N : D(C)(_RN) \to D(B)(_RN)$$

is defined to be $\mathrm{Hom}_{\mathscr{C}_R}(\eta, - \otimes_R N)$. For $M_R \in \mathrm{mod}\,R$ and $_RN \in R\mathrm{mod}$ we have that

$$D(M_R, -) \cong M \otimes_R - \quad \text{and} \quad D(- \otimes_R N) \cong (_RN, -).$$

We shall refer to this duality as the *Auslander–Gruson–Jensen Duality*.

Suppose now R is commutative. Then the category $\mathscr{C}_R$ is closed symmetric monoidal for the same reasons that $\mathrm{Ch}\,\mathscr{C}_R$ is. The monoidal product $C \odot C'$ and internal Hom-object $\underline{\mathrm{Hom}}(C, C')$ are computed by formulas, which are similar to (15)-(16). It can also be shown that the Auslander–Gruson–Jensen Duality D defined above is isomorphic to the internal Hom-functor

$$\underline{\mathrm{Hom}}(-, (R, -)) \cong \underline{\mathrm{Hom}}(-, - \otimes_R R)$$

(we refer the reader to [3] for further details).

The following example shows that compact objects of $\mathscr{D}(\mathscr{C}_R)$ are not strongly dualizable in general.

Example 7.5. There are objects $C \in \mathscr{D}(\mathscr{C}_R)^c$ and $X \in \mathscr{D}(\mathscr{C}_R)$ such that the natural arrow

$$C^\vee \odot^L X \to R\underline{\mathrm{Hom}}(C, X) \tag{17}$$

is not an isomorphism, where

$$C^\vee := R\underline{\mathrm{Hom}}(C, - \otimes_R R) = R\underline{\mathrm{Hom}}(C, (R, -)).$$

Let $R = \mathbb{Z}$ and $M = N = \mathbb{Z}_2 \in \mathrm{mod}\,\mathbb{Z}$. We have an exact sequence

$$0 \longrightarrow \mathbb{Z} \stackrel{2}{\longrightarrow} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

We want to compute $(- \otimes M)^\vee \odot^L - \otimes N = (- \otimes \mathbb{Z}_2)^\vee \odot^L - \otimes \mathbb{Z}_2$.

To compute $(- \otimes \mathbb{Z}_2)^\vee$, consider a projective resolution for $- \otimes \mathbb{Z}_2$ in $\mathscr{C}_\mathbb{Z}$

$$0 \longrightarrow (\mathbb{Z}_2, -) \longrightarrow (\mathbb{Z}, -) \stackrel{(2,-)}{\longrightarrow} (\mathbb{Z}, -) \longrightarrow - \otimes \mathbb{Z}_2 \longrightarrow 0.$$

Then $(- \otimes \mathbb{Z}_2)^\vee$ is the value of $\underline{\mathrm{Hom}}(-, (\mathbb{Z}, -))$ at the projective resolution of $- \otimes \mathbb{Z}_2$. Since $\underline{\mathrm{Hom}}(-, (\mathbb{Z}, -))$ puts the category of coherent objects $\mathrm{coh}\,\mathscr{C}_\mathbb{Z}$ in duality and takes representable functors $(L, -)$ to $- \otimes L$, then $(- \otimes \mathbb{Z}_2)^\vee$ is the complex

$$\cdots \to 0 \to - \otimes \mathbb{Z} \to - \otimes \mathbb{Z} \to - \otimes \mathbb{Z}_2 \to 0 \to \cdots$$

This complex has only one non-zero homology group $(\mathbb{Z}_2, -)$, which we place in zeroth degree as a complex. We see that $(- \otimes \mathbb{Z}_2)^\vee \cong (\mathbb{Z}_2, -)$.

