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Relative Homological Algebra for the Proper Class ω_f#

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ABSTRACT

Homological algebra based on fp-injective and fp-flat modules is studied. It can be realized as relative homological algebra corresponding to the proper class of monomorphisms

$$\omega_f = \{\mu \mid R^I \otimes \mu \text{ is a monomorphism for any power } R^I \text{ of a ring } R\}.$$

Also, the corresponding homological functors Ext_f and Tor^f as well as various homological dimensions are investigated.

Key Words: Relative homological algebra; Homological dimensions; Homological functors on modules.

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The main object of this paper is to study homological aspects as well as further properties of fp-injective and fp-flat modules. These were introduced in Garkusha and Generalov (1999) to characterize FP-injective and weakly quasi-Frobenius rings. Both classes of modules are definable (=elementary) in the first order language of modules and naturally generalize the corresponding classes of FP-injective and flat modules.

Homological algebra based on FP-injective and flat modules is well studied and was developed in the 1970s (see e.g., Jain, 1973; Sklyarenko, 1978b; Stenström, 1970). One would like to construct homological algebra based on fp-injective and fp-flat modules. In this paper, we shall show that such homological algebra can be realized as relative homological algebra corresponding to the proper class of monomorphisms

$$\omega_f = \{\mu \mid R^I \otimes \mu \text{ is a monomorphism for any power } R^I \text{ of a ring } R\}.$$

The corresponding relative injective and relative flat modules, we call them f-injective and f-flat modules respectively, correspond to fp-injective pure-injective modules and fp-flat modules.

There are many results of a homological nature which may be generalized from Noetherian rings to coherent rings. In this direction, finitely generated modules are replaced by finitely presented modules and injective modules should be replaced by FP-injective modules (see Sklyarenko, 1978b; Stenström, 1970). In our situation analogous results, which are related to the relative homological algebra we construct in this paper, are valid for all rings. This phenomenon can be explained as follows. The classical Chase theorem (see Stenström, 1975) states that a ring R is left coherent iff each power R^I of I copies of the ring R is a flat right R-module. But if we consider only those exact sequences under which the functor $R^I \otimes_R -$ is exact, we shall arrive at the proper class ω_f.

Relative homological algebra is a classical subject, thanks to works by Eilenberg–Moore (1965) and Butler–Horrocks (1961). However these give us little information for our case. The idea is to use some localization of the category of generalized modules

$$_R\mathscr{C} = (\operatorname{mod} R, \mathrm{Ab})$$

to turn the relative homological algebra with respect to ω_f into an absolute one.

Every time we deal with definable subcategories the category $_R\mathscr{C}$ is of great utility in this context as well as torsion theories of finite type of $_R\mathscr{C}$ (consult e.g., the works Herzog, 1997; Krause, 2001). For this reason the majority of statements of the paper is proved by using the technique of torsion/localizing functors in $_R\mathscr{C}$.

The paper is organized as follows. The first two sections are preliminary. There we present the necessary category-theoretic background and introduce the proper class ω_f as a class of monomorphisms of a certain quotient category in $_R\mathscr{C}$. In Sec. 3 the classes of f-injective and f-flat modules are investigated. In Sec. 4 we study basic properties for the functors Ext_f and Tor^f which are related to f-injective and f-flat resolutions. Also, certain homological dimensions are discussed in this section. In Sec. 5 we describe the relative derived category $D_f(R)$ and the functor Ext_f as

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Hom-sets in $D_f(R)$. In the remaining section we study the relationship of the G-theory of a ring with the K-theory of some abelian subcategories of ${}_R\mathscr{C}$.

Throughout the paper R denotes a ring with identity. The category of left (respectively right) R-modules is denoted by $R\,\mathrm{Mod}$ (respectively $\mathrm{Mod}\,R$) and the category of finitely presented left (respectively right) R-modules by $R\,\mathrm{mod}$ (respectively $\mathrm{mod}\,R$).

1. THE CATEGORY OF GENERALIZED MODULES

The *category of generalized left R-modules*

$$ {}_R\mathscr{C} = (\mathrm{mod}\,R, \mathrm{Ab}) $$

consists of additive covariant functors defined on the category of finitely presented right R-modules mod R with values in the category of abelian groups Ab. In this section we collect basic facts about the category ${}_R\mathscr{C}$. For details and proofs we refer the reader to Herzog (1997) and Krause (2001), for example. All subcategories considered are assumed to be full.

We say that a subcategory $\mathscr{S}$ of an abelian category $\mathscr{C}$ is a *Serre subcategory* if for any short exact sequence

$$ 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0 $$

in $\mathscr{C}$ an object $Y \in \mathscr{S}$ if and only if $X,\ Z \in \mathscr{S}$. A Serre subcategory $\mathscr{S}$ of a Grothendieck category $\mathscr{C}$ is *localizing* if it is closed under taking direct limits. Equivalently, the inclusion functor $i : \mathscr{S} \to \mathscr{C}$ admits the right adjoint $t = t_{\mathscr{S}} : \mathscr{C} \to \mathscr{S}$ which takes every object $X \in \mathscr{C}$ to the maximal subobject $t(X)$ of X belonging to $\mathscr{S}$ (Krause, 1997). The functor t we call the *torsion functor*. An object C of ${}_R\mathscr{C}$ is said to be *$\mathscr{S}$-torsionfree* if $t(C) = 0$. Given a localizing subcategory $\mathscr{S}$ of ${}_R\mathscr{C}$ the *quotient category* ${}_R\mathscr{C}/\mathscr{S}$ consists of $C \in {}_R\mathscr{C}$ such that $t(C) = t^1(C) = 0$. The objects from ${}_R\mathscr{C}/\mathscr{S}$ we call *$\mathscr{S}$-closed objects*. Given $C \in {}_R\mathscr{C}$ there exists a canonical exact sequence

$$ 0 \longrightarrow A' \longrightarrow C \xrightarrow{\lambda_C} C_{\mathscr{S}} \longrightarrow A'' \longrightarrow 0 $$

with $A' = t(C)$, $A'' \in \mathscr{S}$, and where $C_{\mathscr{S}} \in {}_R\mathscr{C}/\mathscr{S}$ is the maximal essential extension of $\widetilde{C} = C/t(C)$ such that $C_{\mathscr{S}}/\widetilde{C} \in \mathscr{S}$. The object $C_{\mathscr{S}}$ is uniquely defined and is called the *$\mathscr{S}$-envelope* of C. Moreover, the inclusion functor $i : {}_R\mathscr{C}/\mathscr{S} \to {}_R\mathscr{C}$ has the left adjoint *localizing functor* $(-)_{\mathscr{S}} : {}_R\mathscr{C} \to {}_R\mathscr{C}/\mathscr{S}$, which is also exact. It takes each $C \in {}_R\mathscr{C}$ to $C_{\mathscr{S}} \in {}_R\mathscr{C}/\mathscr{S}$. Then,

$$ \mathrm{Hom}_{{}_R\mathscr{C}}(X, Y) \simeq \mathrm{Hom}_{{}_R\mathscr{C}/\mathscr{S}}(X_{\mathscr{S}}, Y) $$

for all $X \in {}_R\mathscr{C}$ and $Y \in {}_R\mathscr{C}/\mathscr{S}$.

An object X of a Grothendieck category $\mathscr{C}$ is *finitely generated* if whenever there are subobjects $X_i \subseteq X$ with $i \in I$ satisfying $X = \sum_{i\in I} X_i$, then there is a finite subset $J \subset I$ such that $X = \sum_{i\in J} X_i$. The subcategory of finitely generated objects is denoted by fg $\mathscr{C}$. A finitely generated object X is said to be *finitely presented* if every epimorphism $\gamma : Y \to X$ with $Y \in \mathrm{fg}\,\mathscr{C}$ has the finitely generated kernel $\mathrm{Ker}\,\gamma$. By fp $\mathscr{C}$ we denote the subcategory consisting of finitely presented objects. Finally, we refer to a finitely presented object $X \in \mathscr{C}$ as *coherent* if every finitely generated subobject of X is finitely presented. The corresponding subcategory of coherent objects will be denoted by coh $\mathscr{C}$.

The category ${}_R\mathscr{C}$ is a locally coherent Grothendieck category, i.e., every object $C \in {}_R\mathscr{C}$ is a direct limit $C = \varinjlim_I C_i$ of coherent objects $C_i \in \mathrm{coh}\,{}_R\mathscr{C}$. The category $\mathrm{coh}\,{}_R\mathscr{C}$ is abelian. Moreover, ${}_R\mathscr{C}$ has enough coherent projective generators $\{(M,-)\}_{M \in \mathrm{mod}\,R}$. Thus, every coherent object $C \in \mathrm{coh}\,{}_R\mathscr{C}$ has a projective presentation

$$(N,-) \longrightarrow (M,-) \longrightarrow C \longrightarrow 0,$$

where $M, N \in \mathrm{mod}\,R$.

A localizing subcategory $\mathscr{S}$ of ${}_R\mathscr{C}$ is of *finite type* if the inclusion functor $i : {}_R\mathscr{C}/\mathscr{S} \to {}_R\mathscr{C}$ commutes with direct limits. Then the quotient category ${}_R\mathscr{C}/\mathscr{S}$ is locally coherent and $\mathrm{coh}\,{}_R\mathscr{C}/\mathscr{S} = \{C_{\mathscr{S}} \mid C \in \mathrm{coh}\,{}_R\mathscr{C}\}$.

Proposition 1.1. *For a localizing subcategory $\mathscr{S}$ of ${}_R\mathscr{C}$ the following are equivalent*:

(1) *$\mathscr{S}$ is of finite type.*
(2) *The inclusion functor $i : {}_R\mathscr{C}/\mathscr{S} \to {}_R\mathscr{C}$ commutes with direct unions.*
(3) *The torsion functor $t_{\mathscr{S}}$ commutes with direct limits.*

Proof. The equivalence (1)$\Longleftrightarrow$(2) is a consequence of Garkusha (2001a, Theorem 5.14); (1)$\Longleftrightarrow$(3) follows from Krause (1997, Lemma 2.4). □

By a result of Herzog and Krause (Herzog, 1997; Krause, 1997) a localizing subcategory in ${}_R\mathscr{C}$ is of finite type iff it has the form $\vec{\mathscr{S}}$ with $\mathscr{S}$ some Serre subcategory of $\mathrm{coh}\,{}_R\mathscr{C}$ and

$$\vec{\mathscr{S}} = \left\{ \varinjlim_{i \in I} C_i \mid C_i \in \mathscr{S} \right\}.$$

We say that $M \in {}_R\mathscr{C}$ is a *coh-injective object* if $\mathrm{Ext}^1_{{}_R\mathscr{C}}(C, M) = 0$ for any $C \in \mathrm{coh}\,{}_R\mathscr{C}$. The fully faithful functor $-\otimes_R ? : R\,\mathrm{Mod} \to {}_R\mathscr{C}$, $M \mapsto -\otimes_R M$, identifies $R\,\mathrm{Mod}$ with the subcategory of coh-injective objects of ${}_R\mathscr{C}$. Moreover, the functor $-\otimes_R M \in \mathrm{coh}\,{}_R\mathscr{C}$ iff $M \in R\,\mathrm{mod}$. Given $C \in \mathrm{coh}\,{}_R\mathscr{C}$ there is an exact sequence

$$0 \longrightarrow C \longrightarrow -\otimes_R M \longrightarrow -\otimes_R N$$

in $\mathrm{coh}\,{}_R\mathscr{C}$ with $M, N \in R\,\mathrm{mod}$.

A monomorphism $\mu : M \to N$ in $R\,\mathrm{Mod}$ is a *pure monomorphim* if for any $K \in \mathrm{Mod}\,R$ the morphism $K \otimes \mu$ is a monomorphism. Equivalently, the ${}_R\mathscr{C}$-morphism $-\otimes \mu$ is a monomorphism. A module Q is *pure-injective* if the functor $\mathrm{Hom}_R(-, Q)$ takes the pure monomorphisms to epimorphisms. The injective objects of ${}_R\mathscr{C}$ are precisely the objects of the form $-\otimes_R Q$ with Q a pure-injective module.