We take a projective resolution for $- \otimes N$ in $\mathrm{coh}\,\mathscr{C}_\mathbb{Z}$ as above

$$0 \longrightarrow (\mathbb{Z}_2, -) \longrightarrow (\mathbb{Z}, -) \stackrel{(2,-)}{\longrightarrow} (\mathbb{Z}, -) \longrightarrow - \otimes \mathbb{Z}_2 \longrightarrow 0.$$

Tensoring it with $(M, -) = (\mathbb{Z}_2, -)$ we get $(M, -) \odot^L - \otimes N$ which is the complex

$$\cdots \to 0 \to (\mathbb{Z}_2, -) \odot (\mathbb{Z}_2, -) \to (\mathbb{Z}_2, -) \odot (\mathbb{Z}, -) \to (\mathbb{Z}_2, -) \odot (\mathbb{Z}, -) \to 0 \to \cdots$$

By Lemma 2.12 it equals the complex

$$\cdots \longrightarrow 0 \longrightarrow (\mathbb{Z}_2, -) \longrightarrow (\mathbb{Z}_2, -) \stackrel{0}{\longrightarrow} (\mathbb{Z}_2, -) \longrightarrow 0 \longrightarrow \cdots$$

Note that evaluation of this complex at $\mathbb{Z}$ is zero.

Now compute $R\underline{\mathrm{Hom}}(- \otimes M, - \otimes N)$. It is the value of $\underline{\mathrm{Hom}}(-, - \otimes N)$ at the projective resolution of $- \otimes M$

$$0 \longrightarrow (\mathbb{Z}_2, -) \longrightarrow (\mathbb{Z}, -) \stackrel{(2,-)}{\longrightarrow} (\mathbb{Z}, -) \longrightarrow - \otimes M \longrightarrow 0.$$

Applying $\underline{\mathrm{Hom}}(-,-\otimes N)$ to the complex

$$0 \longrightarrow (\mathbb{Z}_2,-) \longrightarrow (\mathbb{Z},-) \xrightarrow{(2,-)} (\mathbb{Z},-) \longrightarrow 0$$
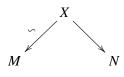
we get a complex

$$0 \to \underline{\mathrm{Hom}}((\mathbb{Z},-),-\otimes\mathbb{Z}_2) \to \underline{\mathrm{Hom}}((\mathbb{Z},-),-\otimes\mathbb{Z}_2) \to \underline{\mathrm{Hom}}((\mathbb{Z}_2,-),-\otimes\mathbb{Z}_2) \to 0$$

Using The Enriched Yoneda Lemma 2.8, it is equal to

$$0 \longrightarrow -\otimes\mathbb{Z}_2 \xrightarrow{0} -\otimes\mathbb{Z}_2 \xrightarrow{\cong} -\otimes\mathbb{Z}_2 \longrightarrow 0.$$

The value of this complex at $\mathbb{Z}$ has non-trivial homology. We conclude that (17) cannot be an isomorphism in general. We conclude that compact objects of $\mathscr{D}(\mathscr{C}_R)$ are not strongly dualizable.

Lemma 7.6. *The triangulated category $\mathscr{D}(\mathscr{C}_R)^c$ of compact objects of $\mathscr{D}(\mathscr{C}_R)$ is triangle equivalent to the derived category $\mathscr{D}^b(\mathrm{coh}\,\mathscr{C}_R)$ of bounded complexes in $\mathrm{coh}\,\mathscr{C}_R$.*

Proof. By the proof of Theorem 7.2 $\mathscr{D}(\mathscr{C}_R)^c$ is triangle equivalent to the full subcategory $\widetilde{\mathscr{D}}(\mathscr{C}_R)^c$ of bounded complexes in $\mathrm{coh}\,\mathscr{C}_R$. Let

$$
\begin{array}{ccc}
 & X & \\
\swarrow & & \searrow \\
M & & N
\end{array}
$$

be a morphism in $\mathscr{D}(\mathscr{C}_R)$ with $M,N \in \widetilde{\mathscr{D}}(\mathscr{C}_R)^c$. Let $P \to M$ be a projective resolution of M. Then P is in $\widetilde{\mathscr{D}}(\mathscr{C}_R)^c$ and P is isomorphic to X in $\mathscr{D}(\mathscr{C}_R)$. But P is a bounded complex of projectives in $\mathscr{C}_R$. Therefore there is a quasi-isomorphism $P \to X$. Then we get a diagram in $\widetilde{\mathscr{D}}(\mathscr{C}_R)^c$

$$
\begin{array}{c}
P \\
\downarrow \\
X \\
\swarrow \quad \searrow \\
M \qquad N
\end{array}
$$

By [16, 9.1] the natural functor

$$\mathscr{D}^b(\mathrm{coh}\,\mathscr{C}_R) \to \widetilde{\mathscr{D}}(\mathscr{C}_R)^c$$

is fully faithful. But the objects are the same, and therefore these subcategories of $\mathscr{D}(\mathscr{C}_R)$ coincide. $\qquad\square$

Lemma 7.7. *There is a triangle equivalence of triangulated categories $\mathscr{D}^b(\mathrm{coh}\,\mathscr{C}_R)$ and $\mathscr{D}^b((\mathrm{coh}\,\mathscr{C}_R)^{\mathrm{op}})$ taking $X \in \mathscr{D}^b(\mathrm{coh}\,\mathscr{C}_R)$ to $\underline{\mathrm{Hom}}(X,-\otimes R)$.*

Proof. By [3] $\underline{\mathrm{Hom}}(-,-\otimes R) : \mathrm{coh}\,\mathscr{C}_R \to (\mathrm{coh}\,\mathscr{C}_R)^{\mathrm{op}}$ is an equivalence of abelian categories. Moreover, this functor is isomorphic to the Auslander–Gruson–Jensen duality (see [3, 4.6]). The fact that equivalent abelian categories have equivalent derived categories finishes the proof. $\quad\square$

Corollary 7.8. $\mathscr{D}^b(\mathrm{coh}\,\mathscr{C}_R)$ *is triangle equivalent to* $(\mathscr{D}^b(\mathrm{coh}\,\mathscr{C}_R))^{\mathrm{op}}$.

Proof. This follows from the previous lemma and the fact that $\mathscr{D}^b(\mathscr{A}^{\mathrm{op}})$ is triangle equivalent to $(\mathscr{D}^c(\mathscr{A}))^{\mathrm{op}}$ for any abelian category $\mathscr{A}$. $\qquad\square$

We are now in a position to prove the following

Theorem 7.9 (Auslander–Gruson–Jensen Duality for compact objects). *Let $\mathscr{D}(\mathscr{C}_R)^c$ be the full triangulated subcategory of $\mathscr{D}(\mathscr{C}_R)$ of compact objects. Then there is a duality*

$$D : (\mathscr{D}(\mathscr{C}_R)^c)^{\mathrm{op}} \to \mathscr{D}(\mathscr{C}_R)^c$$

that takes a compact object $C_\bullet$ to

$$D\mathscr{C}_\bullet := R\underline{\mathrm{Hom}}(C_\bullet, - \otimes_R R).$$

Proof. By Lemma 7.6 $\mathscr{D}(\mathscr{C}_R)^c \simeq \mathscr{D}^b(\mathrm{coh}\,\mathscr{C}_R)$. Let $\mathscr{D}^b(\mathrm{proj}\,\mathscr{C}_R)$ be a full subcategory of $\mathscr{D}(\mathscr{C}_R)^c$ consisting of bounded complexes of representable coherent functors. The composition

$$\mathscr{D}^b(\mathrm{proj}\,\mathscr{C}_R) \longrightarrow \mathscr{D}(\mathscr{C}_R)^c \xrightarrow{\;\simeq\;} \mathscr{D}^b(\mathrm{coh}\,\mathscr{C}_R)$$

is an equivalence of triangulated categories.

Lemma 7.7 and Corollary 7.8 imply $\underline{\mathrm{Hom}}(-, - \otimes R)$ is an equivalence of triangulated categories $\mathscr{D}(\mathrm{coh}\,\mathscr{C}_R) \simeq (\mathscr{D}^b(\mathrm{coh}\,\mathscr{C}_R))^{\mathrm{op}}$.