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Let M be a left module; then

$$\mathscr{S}_M = \{C \in \operatorname{coh}{}_R\mathscr{C} \mid (C, - \otimes_R M) = 0\}$$

is Serre. Furthermore, every Serre subcategory of $\operatorname{coh}{}_R\mathscr{C}$ arises in this fashion.

Proposition 1.2. *Let $\mathscr{S}_M$ be the Serre subcategory of $\operatorname{coh}{}_R\mathscr{C}$ cogenerated by a left R-module M. Denote by $\mathscr{X}$ the subcategory of $R\,\mathrm{Mod}$ consisting of the modules of the form $\varinjlim_{k\in K} M^{I_k}$ with some sets of indices K and $I_k, k \in K$, and M^{I_k} the product of I_k copies of M. Then an object F of ${}_R\mathscr{C}$ is $\mathscr{S}_M$-torsionfree iff it is a subobject of $- \otimes_R L$ with $L \in \mathscr{X}$. Moreover,*

$$\vec{\mathscr{S}}_M = \{F \in {}_R\mathscr{C} \mid (F, - \otimes_R L) = 0 \text{ for all } L \in \mathscr{X}\}.$$

Proof. The proof is like that of Krause (2001, Theorem 2.1). Let $C \in \operatorname{coh}{}_R\mathscr{C}$ and I_C denote the set $(C, - \otimes_R M)$. We have an exact sequence

$$0 \longrightarrow X \longrightarrow C \xrightarrow{\rho} - \otimes_R M^{I_C}. \tag{1.1}$$

Here we use the relation $(- \otimes_R M)^{I_C} = - \otimes_R M^{I_C}$. We claim that $X = t(C)$. Indeed, if we apply the left exact $\mathscr{S}_M$-torsion functor to (1.1), we shall obtain

$$0 \longrightarrow t(X) \longrightarrow t(C) \longrightarrow t(- \otimes_R M^{I_C}) = 0.$$

Therefore $t(C) = t(X) \subseteq X$. Let Y be a finitely generated subobject of X such that there exists a non-zero morphism $\varphi : Y \to - \otimes_R M$. Since $- \otimes_R M$ is coh-injective, φ can be extended to a non-zero morphism $\psi : C \to - \otimes_R M$. But ψ factors through ρ. Hence $\varphi = 0$, a contradiction to our assumption. Thus $X = t(C)$.

Consider a direct limit of coherent objects $C = \varinjlim_{k\in K} C_k$. Then the sequence

$$0 \longrightarrow \varinjlim t(C_k) = t(C) \longrightarrow C \longrightarrow \varinjlim C_k/t(C_k) = C/t(C) \longrightarrow 0$$

is exact. Given $k \in K$ fix the monomorphism $C_k/t(C_k) \to - \otimes_R M^{I_{C_k}}$ constructed as above. Let $J_k = \{j \in K \mid k \leq j\}$ and $I_k = I_{C_k} \times J_k$. The canonically defined monomorphisms $C_k/t(C_k) \to - \otimes_R M^{I_k}$, $k \in K$, induce a monomorphism $C/t(C) \to - \otimes_R L$ with $L = \varinjlim_{k\in K} M^{I_k} \in \mathscr{X}$. Thus the $\mathscr{S}_M$-torsionfree objects of ${}_R\mathscr{C}$ are precisely the subobjects of the functors $- \otimes_R L$ with $L \in \mathscr{X}$.

Next, it is easy to see that $(C, - \otimes_R L) = 0$ for $C \in \vec{\mathscr{S}}_M$ and $L \in \mathscr{X}$. Given an object $F \in {}_R\mathscr{C}$ the relation $(F, - \otimes_R L) = 0$, for all $L \in \mathscr{X}$, implies the relation $(F/t(F), - \otimes_R L) = 0$. Therefore $F = t(F)$, as claimed. □

We refer to the subcategory

$$\mathscr{Z} = \{N \in R\ \mathrm{Mod} \mid - \otimes_R N \text{ is } \mathscr{S}_M\text{-torsionfree}\}$$

as a *definable subcategory*.

Corollary 1.3. *A left R-module belongs to* $\mathscr{Z}$ *iff it is a pure submodule of a module* L *from* $\mathscr{X}$.

In the sequel, we use the following Serre subcategories of $\operatorname{coh}_R\mathscr{C}$:

$$\mathscr{S}^R = \{C \in \operatorname{coh}_R\mathscr{C} \mid C(R) = 0\}$$
$$\mathscr{S}_R = \{C \in \operatorname{coh}_R\mathscr{C} \mid (C, -\otimes_R R) = 0\},$$

as well as the localizing subcategories of finite type $\vec{\mathscr{S}}^R$ and $\vec{\mathscr{S}}_R$

$$\vec{\mathscr{S}}^R = \{C \in {}_R\mathscr{C} \mid C = \varinjlim C_i, C_i \in \mathscr{S}^R\}$$
$$\vec{\mathscr{S}}_R = \{C \in {}_R\mathscr{C} \mid C = \varinjlim C_i, C_i \in \mathscr{S}_R\}.$$

The corresponding $\vec{\mathscr{S}}^R$-torsion and $\vec{\mathscr{S}}_R$-torsion functors will be denoted by $t_{\mathscr{S}^R}$ and $t_{\mathscr{S}_R}$ respectively.

2. PROPER CLASSES

Let T denote the functor that takes a left R-module M to the object $(-\otimes_R M)_{\mathscr{S}^R}$ in ${}_R\mathscr{C}/\vec{\mathscr{S}}^R$.

Lemma 2.1. *The functor T is fully faithful, right exact and preserves direct limits.*

Proof. Let $\mathscr{P}^R$ denote the localizing subcategory $\{F \in {}_R\mathscr{C} \mid F(R) = 0\}$ in ${}_R\mathscr{C}$. Then the functor $M \mapsto P(M) = (-\otimes_R M)_{\mathscr{P}^R}$ induces an equivalence between $R\operatorname{Mod}$ and ${}_R\mathscr{C}/\mathscr{P}^R$ (Garkusha and Generalov, 2001). A quasi-inverse functor to P is induced by the functor ${}_R\mathscr{C} \to \operatorname{Mod} R$ taking an object F of ${}_R\mathscr{C}$ to $F(R)$. Since $\vec{\mathscr{S}}^R \subseteq \mathscr{P}^R$ the functor P factors as $P = L \circ T$, where $L : {}_R\mathscr{C}/\vec{\mathscr{S}}^R \to {}_R\mathscr{C}/\mathscr{P}^R$ is the localization functor with respect to the localizing subcategory $\mathscr{P}^R/\vec{\mathscr{S}}^R$. It follows that T is faithful.

To show that T is full, it suffices to prove that the map $L(\mu) : LT(M) \to LT(N)$ with $0 \neq \mu : T(M) \to T(N)$ is non-zero. Assume the converse. Then $\mu(R) = 0$ and, hence, $D = \operatorname{Coker}_{{}_R\mathscr{C}}\mu$ belongs to $\mathscr{P}^R$.

Let $\lambda : T(N) \to LT(N)$ denote the $\mathscr{P}^R$-envelope of $T(N)$ in ${}_R\mathscr{C}$. Then $\lambda\mu = 0$ and therefore λ factors through D. Since $LT(N)$ is $\mathscr{P}^R$-torsionfree, wee see that $\lambda = 0$. This implies that $T(N)$ belongs to $\mathscr{P}^R$ and thus $N = 0$. It follows that $T(N) = 0$, a contradiction.

Since both the tensor functor and the localizing functor preserve direct limits (right exact), then so does T. □

A class of monomorphisms ω of $R\operatorname{Mod}$ is *proper* if it satisfies the following axioms:

- $(P0)$ ω contains all split monomorphisms.
- $(P1)$ The composition of two monomorphisms in ω, if defined, is also in ω.

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$(P2)$ If a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & B \\ \mu\big\downarrow & & \big\downarrow\mu' \\ A' & \xrightarrow[\sigma']{} & B' \end{array} \tag{2.1}$$

has $\sigma \in \omega$, then $\sigma' \in \omega$.

$(P3)$ If $\sigma\tau \in \omega$, then $\tau \in \omega$.

Proposition 2.2. *The class of R-monomorphisms*

$$\omega_f = \{\mu \mid T\mu \text{ is a monomorphism}\}$$

satisfies the axioms $P0-P3$.

Proof. The axioms $P0-P1$ and $P3$ are obvious. If the square (2.1) is pushout, then the sequence

$$A \xrightarrow{(-\mu,\sigma)^T} A' \oplus B \xrightarrow{(\sigma',\mu')} B' \longrightarrow 0 \quad \text{with } (\sigma',\mu') = \text{coker}(-\mu,\sigma)^T \text{ is exact.}$$

By Lemma 2.1 the sequence

$$TA \xrightarrow{(-T\mu,T\sigma)^T} TA' \oplus TB \xrightarrow{(T\sigma',T\mu')} TB' \longrightarrow 0$$

is exact. Thus $TB' = TA' \coprod_{TA} TB$. Since $T\sigma$ is a monomorphism, it follows that $T\sigma'$ is a monomorphism. This implies the claim. □

By Lemma 2.1 the proper class ω_f is inductively closed, i.e., given any direct system $\{\mu_i\}_{i\in I}$ of monomorphisms from ω_f the morphism $\varinjlim \mu_i$ is in ω_f. Below we shall show (see Proposition 3.10) that

$$\omega_f = \{\mu \mid R^I \otimes \mu \text{ is a monomorphism for any power } R^I \text{of } R\}.$$

The most obvious example of a monomorphism in ω_f is a pure monomorphism, because it is a direct limit of split monomorphisms (Sklyarenko, 1978a, Theorem 6.2). We say that a module M is *coinjective* if any extension of M belongs to ω_f. As an example, every extension $\mu: M \to E$ of a finitely presented module $M \in R\,\text{mod}$ belongs to ω_f. Indeed, if X is a finitely generated subobject of $\text{Ker}(-\otimes\mu)$, it is coherent and $X(R) = 0$. Hence $\text{Ker}(-\otimes\mu) \in \vec{\mathcal{S}}^R$. Therefore the morphism $T\mu$ is a monomorphism, i.e., $\mu \in \omega_f$.

Proposition 2.3. *A ring R is left coherent iff every left R-module is coinjective.*

Proof. Let R be left coherent. Then any monomorphism $\mu : M \to N$ is a direct limit $\mu = \varinjlim_I \mu_i$ of monomorphisms μ_i in $R\,\mathrm{mod}$ (Krause, 1998, Lemma 5.9). Since each $\mu_i \in \omega_f$, it follows that $\mu = \varinjlim \mu_i$ is also in ω_f. Therefore M is coinjective.

Assume the converse. Since $\mu \in \omega_f$ iff $\mathrm{Ker}(-\otimes \mu) \in \vec{\mathscr{S}}^R$ and since any monomorphism belongs to ω_f, our assertion follows from Garkusha and Generalov (2001, Theorem 2.5). □

A module $M \in R\,\mathrm{Mod}$ is a *f-submodule* of $N \in R\,\mathrm{Mod}$ (respectively *f-quotient module*) if there is a monomorphism $\mu : M \to N$ with $\mu \in \omega_f$ (respectively if there is an epimorphism $\mu : N \to M$ with $\ker \mu \in \omega_f$). A short exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0 \tag{2.2}$$

is *f-exact* if M is a f-submodule of N. Equivalently, if L is a f-quotient module of N. Clearly, the exact sequence (2.2) is f-exact iff the sequence

$$0 \longrightarrow TM \longrightarrow TN \longrightarrow TL \longrightarrow 0$$

is exact in ${}_R\mathscr{C}/\vec{\mathscr{S}}^R$.

We refer to a morphism $\mu : M \to N$ as an *f-epimorphism* if N is a f-quotient of M. The class of f-epimorphisms will be denoted by ω^f. A morphism $\mu : M \to N$ is an *f-homomorphism* if $\mu = \sigma\tau$ with $\sigma \in \omega_f$ and $\tau \in \omega^f$.

3. f-INJECTIVE AND f-FLAT MODULES

In this section we study relative injective and relative flat modules for the proper class ω_f.

Definitions. (1) A left R-module M is said to be *FP-injective* (or *absolutely pure*) if every monomorphism $\mu : M \to N$ is pure. Equivalently, for all $F \in R\,\mathrm{mod}$ we have: $\mathrm{Ext}^1_R(F, M) = 0$ (Stenström, 1970). A ring R is *left FP-injective* if the module ${}_RR$ is *FP*-injective.

(2) M is an *fp-injective module* if for every monomorphism $\mu : K \to L$ in $R\,\mathrm{mod}$ the morphism (μ, M) is an epimorphism. Clearly, *FP*-injective modules are *fp*-injective and every finitely presented *fp*-injective module is *FP*-injective.

(3) M is said to be *fp-flat* if for every monomorphism $\mu : K \to L$ in $\mathrm{mod}\,R$ the morphism $\mu \otimes M$ is a monomorphism. Clearly, every flat left R-module is *fp*-flat. The converse holds iff the ring R is right coherent (Garkusha and Generalov, 1999, Theorem 2.4).

Let the functor S take a module $M \in R\,\mathrm{Mod}$ to the functor $SM = (-\otimes_R M)_{\mathscr{S}_R}$.

Proposition 3.1. *The following statements are true*:

(1) *A module M is fp-injective iff $-\otimes_R M = TM$.*
(2) *A module M is fp-flat iff $-\otimes_R M = SM$.*
(3) *The subcategory of fp-injective (fp-flat) modules is definible.*

Moreover, every object TM (SM) *with* M *fp-injective* (*fp-flat*) *is coh-injective in* ${}_R\mathscr{C}/\vec{\mathscr{S}}^R$ $({}_R\mathscr{C}/\vec{\mathscr{S}}_R)$.

Proof. By Garkusha and Generalov (1999, Proposition 2.2) M is fp-injective (fp-flat) iff $-\otimes_R M$ is $\mathscr{S}^R$-torsionfree ($\mathscr{S}_R$-torsionfree). This implies (3). The rest of the proof is a consequence of Herzog (1997, Proposition 3.10). □

The *Auslander–Gruson–Jensen duality* D (see Auslander, 1986; Gruson and Jensen, 1981) takes each object $C \in \operatorname{coh} {}_R\mathscr{C}$ to the object $DC \in \operatorname{coh} \mathscr{C}_R$ defined as follows:

$$(DC)({}_RN) = \operatorname{Hom}_{{}_R\mathscr{C}}(C, -\otimes_R N)$$

and a morphism α of $\operatorname{coh} {}_R\mathscr{C}$ to the morphism $D\alpha$ of $\operatorname{coh} \mathscr{C}_R$ defined by the rule:

$$(D\alpha)({}_RN) = \operatorname{Hom}_{{}_R\mathscr{C}}(\alpha, -\otimes_R N).$$

Proposition 3.2 (Herzog, 1997, Proposition 5.6; Zimmermann-Huisgen and Zimmermann, 1990, Lemma 2). *Let* ${}_SM_R$ *be an* (S,R)*-bimodule and let* ${}_SE$ *be an injective S-module. Then for each* $C \in \operatorname{coh} \mathscr{C}_R$ *there is an isomorphism*

$$\operatorname{Hom}_S({}_S(C, M\otimes_R -), {}_SE) \simeq \operatorname{Hom}_{{}_R\mathscr{C}}(DC, -\otimes_R ({}_SM_R, {}_SE))$$

natural in C.

Throughout the paper the character module $\operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of a module M is denoted by $\widehat{M}$.

Since

$$D\mathscr{S}^R = {}_R\mathscr{S} = \{C \in \operatorname{coh} \mathscr{C}_R \mid (C, R\otimes_R -) = 0\}$$

and

$$D\mathscr{S}_R = {}^R\mathscr{S} = \{C \in \operatorname{coh} \mathscr{C}_R \mid C(R) = 0\}$$

it follows from Garkusha and Generalov (1999, Proposition 2.2) the following.

Corollary 3.3. *A left R-module* M *is fp-injective* (*fp-flat*) *iff its character module* $\widehat{M}$ *is fp-flat* (*fp-injective*).

Observe that the Serre subcategory $\mathscr{S}^R$ of $\operatorname{coh} {}_R\mathscr{C}$ is cogenerated by any injective cogenerator, say $\widehat{R}$, of $R\,\mathrm{Mod}$.

Proposition 3.4. *A left R-module is fp-injective* (*fp-flat*) *iff it is a pure submodule of* $\varinjlim_{k\in K} \widehat{R}^{I_k}$ $(\varinjlim_{k\in K} R^{I_k})$ *for some sets of indices* K *and* I_k, $k \in K$.

Proof. This follows from Proposition 1.2 and Corollary 1.3. □

A left R-module Q is said to be *f-injective* (respectively *f-projective*) if the functor $\mathrm{Hom}_R(-,Q)$ (respectively $\mathrm{Hom}_R(Q,-)$) takes the monomorphisms of ω_f (respectively the epimorphisms of ω^f) to epimorphisms.

Proposition 3.5. *An R-module Q is f-injective iff it is a fp-injective pure-injective module.*

Proof. Since an injective object $-\otimes_R Q$ of ${}_R\mathscr{C}$ is injective in ${}_R\mathscr{C}/\vec{\mathscr{S}}^R$ iff it is $\mathscr{S}^R$-closed, Proposition 3.1 implies that the ${}_R\mathscr{C}/\vec{\mathscr{S}}^R$-injective objects are precisely the fp-injective pure-injective modules.

Suppose that M is f-injective; then the ${}_R\mathscr{C}/\vec{\mathscr{S}}^R$-monomorphism $TM \to -\otimes_R Q = TQ$ with $-\otimes_R Q = E(TM)$ splits. Therefore, M is a pure-injective fp-injective module. The converse is easy. □

A f-monomorphism $\mu: M \to E = E_f(M)$ with E a f-injective module is called a *f-injective envelope* of M if any endomorphism φ of E such that $\varphi\mu = \mu$ is an isomorphism.

Lemma 3.6. *A f-monomorphism $\mu: N \to E$ is a f-injective envelope iff $T\mu: TN \to TE$ is an injective envelope of TN in ${}_R\mathscr{C}/\vec{\mathscr{S}}^R$. If $\mu': N \to E'$ is another f-injective envelope, then there exists an isomorphism $\psi: E \to E'$ such that $\psi\mu = \mu'$.*

Proof. Straightforward. □

The existence of injective envelopes in ${}_R\mathscr{C}/\vec{\mathscr{S}}^R$ implies the following.

Corollary 3.7. *A f-injective envelope of a module always exists.*

So we obtain that the proper class ω_f is *injective*, that is, every module is a f-submodule of a f-injective module. The *projective* proper classes are dually defined.

We refer to a right R-module M as a *f-flat module* if the tensor functor $M\otimes_R -$ preserves f-exact sequences.

Theorem 3.8. *A right R-module M is f-flat iff it is fp-flat. Moreover, the class of f-flat R-modules is closed under taking products and direct limits.*

Proof. Since every monomorphism μ in $R\,\mathrm{mod}$ belongs to ω_f, it follows that the f-flat R-modules are fp-flat R-modules.

Consider a fp-flat right R-module M and a f-monomorphism $\mu: K \to L$. We want to show that $1\otimes\mu: M\otimes_R K \to M\otimes_R L$ is a monomorphism. From Herzog (1997, Proposition 4.3) it follows that the character module $\widehat{M}$ is pure-injective. On the other hand, $\widehat{M}$ is fp-injective by Corollary 3.3. Proposition 3.5 implies that $\widehat{M}$ is f-injective. We see that the map

$$\mathrm{Hom}_R(L,\widehat{M}) \longrightarrow \mathrm{Hom}_R(K,\widehat{M}) \tag{3.1}$$

is an epimorphism. This map is isomorphic to the map

$$\mathrm{Hom}_{\mathbb{Z}}(M\otimes_R L, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(M\otimes_R K, \mathbb{Q}/\mathbb{Z}).$$

Since $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator in Ab, it follows that $1 \otimes \mu : M \otimes_R K \to M \otimes_R L$ is a monomorphism.

The fact that f-flat modules are closed under products and direct limits follows from Garkusha and Generalov (1999, Proposition 2.3). □

Corollary 3.9. *A module M is fp-flat iff its character module $\widehat{M}$ is an fp-injective pure-injective module. In other words, a module is f-flat iff its character module is a f-injective module.*

As for the class of flat R-modules by the Chase theorem (Stenström, 1975, Proposition I.13.3) every product of flat right R-modules is flat iff the ring R is left coherent.

The class ω_f is *flatly generated* by fp-flat modules, that is, $\mu \in \omega_f$ iff $M \otimes \mu$ is a monomorphism for any fp-flat module M. Indeed, if $\mu \in \omega_f$ and M is fp-flat, Theorem 3.8 implies that $M \otimes \mu$ is a monomorphism.

On the other hand, if $M \otimes \mu$ with $\mu : K \to L$ is a monomorphism for each fp-flat module M, the map (3.1) is an epimorphism. In particular, it is an epimorphism for $M = \widehat{Q}$ with Q a f-injective module and, hence, $\mathrm{Hom}_{{}_R\mathscr{C}/\vec{\mathscr{S}}^R}(\mathrm{Ker}\, T\mu, - \otimes_R Q^{\wedge\wedge}) = 0$. Since Q is a pure submodule of $Q^{\wedge\wedge}$, it follows that $\mathrm{Hom}_{{}_R\mathscr{C}/\vec{\mathscr{S}}^R}(\mathrm{Ker}\, T\mu, - \otimes_R Q) = 0$. As f-injectives cogenerate ${}_R\mathscr{C}/\vec{\mathscr{S}}^R$ this implies $\mathrm{Ker}\, T\mu = 0$, and, hence, $\mu \in \omega_f$.

Proposition 3.10. *ω_f is flatly generated by modules R^I with I a set of indices, R^I the product of I copies of R.*

Proof. Let F be a fp-flat module. By Proposition 3.4 it is a pure submodule of $\varinjlim_{k \in K} R^{I_k}$ with K, I_k, $k \in K$, some sets of indices. Let $0 \to M \xrightarrow{\mu} N \to L \to 0$ be an exact sequence and the map $R^I \otimes \mu$ is a monomorphism for any set I. By the above arguments it suffices to show that $F \otimes M$ is a monomorphism. One has the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & F \otimes_R M & \xrightarrow{F\otimes\mu} & F \otimes_R N & \longrightarrow & F \otimes_R L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \varinjlim_k R^{I_k} \otimes_R M & \longrightarrow & \varinjlim_k R^{I_k} \otimes_R N & \longrightarrow & \varinjlim_k R^{I_k} \otimes_R L & \longrightarrow & 0. \end{array}$$

Since the vertical arrows are monomorphisms, $F \otimes \mu$ is a monomorphism. Thus $\mu \in \omega_f$, as claimed. □

Below we discuss some properties of fp-injective and fp-flat modules we shall use later.

Proposition 3.11. *For a left R-module M the following statements are equivalent:*

(1) *M is fp-injective.*
(2) *Any f-exact sequence $0 \to M \to M' \to M'' \to 0$ is pure.*
(3) *There is a pure-exact sequence $0 \to M \to M' \to M'' \to 0$ with M' a fp-injective module.*
(4) $\mathrm{Ext}^1_{{}_R\mathscr{C}/\vec{\mathscr{S}}^R}(TF, TM) = 0$ *for all $F \in R$* mod.

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Moreover, any product and any direct limit of fp-injective modules is also a fp-injective module.