Now the composite of equivalences

$$\mathscr{D}^b(\mathrm{proj}\,\mathscr{C}_R) \to \mathscr{D}(\mathscr{C}_R)^c \to \mathscr{D}^b(\mathrm{coh}\,\mathscr{C}_R) \to (\mathscr{D}^b(\mathrm{coh}\,\mathscr{C}_R))^{\mathrm{op}} \to (\mathscr{D}(\mathscr{C}_R)^c)^{\mathrm{op}}$$

computes the desired equivalence $R\underline{\mathrm{Hom}}(-, - \otimes R)$ of triangulated categories. $\square$

Definition 7.10 (Hovey–Palmieri–Strickland [14]). A *stable homotopy category* is a category $\mathscr{C}$ with the following extra structure:

(1) A triangulation.
(2) A closed symmetric monoidal structure, compatible with the triangulation.
(3) A set $\mathscr{G}$ of strongly dualizable objects of $\mathscr{C}$, such that the only localizing subcategory of $\mathscr{C}$ containing $\mathscr{G}$ is $\mathscr{C}$ itself.
We also assume that $\mathscr{C}$ satisfies the following:
(4) Arbitrary coproducts of objects of $\mathscr{C}$ exist.
(5) Every cohomology functor on $\mathscr{C}$ is representable.

We shall say that such a category $\mathscr{C}$ is *algebraic* if the objects of $\mathscr{G}$ are compact. If, in addition, the unit object e is compact, we say that $\mathscr{C}$ is *unital algebraic*.

Definition 7.11. An *almost stable homotopy category* is a category $\mathscr{C}$ which satisfies axioms (1)-(2) and (4)-(5) for a stable homotopy category and the following axiom:

(3$'$) There is a full small subcategory $\mathscr{G}$ of $\mathscr{C}$ with duality $D : \mathscr{G}^{\mathrm{op}} \to \mathscr{G}$, such that the only localizing subcategory of $\mathscr{C}$ containing $\mathscr{G}$ is $\mathscr{C}$ itself.

Algebraic and *unital algebraic* almost stable homotopy categories are defined as in Definition 7.10.

Remark 7.12. Every stable homotopy category $\mathscr{C}$ with generating set of strongly dualizable objects $\mathscr{G}$ is an almost stable homotopy category. Indeed, we can assume without loss of generality that $x^\vee \in \mathscr{G}$ for every $x \in \mathscr{G}$, and then the full subcategory of $\mathscr{C}$ whose objects are those of $\mathscr{G}$ has a duality $x \mapsto x^\vee$ and generates $\mathscr{C}$. The following theorem shows that there are algebraic almost stable homotopy categories which are not stable homotopy categories.

Theorem 7.13. *Let R be a commutative ring. Then $\mathscr{D}(\mathscr{C}_R)$ is a unital algebraic almost stable homotopy category, which is not an algebraic stable homotopy category in the sense of Definition 7.10.*

Proof. Let $\mathscr{G}$ be the full subcategory of compact objects of $\mathscr{D}(\mathscr{C}_R)$. The fact that $\mathscr{D}(\mathscr{C}_R)$ is a unital algebraic almost stable homotopy category follows from Theorems 7.2 and 7.9. We also use here the fact that every cohomology functor on a compactly generated triangulated category is representable by a theorem of Neeman [19, Theorem 3.1].

Suppose $\mathscr{D}(\mathscr{C}_R)$ is generated by compact strongly dualizable objects $\mathscr{G}$ as required for an algebraic stable homotopy category. By [14, Theorem A.2.5] we may assume without loss of generality that $\mathscr{G}$ is a thick subcategory in the triangulated category of compact objects. If $\mathscr{G}$ generated $\mathscr{D}(\mathscr{C}_R)$, then another theorem of Neeman [19, Theorem 2.1] would imply that $\mathscr{G}$ contains all compact objects. But Example 7.5 shows that compact objects of $\mathscr{D}(\mathscr{C}_R)$ are not strongly dualizable in general. This contradiction finishes the proof. $\square$

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