Proof. $(1) \Longrightarrow (2)$. Let $F \in R\,\mathrm{mod}$ then $TF \in \mathrm{coh}\,{}_R\mathscr{C}/\vec{\mathscr{S}}^R$. Since TM is coh-injective in ${}_R\mathscr{C}/\vec{\mathscr{S}}^R$ by Proposition 3.1, it follows that the sequence

$$0 \longrightarrow (TF, TM) \longrightarrow (TF, TM') \longrightarrow (TF, TM'') \longrightarrow 0$$

is exact. By Lemma 2.1 the sequence

$$0 \longrightarrow (F, M) \longrightarrow (F, M') \longrightarrow (F, M'') \longrightarrow 0$$

is exact. So $0 \to M \to M' \to M'' \to 0$ is pure-exact.

$(2) \Longrightarrow (3)$. We may take M' to be a f-injective envelope of M.

$(3) \Longrightarrow (4)$. Since $-\otimes_R M$ is a subobject of the $\mathscr{S}^R$-torsionfree object $-\otimes_R M'$, it is also $\mathscr{S}^R$-torsionfree. From Herzog (1997, Proposition 3.10) it follows that $TM = -\otimes_R M$ is a coh-injective object for ${}_R\mathscr{C}/\vec{\mathscr{S}}^R$.

$(4) \Longrightarrow (1)$. Let μ be a monomorphism in $R\,\mathrm{mod}$; then $T\mu$ is a monomorphism. By assumption, $(T\mu, TM)$ is an epimorphism. Since T is full and faithful by Lemma 2.1, the morphism (μ, M) is a monomorphism.

The fact that the class of fp-injective modules is closed under taking products and direct limits follows from Garkusha and Generalov (1999, Proposition 2.3). □

It is well known that a direct limit of *FP*-injective left *R*-modules is *FP*-injective iff the ring *R* is left coherent (see, e.g., Stenström, 1970).

Corollary 3.12. *If in a short f-exact sequence* $0 \to M \to M' \to M'' \to 0$ *modules* M *and* M' *are fp-injective, then so is* M''.

Proof. By the preceding proposition the sequence of the corollary is pure-exact. If we apply the right exact $\mathscr{S}^R$-torsion functor $t = t_{\mathscr{S}^R}$ to the ${}_R\mathscr{C}$-exact sequence

$$0 \longrightarrow -\otimes_R M \longrightarrow -\otimes_R M' \longrightarrow -\otimes_R M'' \longrightarrow 0$$

we shall get an exact sequence

$$0 = t(-\otimes_R M') \longrightarrow t(-\otimes_R M'') \longrightarrow t^1(-\otimes_R M) = 0.$$

Hence M'' is fp-injective by Garkusha and Generalov (1999, Proposition 2.2). □

The following two propositions extend the list of properties characterizing the coherent and Noetherian rings respectively (cf. Sklyarenko, 1978b).

Proposition 3.13. *The following assertions are equivalent for a ring R:*

(1) *R is left coherent.*
(2) *The character module of a fp-injective left module is a flat module.*
(3) *A left module is fp-injective iff its character module is a flat module.*
(4) *A pure-injective envelope of a right f-flat module is a flat module.*

Proof. (1) $\Longrightarrow$ (2). Over a left coherent ring every fp-flat right module is flat Garkusha and Generalov (1999, Theorem 2.4). Now our implication follows from Corollary 3.3.

(2) $\Longrightarrow$ (3). This follows from Corollary 3.3.

(3) $\Longrightarrow$ (1). Suppose that the character module of an fp-injective left module M is flat. Then M is a pure submodule of the injective module $M^{\wedge\wedge}$. Hence M is a FP-injective module. Now our assertion follows from Garkusha and Generalov (1999, Theorem 2.4).

(1)$\Longrightarrow$(4). A pure-injective envelope Q of a f-flat module M is always a f-flat module. Indeed, the $\mathscr{C}_R$-injective functor $Q \otimes_R -$ is an injective invelope of $M \otimes_R -$ in $\mathscr{C}_R$. Since $M \otimes_R -$ is $\mathscr{S}_R$-torsionfree, then so is $Q \otimes_R -$. Proposition 3.1 implies that Q is f-flat. By assumption, R is left coherent and therefore Q is flat by Garkusha and Generalov (1999, Theorem 2.4).

(4) $\Longrightarrow$ (1). It is directly verified that given a pure-exact sequence

$$0 \longrightarrow M \longrightarrow Q \longrightarrow N \longrightarrow 0$$

with Q flat, the module M is also flat.

Suppose that a right R-module M is f-flat. Then its pure-injective envelope Q is flat by assumption. Hence M is so. Therefore R is left coherent by Garkusha and Generalov (1999, Theorem 2.4). □

Remark. By a recent result of Rothmaler (2002) there are non-coherent rings over which pure-injective envelopes of flat modules are flat.

Proposition 3.14. *The following assertions are equivalent for a ring R:*

(1) *R is left Noetherian.*
(2) *Every left fp-injective module is an injective module.*
(3) *A left module is f-injective iff its character module is a flat module.*

Proof. (1) $\Longrightarrow$ (2). Since every left ideal I of R is finitely presented, then the representable functor $\mathrm{Hom}_R(-, M)$ takes the inclusion $I \subset R$ to an epimorphism whenever M is fp-injective. Therefore M is an injective module by the Baer criterion.

(2) $\Longrightarrow$ (1). It suffices to observe that every direct limit of fp-injective modules is a fp-injective module.

(1), (2) $\Longrightarrow$ (3). This implication follows from Proposition 3.13.

$(3) \Longrightarrow (2)$. Let E be a f-injective envelope of a fp-injective module M. By Proposition 3.11 M is a pure submodule of E. Therefore $\widehat{M}$ is a direct summand of the flat module $\widehat{E}$. Then M is f-injective by assumption. By the preceding proposition the ring R is left coherent. Now our assertion follows from the fact that over the left coherent rings the classes of f-injective and injective left modules coincide (see Garkusha and Generalov, 1999, Theorem 2.4). □

We conclude the section by characterizing the class of *FP*-injective rings. We give the most interesting criteria for this class of rings in terms of f-flat and f-injective modules. For details and proofs we refer the reader to Garkusha (1999) and Garkusha and Generalov (1999).

Recall that $M \in R\,\mathrm{Mod}$ is a *FP-cogenerator* if for any non-zero homomorphism $\rho : K \to L$ with K a finitely generated R-module and L a finitely presented module there exists a homomorphism $\mu : L \to M$ such that $\mu\rho \neq 0$. Equivalently, every finitely presented left R-module is a submodule of a product M^I of some copies of the module M (Garkusha, 1999).

Lemma 3.15. *M is a FP-cogenerator iff for any non-zero morphism $\rho : K \to L$ in $R\,\mathrm{mod}$ there exists a morphism $\mu : L \to M$ such that $\mu\rho \neq 0$.*

Proof. The necessary condition is trivial. Assume the converse. Let $\rho : K \to L$ be a non-zero homomorphism from a finitely generated R-module K to a finitely presented module L. Then there exists an epimorphism $\pi : R^n \to K$. By assumption, there exists a homomorphism $\mu : L \to M$ such that $\mu(\rho\pi) = (\mu\rho)\pi \neq 0$. Hence $\mu\rho \neq 0$. □

Theorem 3.16 (Garkusha, 1999; Garkusha and Generalov, 1999)**.** *For a ring R the following conditions are equivalent*:

(1) *The module R_R is FP-injective.*
(2) *The module ${}_RR$ is a FP-cogenerator.*
(3) *There is a f-flat cogenerator in $R\,\mathrm{Mod}$.*
(4) *Every left R-module is a $(f$-$)$submodule of a f-flat module.*
(5) *Every fp-injective left R-module is a f-flat module.*
(6) *Every f-injective left R-module is a f-flat module.*
(7) *Every f-flat right R-module is a fp-injective module.*
(8) *Every pure-injective f-flat right R-module is an f-injective module.*

Remark. The most difficult statements for *FP*-injective rings are proved inside of the category ${}_R\mathscr{C}$ and use localization theory in ${}_R\mathscr{C}$. It would be interesting to have a proof (especially of the equivalence $(1) \Longleftrightarrow (4)$) by using usual module-theoretic technique. The author does not know such a proof.

Rings over which any module is embedded into a projective module are quasi-Frobenius. This is the classical Faith–Walker theorem. Rings over which any module is embedded into a flat module, the IF-rings, were investigated in the early 1970s (see, e.g., Colby, 1975; Jain, 1973; Stenström, 1970). The equivalent statements of the preceding theorem are, in a certain sense, similar to properties both for QF-rings and for IF-rings. We have thus the following proper inclusions:

$$\left\{\begin{array}{c}\text{QF-rings}\\ \scriptstyle\text{modules are submodules}\\ \scriptstyle\text{of projective modules}\end{array}\right\} \subsetneq \left\{\begin{array}{c}\text{IF-rings}\\ \scriptstyle\text{modules are submodules}\\ \scriptstyle\text{of flat modules}\end{array}\right\} \subsetneq \left\{\begin{array}{c}\text{FP-injective rings}\\ \scriptstyle\text{modules are submodules}\\ \scriptstyle\text{of } f\text{-flat modules}\end{array}\right\}.$$

The class of FP-injective rings is big enough. For example, given an arbitrary left self-injective ring R and a locally finite group G, the group ring $R(G)$ is left FP-injective (Garkusha, 1999, Theorem 3.2). Moreover, if $|G| = \infty$, the group ring $R(G)$ is left FP-injective but not left self-injective (Garkusha, 1999, Corollary 3.3).

4. THE FUNCTORS Ext_f AND Tor^f

Definitions. Let N be a left R-module.

(1) The *f-injective dimension* $\text{inj.dim}_f N$ is the minimum integer n (if it exists) such that there is a f-exact sequence

$$0 \to N \to E^0 \to E^1 \to \cdots \to E^n \to 0$$

with $E^0, \ldots, E^n$ being f-injective modules. We call such a sequence a *f-resolution* of N by f-injective modules.

(2) The *left global f-dimension* $\text{l.gl.dim}_f R$ of a ring R is $\sup\{\text{inj.dim}_f N \mid N \in R\,\text{Mod}\}$.

(3) The *fp-injective dimension* $\text{fp-inj.dim}\, M$ is the minimum integer n (if it exists) such that there is a f-resolution

$$0 \to N \to E^0 \to E^1 \to \cdots \to E^n \to 0$$

of N by fp-injective modules $E^0, \ldots, E^n$.

If no finite resolution exists, we set $\text{inj.dim}_f N$, $\text{fp-inj.dim}\, N$ equal to ∞.

(4) The *left global fp-dimension* $\text{l.gl.fp-dim}\, R$ of a ring R is $\sup\{\text{fp-inj.dim}\, N \mid N \in R\,\text{Mod}\}$.

(5) Let

$$0 \to N \to E^0 \to E^1 \to \cdots$$

be a f-injective f-resolution for N. Then $\text{Ext}^n_f(M, N)$, $M \in R\,\text{Mod}$, $n \geq 0$, denote the cohomology groups for the complex

$$0 \to \text{Hom}_R(M, E^0) \to \text{Hom}_R(M, E^1) \to \cdots$$

Note that

$$0 \to TN \to TE^0 \to TE^1 \to \cdots$$

is an injective resolution in ${}_R\mathscr{C}/\vec{\mathscr{S}}^R$ for TN by $TE^i = -\otimes_R E^i$ and the cohomology groups for the complex

$$0 \to (TM, TE^0) \to (TM, TE^1) \to \cdots$$

are the $\mathrm{Ext}_f^n(M, N)$, $n \geq 0$. It follows that

$$\mathrm{Ext}_f^n(M, N) = \mathrm{Ext}^n_{{}_R\mathscr{C}/\vec{\mathscr{S}}^R}(TM, TN).$$

Let us remark that any f-exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

yields a long exact sequence for any $F \in R\,\mathrm{Mod}$

$$\cdots \to \mathrm{Ext}_f^{n-1}(F, N) \to \mathrm{Ext}_f^n(F, L) \to \mathrm{Ext}_f^n(F, M) \to \mathrm{Ext}_f^n(F, N) \to \cdots$$

Lemma 4.1. *The following are equivalent for a left R-module N*:

(1) $\mathrm{inj.dim}_f\, N \leq n$.
(2) $\mathrm{Ext}_f^p(M, N) = 0$ *for all* $p > n$ *and all R-modules M*.
(3) $\mathrm{Ext}_f^{n+1}(M, N) = 0$ *for all R-modules M*.
(4) *If* $0 \to N \to E^0 \to \cdots \to E^{n-1} \to L^n \to 0$ *is a f-resolution of N by f-injective modules* E^i, *then* L^n *is also f-injective.*

Proof. Easy. □

Corollary 4.2. $\mathrm{l.gl.dim}_f\, R = \sup\{n \mid \mathrm{Ext}_f^n(M, N) \neq 0$ *for some R-modules M and N*$\}$.

An acyclic complex $M^* = (M^n, d^n)$ is said to be *f-acyclic* if each d^n, $n \in \mathbb{Z}$, is an f-homomorphism. Similarly to the absolute case, the elements of $\mathrm{Ext}_f^n(M, N)$, $n \geq 1$, are represented by f-acyclic complexes of the form (see Generalov, 1992):

$$0 \to N \to M^{n-1} \to M^{n-2} \to \cdots \to M^0 \to M \to 0.$$

The proof of the following lemma is straightforward (cf. Stenström, 1970, Lemma 3.1).

Lemma 4.3. *The following are equivalent for a left R-module N*:

(1) $\text{fp-inj.dim}\, N \leq n$.
(2) $\mathrm{Ext}_f^p(M, N) = 0$ *for all* $p > n$ *and all finitely presented R-modules M*.
(3) $\mathrm{Ext}_f^{n+1}(M, N) = 0$ *for all finitely presented R-modules M*.
(4) *If* $0 \to N \to E^0 \to \cdots \to E^{n-1} \to L^n \to 0$ *is a f-resolution of N by fp-injective modules* E^i, *then* L^n *is also fp-injective.*

It is well known that a ring R is left Noetherian iff every direct limit of left modules of injective dimension $\leq n$ has injective dimension $\leq n$. In turn, let FP-inj. dim N denote the smallest integer $n \geq 0$ such that there is a resolution

$$0 \to N \to E^0 \to E^1 \to \cdots \to E^n \to 0$$

of N by FP-injective modules E^i. Then R is left coherent iff every direct limit of left modules of FP-injective dimension $\leq n$ has FP-injective dimension $\leq n$ (Stenström, 1970, Theorem 3.2).

Corollary 4.4. *Let R be an arbitrary ring. Then every direct limit of left modules of fp-injective dimension $\leq n$ has fp-injective dimension $\leq n$.*

Proof. Let $\{M_i\}_I$ be a direct system of left modules of fp-injective dimension $\leq n$ and let

$$0 \to M_i \to E_i^0 \to E_i^1 \to \cdots$$

be the f-injective f-resolution for M_i, $i \in I$, constructed as follows. Choose a direct system of f-injective modules $E_i^0 \supseteq M_i$ in the following way. Let Q be a f-injective cogenerator. Such a module exists: we can take an injective cogenerator $-\otimes_R Q$ in ${}_R\mathscr{C}/\vec{\mathscr{S}}^R$ and then Q is a f-injective cogenerator. For every $i \in I$, we set Q_i equal to $Q^{\mathrm{Hom}(M_i,Q)}$, $J_i = \{j \in I \mid i \leq j\}$ and $E_i^0 = \prod_{j \in J_i} Q_j$. The canonically defined f-monomorphisms $M_i \to E_i^0$, $i \in I$, induce a f-monomorphism $M \to \varinjlim E_i^0$. The modules $E_i^{k \geq 1}$ are similarly constructed.

By Lemma 4.3

$$0 \to M_i \to E_i^0 \to \cdots \to E_i^{n-1} \to L_i^n \to 0$$

is a f-resolution of M_i by fp-injective modules. Since every direct limit of fp-injective modules is fp-injective, M has a f-resolution

$$0 \to M \to \varinjlim E_i^0 \to \cdots \to \varinjlim E_i^{n-1} \to \varinjlim L_i^n \to 0$$

by fp-injective modules. Hence fp-inj. dim $M \leq n$. □

For a left finitely presented module F and a direct system $\{M_i\}_I$ of left modules, we consider the canonical homomorphism

$$\xi_n : \varinjlim_I \mathrm{Ext}_f^n(F, M_i) \longrightarrow \mathrm{Ext}_f^n(F, \varinjlim_I M_i).$$

Recall that finite presentation of F is equivalent to ξ_0 being an isomorphism for every $\{M_i\}_I$ (Stenström, 1975, Proposition V.3.4).

Theorem 4.5. $\xi_n : \varinjlim_I \mathrm{Ext}_f^n(F, M_i) \longrightarrow \mathrm{Ext}_f^n(F, \varinjlim_I M_i)$ *are isomorphisms for all* $n \geq 0$, *for every finitely presented module F and direct system* $\{M_i\}_I$.

Proof. ξ_0 is an isomorphim. First we shall show that ξ_1 is an isomorphism. Let us consider the f-injective f-resolution

$$0 \to M_i \to E_i^0 \xrightarrow{d_i^0} E_i^1 \xrightarrow{d_i^1} \cdots$$

for each M_i constructed above.

Let $L_i^0 = E_i^0/M_i$. The module $\varinjlim_I E_i^0$ is fp-injective and, hence, $\mathrm{Ext}_f^1(F, \varinjlim_I E_i^0) = 0$. We get then the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\varinjlim(F, E_i^0) & \longrightarrow & \varinjlim(F, L_i^0) & \longrightarrow & \varinjlim \mathrm{Ext}_f^1(F, M_i) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \xi_0} & & \downarrow{\scriptstyle \xi_0} & & \downarrow{\scriptstyle \xi_1} & & \\
(F, \varinjlim E_i^0) & \longrightarrow & (F, \varinjlim L_i^0) & \longrightarrow & \mathrm{Ext}_f^1(F, \varinjlim M_i) & \longrightarrow & 0.
\end{array}$$

Since the first two arrows of the diagram are isomorphisms, ξ_1 is an isomorphism as well.

Let $L_i^n = \mathrm{Im}\, d_i^n$. Then $\mathrm{Ext}_f^n(F, M_i) = \mathrm{Ext}_f^1(F, L_i^{n-2})$, $n \geq 2$. The sequence

$$0 \to \varinjlim M_i \to \varinjlim E_i^0 \to \varinjlim E_i^1 \to \cdots$$

is a f-resolution of $\varinjlim M_i$ by fp-injective modules. Then

$$\varinjlim \mathrm{Ext}_f^n(F, M_i) = \varinjlim \mathrm{Ext}_f^1(F, L_i^{n-2}) \xrightarrow{\xi_1} \mathrm{Ext}_f^1(F, \varinjlim L_i^{n-2}) = \mathrm{Ext}_f^n(F, \varinjlim M_i).$$

Since ξ_1 is an isomorphism, it follows that ξ_n is an isomorphism. □

Stenström (1970) showed that $\xi_n : \varinjlim_I \mathrm{Ext}_R^n(F, M_i) \longrightarrow \mathrm{Ext}_R^n(F, \varinjlim_I M_i)$ are isomorphisms for all $n \geq 0$, for every finitely presented module F and direct system $\{M_i\}_I$ iff a ring R is left coherent.

Corollary 4.6. *Let M be a left module and $M = \varinjlim_I M_i$ with $M_i \in R\,\mathrm{mod}$. Then $\mathrm{Ext}_f^n(F, M) = \varinjlim_I \mathrm{Ext}_f^n(F, M_i)$ for any finitely presented left module F.*

The following is an immediate consequence of the preceding statements.

Corollary 4.7. *The following numbers are the same for any ring R:*

(1) $\mathrm{l.\,gl.\,fp\text{-}dim}\, R$.
(2) $\sup\{n \mid \mathrm{Ext}_f^n(F, M) \neq 0$ *for some* $F \in R\,\mathrm{mod}$ *and* $M \in R\,\mathrm{Mod}\}$.
(3) $\sup\{\mathrm{fp\text{-}inj.\,dim}\, M \mid M \in R\,\mathrm{mod}\}$.

A ring R is said to be *almost regular* if every (both left and right) module is fp-injective; equivalently, f-flat (see Garkusha and Generalov, 1999). Since every (both left and right) module is fp-injective, every f-exact sequence is pure by

Proposition 3.11. An almost regular ring is (von Neumann) regular iff it is left or right coherent (Garkusha and Generalov, 1999). Let $\mathrm{l.\,gl.\,FP\text{-}dim}\,R$ denote $\sup\{\mathrm{FP\text{-}inj.\,dim}M \mid M \in {}_R\mathrm{Mod}\}$.

Obviously, $\mathrm{l.\,gl.\,FP\text{-}dim}\,R = \mathrm{l.\,gl.\,fp\text{-}dim}\,R$ when R is left coherent. Also, $\mathrm{l.\,gl.\,fp\text{-}dim}\,R = \mathrm{l.\,gl.\,dim}\,R$ when R is left Noetherian. By Stenström (1970, Proposition 3.6) $\mathrm{l.\,gl.\,FP\text{-}dim}\,R = 0$ iff R is (von Neumann) regular. In turn, it follows that $\mathrm{l.\,gl.\,fp\text{-}dim}\,R = \mathrm{r.\,gl.\,fp\text{-}dim}\,R = 0$ iff R is almost regular. The class of non-regular almost regular rings is big enough. Simple almost regular rings we call *indiscrete rings* (Garkusha and Generalov, 1999; Prest et al., 1995). Given a non-regular finite dimensional algebra of finite representation type one can construct a non-regular indiscrete ring as a twisted limit of matrix rings (Prest et al., 1995, Sec. 2.4). In turn, if R is a non-regular almost regular ring, G a non-trivial locally finite group and the order $|H|$ of every finite subgroup H of G is invertible in R, then the group ring $R(G)$ is a non-regular non-indiscrete almost regular ring (Garkusha, 2001b).

There is an interesting problem in our context. A *f-flat precover* of a right module M is a f-epimorphism $\varphi : F \to M$ with F a f-flat module such that the induced map $\varphi^* : \mathrm{Hom}_R(F', F) \to \mathrm{Hom}_R(F', M)$ is an epimorphism for any f-flat module F'. We refer to φ as a *f-cover* if for every endomorphism $\psi : F \to F$ the relation $\varphi\psi = \varphi$ implies ψ is an isomorphism.

Question. Every right R-module has a f-flat (pre-)cover.

Rings over which every module has a f-flat cover exist. For example, over an almost regular ring every module is f-flat. This is the easiest case. In the absolute case, every module has a flat cover. This has recently been proved by Bican et al. (2001).

Proposition 4.8. *For a ring R the following conditions are equivalent*:

(1) $\mathrm{l.\,gl.\,fp\text{-}dim}\,R \leq 1$.
(2) *Every f-quotient module of a f-injective module is fp-injective.*
(3) *Every f-quotient module of a fp-injective module is fp-injective.*
(4) *If in a short exact sequence $0 \to M \to E \to N \to 0$ with M a finitely presented module a module E is (FP-)injective, then N is fp-injective.*

Proof. Apply Corollary 4.7. □

A ring R is left *f-semihereditary* if it satisfies the equivalent conditions of the preceding proposition.

Corollary 4.9. *A left f-semihereditary ring R is left semihereditary iff it is left coherent.*

Proof. Any left semihereditary ring is left coherent. Conversely, over a left coherent ring the functors Ext_R^* and Ext_f^* coincide. Now our assertion follows from Sklyarenko (1978b, Proposition 1.22). □

If $M = \varinjlim_{\alpha \in I} M_\alpha$ with M_α finitely presented modules, then there exists a spectral sequence for each module N

$$E_2^{pq} = \varprojlim{}_I^p \operatorname{Ext}_R^q(M_\alpha, N) \Longrightarrow \operatorname{Ext}_R^n(M, N).$$

In Jensen (1972, Théorème 4.2), the spectral sequence is constructed from a double complex. Precisely, we construct the pure-exact resolution of M

$$\cdots \xrightarrow{\partial_3} R_2 \xrightarrow{\partial_2} R_1 \xrightarrow{\partial_1} R_0 \xrightarrow{\partial_0} M,$$

where

$$R_n = \bigoplus_{\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n} M_{\alpha_0, \alpha_1, \ldots, \alpha_n}$$

and $M_{\alpha_0, \alpha_1, \ldots, \alpha_n}$ is a copy of M_{α_0}.

Let

$$N \longrightarrow E^0 \xrightarrow{\partial^0} E^1 \xrightarrow{\partial^1} E^2 \xrightarrow{\partial^2} \cdots$$

be an injective resolution of N. Then the bicomplex is

$$E_0^{pq} = \operatorname{Hom}_R(R_p, E^q),$$

where the vertical and horizontal differentials are

$$d_0^{pq} = (-1)^p \operatorname{Hom}_R(R_p, \partial^q) : E_0^{pq} \longrightarrow E_0^{p,q+1}$$

and

$$d_1^{pq} = (-1)^p \operatorname{Hom}_R(\partial_{p+1}, E^q) : E_0^{pq} \longrightarrow E_0^{p+1,q},$$

respectively.

If we replace the injective resolution $N \to E^*$ by an f-injective f-resolution of N and also apply Corollary 4.6, we shall get the following result expressing Ext_f in terms of abelian groups $\operatorname{Ext}_f^*(M_\alpha, N_\beta)$ with M_α, N_β finitely presented.

Theorem 4.10. *Let M and N be two modules, $M = \varinjlim_\alpha M_\alpha$ and $N = \varinjlim_\beta N_\beta$ with M_α and N_β finitely presented modules. Then the following relation is valid*:

$$\varprojlim{}_I^p \varinjlim{}_J \operatorname{Ext}_f^q(M_\alpha, N_\beta) = \varprojlim{}_I^p \operatorname{Ext}_f^q(M_\alpha, N) \Longrightarrow \operatorname{Ext}_f^n(M, N).$$

Suppose M has a f-resolution by f-flat modules $F_* \to M$. Given a left module N we put

$$\operatorname{Tor}_*^f(M, N) = H_*(F_* \otimes_R N).$$

Any f-exact sequence of left R-modules

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

yields a long exact sequence for any $G \in \operatorname{Mod} R$ having a f-flat f-resolution

$$\cdots \to \operatorname{Tor}^f_{n+1}(G,N) \to \operatorname{Tor}^f_n(G,L) \to \operatorname{Tor}^f_n(G,M) \to \operatorname{Tor}^f_n(G,N) \to \cdots$$

Remark. The definition of the functor Tor^f depends on the choice of a f-flat f-resolution. In the absolute case, the groups Tor^R do not depend on the choice of a flat resolution (see e.g., Weibel, 1995) since every module admits a projective resolution. It is not clear, however, whether the class ω_f is projective. The case of almost regular rings is trivial, because $\operatorname{Tor}^f_n \equiv 0$ for all $n \geq 1$.

We consider the duality homomorphisms

$$\rho : \operatorname{Ext}^n_R(M, \widehat{N}) \longrightarrow \operatorname{Tor}^R_n(M,N)\widehat{\ }$$

and

$$\sigma : \operatorname{Tor}^R_n(\widehat{N}, F) \longrightarrow \operatorname{Ext}^n_R(F,N)\widehat{\ }$$

where $M \in \operatorname{Mod} R$, $F, N \in R \operatorname{Mod}$. The first homomorphism is always an isomorphism (Cartan, 1956, Proposition VI.5.1) whereas σ is an isomorphism for every finitely generated module F whenever R is a left Noetherian (Cartan, 1956, Proposition VI.5.3) or for every finitely presented module F whenever R is a left coherent (Sklyarenko, 1978b, Lemma 1.12).

It is not clear, however, whether a homomorphism $\widehat{\mu}$ is a f-homomorphism if μ is a f-homomorphism. Therefore we specify some classes of modules to construct the analogous duality homomorphisms corresponding to the functors Ext_f and Tor^f. Precisely, let $\mathscr{F}(R)$ ($\mathscr{F}(R^{\mathrm{op}})$) denote the subcategory of right (left) modules M that have a f-flat f-resolution $F_* \to M$ such that $\widehat{M} \to \widehat{F}^*$ is a f-injective f-resolution for the character module $\widehat{M}$. The class $\mathscr{F}(R)$ is non-empty, because every fp-flat module belongs to $\mathscr{F}(R)$. In a similar way, $\mathscr{I}(R^{\mathrm{op}})$ ($\mathscr{I}(R)$) is the subcategory of left right modules M that have f-injective f-resolution $M \to E^*$ such that $\widehat{E}_* \to \widehat{M}$ is a f-flat f-resolution for the character module $\widehat{M}$. Obviously the fp-injective left modules belong to $\mathscr{I}(R^{\mathrm{op}})$.

Proposition 4.11. *For a right FP-injective ring R and $M \in \mathscr{I}(R^{\mathrm{op}})$ the following are equivalent*:

(1) fp-inj. dim $M = 0$.
(2) fp-inj. dim $M \leq n$.

Proof. The implication (1) $\Longrightarrow$ (2) is trivial. Let us show (2) $\Longrightarrow$ (1). Let $M \to E^*$ be a f-injective f-resolution for M such that $\widehat{E}_* \to \widehat{M}$ is a f-flat f-resolution for $\widehat{M}$.

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Then the module $K^n = \operatorname{Im} d^{n-1}$ in the exact sequence

$$0 \to M \to E^0 \to \cdots \to E^{n-1} \xrightarrow{d^{n-1}} \to K^n \to 0$$

is fp-injective. It is enough to show that $L^{n-1} = \operatorname{Ker} d^{n-1}$ is fp-injective. By assumption, the sequence

$$0 \to \widehat{K}^n \to \widehat{E}^{n-1} \to \cdots \to \widehat{E}^0 \to \widehat{M} \to 0$$

is a f-flat f-resolution of $\widehat{M}$. By Theorem 3.16 all the E^i, $i < n$, and the K^n are f-flat modules. Therefore $\widehat{K}^n$ is a direct summand of $\widehat{E}^{n-1}$ by Corollary 3.9. Hence $\widehat{L}^{n-1}$ is a direct summand of $\widehat{E}^{n-1}$ as well. This implies $\widehat{L}^{n-1}$ is f-flat, and L^{n-1} is fp-injective by Corollary 3.3. □

Given modules $M \in \operatorname{Mod} R$ and $N \in \mathscr{F}(R^{\mathrm{op}})$ we consider the duality homomorphism

$$\rho : \operatorname{Ext}^n_f(M, \widehat{N}) \longrightarrow \operatorname{Tor}^f_n(M, N)^\frown.$$

Similar to the absolute case (Cartan, 1956, Proposition VI.5.1) ρ is an isomorphism. It follows that given two f-flat f-resolutions for N we obtain the same functors $\operatorname{Tor}^f_n(-, N)^\frown$. This follows from the fact that the functors $\operatorname{Ext}^n_f(-, \widehat{N})$ do not depend on the choice of a f-injective f-resolution for $\widehat{N}$ and that ρ is an isomorphism.

Let $F \in R\operatorname{Mod}$ and $N \in \mathscr{I}(R^{\mathrm{op}})$. There is a f-injective f-resolution $N \to E^*$ for N such that $\widehat{E}_* \to \widehat{N}$ is a f-flat f-resolution for the character module $\widehat{N}$. In this case we consider the groups $\operatorname{Tor}^f_n(\widehat{N}, F)$ relative to this resolution. There is a duality homomorphism

$$\sigma : \operatorname{Tor}^f_n(\widehat{N}, F) \longrightarrow \operatorname{Ext}^n_f(F, N)^\frown .$$

If F is finitely presented, then each natural homomorphism $\widehat{E}^i \otimes_R F \to \operatorname{Hom}_R(F, E^i)^\frown$ with E^i the ith component of the complex E^*, $i \geq 0$, is an isomorphism. We see that σ is an isomorphism whenever F is finitely presented.

Now let F be a finitely generated left module without finite presentations, $0 \to H \to P \xrightarrow{\varphi} F \to 0$ an exact sequence with P a finitely generated free module. Then $F = \varinjlim F_\alpha$ where $F_\alpha = P/H_\alpha$, H_α are finitely generated submodules of H. Moreover, $\overrightarrow{\varphi} = \varinjlim \varphi_\alpha$ where φ_α are epimorphisms $P \to F_\alpha$. Let G denote the kernel $\operatorname{Ker} T\varphi$, G_α the kernel $\operatorname{Ker} T\varphi_\alpha$ and $D_\alpha = G/G_\alpha$.

Lemma 4.12. *If a left module F is finitely generated without finite presentations and N is a f-injective module, then the kernel of the natural homomorphism $\widehat{N} \otimes_R F \to \operatorname{Hom}_R(F, N)^\frown$ is $\varinjlim \operatorname{Hom}_{_R\mathscr{C}/\vec{\mathscr{S}}^R}(D_\alpha, TN)^\frown = \varinjlim \operatorname{Hom}_{_R\mathscr{C}/\vec{\mathscr{S}}^R}(D_\alpha, - \otimes_R N)^\frown$.*

Proof. Indeed, for each α we have an exact sequence $0 \to D_\alpha \to TF_\alpha \to TF \to 0$. If N is a f-injective module, then the sequence

$$0 \longrightarrow (D_\alpha, TN)^\frown \longrightarrow (TF_\alpha, TN)^\frown \longrightarrow (TF, TN)^\frown \longrightarrow 0$$

is exact. Since the functor T is fully faithful and the functor of direct limit is exact, we get the exact sequence

$$0 \longrightarrow \varinjlim(D_\alpha, TN)^\frown \longrightarrow \varinjlim \operatorname{Hom}_R(F_\alpha, N)^\frown \longrightarrow \operatorname{Hom}_R(F, N)^\frown \longrightarrow 0.$$

All the modules F_α are finitely presented, and so all the groups $\operatorname{Hom}_R(F_\alpha, N)^\frown$ are isomorphic to $\widehat{N} \otimes_R F_\alpha$. It remains to observe that $\varinjlim \widehat{N} \otimes_R F_\alpha = \widehat{N} \otimes_R F$. □

Theorem 4.13 (The Sklyarenko exact sequence)**.** *If F is a finitely generated left module, then for any module $N \in \mathscr{I}(R^{\mathrm{op}})$ the duality homomorphism σ fits into the exact sequence*

$$\cdots \xrightarrow{\sigma} \operatorname{Ext}_f^{n+1}(F, N)^\frown \xrightarrow{\delta} \varinjlim \operatorname{Ext}^n_{{}_R\mathscr{C}/\vec{\mathscr{S}}^R}(D_\alpha, TN)^\frown \to \operatorname{Tor}_n^f(\widehat{N}, F) \to$$
$$\xrightarrow{\sigma} \operatorname{Ext}_f^n(F, N)^\frown \to \cdots \xrightarrow{\delta} \varinjlim \operatorname{Hom}_{{}_R\mathscr{C}/\vec{\mathscr{S}}^R}(D_\alpha, TN)^\frown \to \widehat{N} \otimes_R F \xrightarrow{\sigma} \operatorname{Hom}_R(F, N)^\frown \to 0.$$

Proof. Let $N \to E^*$ be the chosen f-injective f-resolution of N. By the preceding lemma the sequence of complexes

$$0 \longrightarrow \varinjlim(D_\alpha, TE^*)^\frown \longrightarrow \widehat{E}_* \otimes_R F \longrightarrow \operatorname{Hom}_R(F, E^*)^\frown \longrightarrow 0 \qquad (4.1)$$

is exact. The sequence of the theorem is the homological sequence that corresponds to (4.1). Indeed, the homology groups of the complex $\operatorname{Hom}_R(F, E^*)^\frown$ are obviously the groups $\operatorname{Ext}_f^*(F, N)^\frown$, the homology groups of the complex $\widehat{E}_* \otimes_R F$ are $\operatorname{Tor}_*^f(\widehat{N}, F)$. Finally,

$$H_n(\varinjlim(D_\alpha, TE^*)^\frown) = \varinjlim H_n((D_\alpha, TE^*)^\frown) = \varinjlim H^n((D_\alpha, TE^*))^\frown$$
$$= \varinjlim \operatorname{Ext}^n_{{}_R\mathscr{C}/\vec{\mathscr{S}}^R}(D_\alpha, TN)^\frown$$

because the homology functor H_n commutes with the exact functors $\varinjlim$ and $X \to \widehat{X}$. □

Now we suppose that R is a left coherent ring, and let F be a finitely generated left module and $0 \to G \to P \to F \to 0$ an exact sequence with P a finitely generated free module. Then $F = \varinjlim F_\alpha$ where $F_\alpha = P/G_\alpha$ and G_α are finitely presented submodules of P. As above, let D_α denote G/G_α.

Corollary 4.14 (Sklyarenko, 1978b)**.** *If R is a left coherent ring, F is a finitely generated left module, then for any module $N \in R\,\mathrm{Mod}$ the duality homomorphism σ fits into an exact sequence*

$$\cdots \xrightarrow{\sigma} \operatorname{Ext}_R^{n+1}(F, N)^\frown \xrightarrow{\delta} \varinjlim \operatorname{Ext}_R^n(D_\alpha, N)^\frown \to \operatorname{Tor}_n^R(\widehat{N}, F) \to$$
$$\xrightarrow{\sigma} \operatorname{Ext}_R^n(F, N)^\frown \to \cdots \xrightarrow{\delta} \varinjlim \operatorname{Hom}_R(D_\alpha, N)^\frown \to \widehat{N} \otimes_R F \xrightarrow{\sigma} \operatorname{Hom}_R(F, N)^\frown \to 0.$$

Proof. Since over a left coherent ring the character module of any injective module is flat, the proof is similar to that of Theorem 4.13. □

5. THE DERIVED CATEGORIES $D_f(R)$ AND $D({}_R\mathscr{C}/\vec{\mathscr{S}}^R)$

In this section we use Generalov's construction of the relative derived category (see Generalov, 1992) to describe the derived category of the locally coherent Grothendieck category ${}_R\mathscr{C}/\vec{\mathscr{S}}^R$.

Let $K(R)$ and $K({}_R\mathscr{C}/\vec{\mathscr{S}}^R)$ denote the corresponding homotopy categories of $R\,\mathrm{Mod}$ and ${}_R\mathscr{C}/\vec{\mathscr{S}}^R$, i.e., the quotient categories of $\mathrm{Kom}(R)$ and $\mathrm{Kom}({}_R\mathscr{C}/\vec{\mathscr{S}}^R)$ respectively modulo homotopy equivalence. The functor T induces the fully faithful embeddings

$$\mathrm{Kom}(R) \longrightarrow \mathrm{Kom}({}_R\mathscr{C}/\vec{\mathscr{S}}^R)$$

and

$$K(R) \longrightarrow K({}_R\mathscr{C}/\vec{\mathscr{S}}^R)$$

that take a complex $M^* = (M^n, d^n)$ to $TM^* = (TM^n, Td^n)$. We recall that a *mapping cone* of a morphism $\mu : M^* \to N^*$ in $\mathrm{Kom}(R)$ is defined as the complex

$$C(\mu) = (M^{n+1} \oplus N^n, d_{C(\mu)}), \quad d_{C(\mu)=}\begin{pmatrix} -d_M & 0 \\ \mu & d_N \end{pmatrix}$$

We have the following sequence of complexes

$$M \xrightarrow{\mu} N \xrightarrow{\nu} C(\mu) \xrightarrow{\rho} M[1]$$

where $(M[1])^n = M^{n+1}$, $\nu^n : N^n \to C(\mu)^n$ and $\rho^n : C(\mu)^n \to M^{n+1}$ are the canonical injection and projection respectively. The family of such sequences (up to isomorphism the so-called ''distinguish triangles'') defines the structure of a triangulated category on the homotopy category $K(R)$ (see Gelfand and Manin, 1988; Verdier, 1977; Weibel, 1995).

A morphism $\mu : X^* \to Y^*$ in $K(R)$ is called a *f-quasi-isomorphism* if its mapping cone $C(\mu)$ is f-acyclic. Obviously, if μ is a f-quasi-isomorphism then $T\mu$ is a quasi-isomorphism in $K({}_R\mathscr{C}/\vec{\mathscr{S}}^R)$. We denote by $\mathscr{S}_f$ the class of all f-quasi-isomorphisms in $K(R)$. By Generalov (1992) it is localising in $K(R)$. One can thus construct the localization of $K(R)$ with respect to $\mathscr{S}_f$, and we define the relative derived category $D_f(R)$ of $R\,\mathrm{Mod}$ as this localization: $D_f(R) = K(R)[\mathscr{S}_f]$ (see Generalov, 1992). The category $D_f(R)$ inherits the structure of a triangulated category from $K(R)$. It is easy to see that the functor T induces a map from $D_f(R)$ to the derived category $D({}_R\mathscr{C}/\vec{\mathscr{S}}^R)$ of ${}_R\mathscr{C}/\vec{\mathscr{S}}^R$. If we start with the homotopy category $K^+(R)$ (respectively $K^-(R)$ or $K^b(R)$) of complexes bounded from below (respectively bounded from above or bounded complexes), then we get in a similar way the derived categories $D_f^+(R)$ (respectively $D_f^-(R)$ or $D_f^b(R)$).

Theorem 5.1. *The triangulated functor* $T : D_f^+(R) \longrightarrow D^+({}_R\mathscr{C}/\vec{\mathscr{S}}^R)$ *is an equivalence of categories.*

Proof. Let $K^+(\mathscr{I}_f)$ denote the homotopy category of complexes bounded from below over the full subcategory $\mathscr{I}_f$ of $R\,\mathrm{Mod}$ consisting of f-injectives. It is shown in Generalov (1992) that the natural functor $\Phi : K^+(\mathscr{I}_f) \to D_f^+(R)$ taking $E^* \in K^+(\mathscr{I}_f)$ to its image in $D_f^+(R)$ is an equivalence of categories. Obviously, the functor

$$T : K^+(\mathscr{I}_f) \longrightarrow K^+(\mathscr{I}), \quad E^* \mapsto TE^*$$

is an equivalence of $K^+(\mathscr{I}_f)$ and the homotopy category of complexes bounded from below over the full subcategory $\mathscr{I}$ of ${}_R\mathscr{C}/\vec{\mathscr{S}}^R$ consisting of ${}_R\mathscr{C}/\vec{\mathscr{S}}^R$-injectives. Consider the following commutative diagram

$$\begin{array}{ccc} K^+(\mathscr{I}_f) & \xrightarrow{T} & K^+(\mathscr{I}) \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\tilde{\Phi}} \\ D_f^+(R) & \xrightarrow{T} & D^+({}_R\mathscr{C}/\vec{\mathscr{S}}^R) \end{array}$$

in which the natural functor $\widetilde{\Phi} : K^+(\mathscr{I}) \to D^+({}_R\mathscr{C}/\vec{\mathscr{S}}^R)$ is an equivalence of categories (Gelfand and Manin, 1988, Theorem 3.5.21). It follows that the required functor is an equivalence as well. □

Let the functor I_f send a module $M \in R\,\mathrm{Mod}$ to the complex $\cdots 0 \to M \to 0 \cdots$ concentrated at zero degree. Then the following relation holds:

$$\mathrm{Ext}_f^n(M, N) = \mathrm{Hom}_{D_f(R)}(I_f(M), I_f(N)[n])$$

where $M, N \in R\,\mathrm{Mod}$ Generalov (1992). The groups $\mathrm{Ext}_f^n(M, N)$ can also be defined Generalov (1992) by using f-acyclic complexes of the form:

$$0 \to N \to M^{n-1} \to \cdots \to M^0 \to M \to 0.$$

On the other hand, let the functor I take every $M \in R\,\mathrm{Mod}$ to the complex $\cdots 0 \to TM \to 0 \cdots$ Then

$$\mathrm{Ext}_f^n(M, N) = \mathrm{Hom}_{D({}_R\mathscr{C}/\vec{\mathscr{S}}^R)}(I(M), I(N)[n]).$$

6. K-GROUPS FOR THE CATEGORY $\mathrm{coh}\,{}_R\mathscr{C}/\vec{\mathscr{S}}^R$

An *exact category* $\mathscr{C}$ is a full subcategory of an abelian category $\mathscr{A}$ which is closed under extensions and which contains a zero object of $\mathscr{A}$. A sequence $0 \to E' \to E \to E'' \to 0$ in $\mathscr{C}$ is called *exact* if it is exact in $\mathscr{A}$. We say that a map $i : M \to N$ in an exact category $\mathscr{C}$ is an *admissible monomorphism* if there is a short exact sequence $0 \to M \to N \to L \to 0$ in $\mathscr{C}$. Similarly $j : M \to N$ in an exact category $\mathscr{C}$ is an *admissible epimorphism* if there is a short exact sequence $0 \to L \to M \to N \to 0$ in $\mathscr{C}$.

If $\mathscr{M}$ is an exact category define $Q\mathscr{M}$ to be the category with the same objects as $\mathscr{M}$ and with morphisms defined as follows. A morphism from M to N in $Q\mathscr{M}$ is an equivalence class of diagrams of the form

$$M \twoheadleftarrow X \rightarrowtail N$$

in $\mathscr{M}$. Here $\twoheadrightarrow$ denotes an admissible epimorphism and $\rightarrowtail$ denotes an admissible monomorphism. We say that $M \twoheadleftarrow X \rightarrowtail N$ and $M \twoheadleftarrow Y \rightarrowtail N$ are equivalent if there is an isomorphism $X \simeq Y$ making

$$\begin{array}{ccccc} M & \longleftarrow & X & \longrightarrow & N \\ \| & & \downarrow\simeq & & \| \\ M & \longleftarrow & Y & \longrightarrow & N \end{array}$$

commutative (see Quillen, 1973; Swan, 1995). The Quillen K-theory space (Quillen, 1973) of $\mathscr{M}$ is the pointed space

$$K(\mathscr{M}) = \Omega|Q\mathscr{M}|$$

with $|Q\mathscr{M}|$ standing for the geometric realization of the category $Q\mathscr{M}$. Its homotopy groups are, by definition, the K-groups $K_n(\mathscr{M}), n \geq 0$.

Theorem 6.1 (Resolution). *Let $\mathscr{C}$ be an exact category and let $\mathscr{M} \subseteq \mathscr{C}$ be a full subcategory of $\mathscr{C}$ closed under extensions. Assume*

(1) *If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence in $\mathscr{C}$ with M' and M in $\mathscr{M}$, then M'' is in $\mathscr{M}$.*
(2) *For every object C in $\mathscr{C}$ there is a finite resolution*

$$0 \longrightarrow C \longrightarrow M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_n \longrightarrow 0$$

with M_i in $\mathscr{M}$.

Then $K(\mathscr{M}) \longrightarrow K(\mathscr{C})$ is a homotopy equivalence (and thus $K_n(\mathscr{M}) \xrightarrow{\simeq} K_n(\mathscr{C})$).

Lemma 6.2. *For any $C \in \operatorname{coh}_R\mathscr{C}/\vec{\mathscr{S}}^R$ there is an exact sequence*

$$0 \longrightarrow C \xrightarrow{i} TM \xrightarrow{T\mu} TN \longrightarrow TL \longrightarrow 0$$

with M, N and L finitely presented modules.

Proof. By Herzog (1997, Theorem 2.16) there exists a coherent object D of $\operatorname{coh}_R\mathscr{C}$ such that $C = D_{\mathscr{S}^R}$. The object D fits into an exact sequence

$$0 \longrightarrow D \longrightarrow - \otimes_R M \xrightarrow{-\otimes\mu} \otimes_R - N \longrightarrow - \otimes_R L \longrightarrow 0$$

with $M, N, L \in R\,\mathrm{mod}$. If we apply the exact functor of $\mathscr{S}^R$-localization to this sequence, we shall obtain the exact sequence of lemma. □

Lemma 6.3. *Let* $\mathscr{M} = \{TM \mid M \in R\,\mathrm{mod}\}$. *Then* $\mathscr{M}$ *is closed under extensions in* $\mathrm{coh}\,{}_R\mathscr{C}/\vec{\mathscr{S}}^R$.

Proof. Let

$$0 \longrightarrow TM \longrightarrow C \longrightarrow TN \longrightarrow 0$$

be an exact sequence in $\mathrm{coh}\,{}_R\mathscr{C}/\vec{\mathscr{S}}^R$ with M and N finitely presented. Then the module $K = C(R)$ is finitely presented. By the preceding lemma there is an exact sequence

$$0 \longrightarrow C \xrightarrow{i} TL \xrightarrow{T\mu} TE$$

with L and E finitely presented modules. Then $K = \mathrm{Ker}\mu$ and there is a unique morphism $\varphi : TK \to C$ such that $T\rho = i\varphi$ with $\rho = \mathrm{Ker}\mu$. We claim that φ is an isomorphism.

The module $F = L/K$ is finitely presented and the exact sequence

$$0 \longrightarrow K \longrightarrow L \longrightarrow F \longrightarrow 0$$

is f-exact. We have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & TK & \xrightarrow{T\rho} & TL & \xrightarrow{T\nu} & TF & \longrightarrow & 0 \\ & & \varphi\downarrow & & \| & & \downarrow T\alpha & & \\ 0 & \longrightarrow & C & \underset{i}{\longrightarrow} & TL & \underset{T\mu}{\longrightarrow} & TE. & & \end{array}$$

Since F is a finitely presented submodule of E, it follows that $T\alpha$ is a monomorphism. By the snake lemma φ is an isomorphism and, hence, $C \in \mathscr{M}$. $\square$

G-theory of a ring R is, by definition, the K-theory of the exact category of finitely presented modules $R\,\mathrm{mod}$. The preceding two lemmas imply the following.

Proposition 6.4. *The subcategory* $\mathscr{M} = T(R\,\mathrm{mod})$ *of* $\mathscr{C}$ *satisfies all the hypotheses of the Resolution Theorem. In particular, the functor* T *induces an isomorphism of* K*-groups* $G_n(R) \xrightarrow{\simeq} K_n(\mathrm{coh}\,{}_R\mathscr{C}/\vec{\mathscr{S}}^R)$, $n \geq 0$.

Let ${}^R\mathscr{S} = \{C \in \mathrm{coh}\,\mathscr{C}_R \mid C(R) = 0\}$. The Auslander–Gruson–Jensen duality D takes the category ${}^R\mathscr{S}$ to $\mathscr{S}_R$. By Herzog (1997, Theorem 5.5) D induces a duality

$$D : \mathrm{coh}\,\mathscr{C}_R/{}^R\vec{\mathscr{S}} \longrightarrow \mathrm{coh}\,{}_R\mathscr{C}/\vec{\mathscr{S}}_R.$$

Then the classifying spaces for the categories $Q\,\mathrm{coh}\,\mathscr{C}_R/{}^R\vec{\mathscr{S}}$ and $Q\,\mathrm{coh}\,{}_R\mathscr{C}/\vec{\mathscr{S}}_R$ are homeomorphic.

Proposition 6.5. *The functors T and D induce an isomorphism of K-groups $G_n(R^{\mathrm{op}}) \xrightarrow{\simeq} K_n(\mathrm{coh}\,{}_R\mathscr{C}/\vec{\mathscr{S}}_R)$, $n \geq 0$.*

Corollary 6.6. *If R is a left and right FP-injective ring, then the functors T and D induce an isomorphism of K-groups $G_n(R^{\mathrm{op}}) \xrightarrow{\simeq} G_n(R)$, $n \geq 0$.*

Proof. A ring R is left and right FP-injective iff $\mathscr{S}^R = \mathscr{S}_R$ (Garkusha and Generalov, 1999, Theorem 2.5). Therefore our assertion follows from the preceding two propositions. □

The duality D yields isomorophisms of K-groups $K_i(\mathrm{coh}\,{}_R\mathscr{C}) \xrightarrow{\simeq} K_i(\mathrm{coh}\,\mathscr{C}_R)$. Since every coherent object $C \in \mathrm{coh}\,{}_R\mathscr{C}$ fits into an exact sequence

$$0 \to C \to - \otimes_R M \to - \otimes_R N \to - \otimes_R L \to 0$$

the Resolution theorem implies that the K-theory of $\mathrm{coh}\,{}_R\mathscr{C}$ is equivalent to the K-theory of $R\mathrm{mod}^{\oplus}$. Here $R\mathrm{mod}^{\oplus}$ denotes the exact category of finitely presented left R-modules where only short exact sequences are used. Let ${}_R\mathscr{S}$ denote the Serre subcategory $D\mathscr{S}^R = \{C \in \mathrm{coh}\,\mathscr{C}_R \mid (C, R \otimes_R -) = 0\}$ of $\mathrm{coh}\,\mathscr{C}_R$.

Proposition 6.7. *The functors T and D induce two isomorphic long exact sequences*

$$\cdots G_{n+1}(R) \xrightarrow{\delta} K_n(\mathscr{S}^R) \to K_n(R\,\mathrm{mod}^{\oplus}) \to G_n(R) \to \\ \cdots \xrightarrow{\delta} K_0(\mathscr{S}^R) \to K_0(R\,\mathrm{mod}^{\oplus}) \to G_0(R) \to 0$$

and

$$\cdots G_{n+1}(R) \xrightarrow{\delta} K_n({}_R\mathscr{S}) \to K_n(\mathrm{mod}^{\oplus}R) \to G_n(R) \to \\ \cdots \xrightarrow{\delta} K_0({}_R\mathscr{S}) \to K_0(\mathrm{mod}^{\oplus}R) \to G_0(R) \to 0.$$

Proof. Since $\mathscr{S}^R$ is a Serre subcategory of $\mathrm{coh}\,{}_R\mathscr{C}$, there is a long exact sequence Quillen (1973, Theorem 5.5)

$$\cdots \to K_1(\mathrm{coh}\,{}_R\mathscr{C}/\vec{\mathscr{S}}^R) \xrightarrow{\delta} K_0(\mathscr{S}^R) \to K_0(R\,\mathrm{mod}^{\oplus}) \to K_0(\mathrm{coh}\,{}_R\mathscr{C}/\vec{\mathscr{S}}^R) \to 0.$$

By Proposition 6.4 $G_n(R)$ is isomorphic to $K_n(\mathrm{coh}\,{}_R\mathscr{C}/\vec{\mathscr{S}}^R)$ for all $n \geq 0$. We obtain then the first long exact sequence. Proposition 6.5 implies the second long exact sequence. □

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REFERENCES

Auslander, M. (1986). Isolated singularities and almost split sequences. In: *Representation Theory II*. Lecture Notes in Mathematics No. 1178, Springer-Verlag, pp. 194–242.

Bican, L., El Bashir, R., Enochs, E. (2001). All modules have flat covers. *Bull. London Math. Soc.* 33(4):385–390.

Butler, M. C. R., Horrocks, G. (1961). Classes of extensions and resolutions. *Philos. Trans. Roy. Soc. London, Ser. A* 254:155–222.

Cartan, H., Eilenberg, S. (1956). *Homological Algebra*. Princeton University Press.

Colby, R. R. (1975). Rings which have flat injective modules. *J. Algebra* 35:239–252.

Eilenberg, S., Moore, J. C. (1965). Foundations of relative homological algebra. *Mem. Am. Math. Soc.* 55.

Garkusha, G. A. (1999). FP-injective and weakly quasi-Frobenius rings. *Zap. Nauchn. Sem. S.-Peterburg. Otd. Mat. Inst. Steklov.* (*POMI*) 265:110–129 (Russian). English transl. (2002) in *J. Math. Sci.* 112(3) (also available at www.pdmi.ras.ru/znsl).

Garkusha, G. A. (2001a). Grothendieck categories. *Algebra i Analiz* 13(2):1–68, (Russian). English transl. (2002) in *St. Petersburg Math. J.* 13(2).

Garkusha, G. A. (2001b). A note on almost regular group rings. *Zap. Nauchn. Sem. S.-Peterburg. Otd. Mat. Inst. Steklov.* (*POMI*) 281:128–132 (Russian) (also available at www.pdmi.ras.ru/znsl).

Garkusha, G. A., Generalov, A. I. (1999). Duality for categories of finitely presented modules. *Algebra i Analiz* 11(6):139–152, (Russian). English transl. (2000) in *St. Petersburg Math. J.* 11(6).

Garkusha, G. A., Generalov, A. I. (2001). Grothendieck categories as quotient categories of $(R\,\mathrm{mod},\ \mathrm{Ab})$. *Fund. Appl. Math.* 7(4):983–992 (Russian).

Gelfand, S. I., Manin, Yu. I. (1988). *Methods of Homological Algebra. I.* Russia: Nauka (Russian).

Generalov, A. I. (1992). Relative homological algebra in preabelian categories, I: Derived categories. *Algebra i Analiz* 4(1):98–119, (Russian). English transl. in *St. Petersburg Math. J.* 4(1) (1993).

Gruson, L., Jensen, C. U. (1981). Dimensions cohomologiques reliées aux foncteurs $\varprojlim\,(i)$. In: *Paul Dubreil and Marie-Paule Malliavin Algebra Seminar*. Lecture Notes in Mathematics, No. 867. Springer-Verlag, pp. 234–294.

Herzog, I. (1997). The Ziegler spectrum of a locally coherent Grothendieck category. *Proc. London Math. Soc.* 74(3):503–558.

Jain, S. (1973). Flat and FP-injectivity. *Proc. Am. Math. Soc.* 41(2):437–442.

Jensen, C. U. (1972). *Les Foncteurs Dérivés de* $\varprojlim^{(i)}$ *et Leurs Applications en Théorie des Modules*. Lecture Notes in Mathematics, No. 254. Springer-Verlag.

Krause, H. (1997). The spectrum of a locally coherent category. *J. Pure Appl. Algebra* 114:259–271.

Krause, H. (1998). Functors on locally finitely presented additive categories. *Colloq. Math.* 75:105–132.

Krause, H. (2001). The spectrum of a module category. *Mem. Am. Math. Soc.* 149:707.

Prest, M., Rothmaler, Ph., Ziegler, M. (1995). Absolutely pure and flat modules and ‘‘indiscrete’’ rings. *J. Algebra* 174:349–372.

Quillen, D. (1973). Higher algebraic *K*-theory. I. Algebraic *K*-theory I. Lecture Notes in Mathematics, No. 341. Springer-Verlag, pp. 85–147.

Rothmaler, Ph. (2002). When are pure-injective envelopes of flat modules flat. *Comm. Algebra* 30(6):3077–3085.

Sklyarenko, E. G. (1978a). Relative homological algebra in categories of modules. *Uspekhi Mat. Nauk* 33(3):85–120 (Russian). English transl. (1978) in *Russian Math. Surveys* 33(3).

Sklyarenko, E. G. (1978b). Pure and finitely presented modules, duality homomorphisms and the coherence property of a ring. *Mat. Sb.* 105(2):192–206 (Russian). Engl. transl. (1978) in *Math. USSR-Sb.* 34.

Stenström, B. (1970). Coherent rings and FP-injective modules. *J. London Math. Soc.* 22(2):323–329.

Stenström, B. (1975). *Rings of Quotients*. Springer-Verlag.

Swan, R. G. (1995). Higher algebraic *K*-theory. *Proc. Symposia Pure Math.* 58(1):247–293.

Verdier, J.-L. (1977). Catégories dérivées. In: *Séminaire de Géométrie Algébrique Du Bois-Marie* (*SGA* $4\frac{1}{2}$). Lecture Notes in Mathematics, No. 569. Springer-Verlag, pp. 262–311.

Weibel, C. A. (1995). *An Introduction to Homological Algebra*. Cambridge University Press.

Zimmermann-Huisgen, B., Zimmermann, W. (1990). On the sparsity of representations of rings of pure global dimension zero. *Trans. Am. Math. Soc.* 320:695–711.

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