# RECONSTRUCTING PROJECTIVE SCHEMES FROM SERRE SUBCATEGORIES

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Abstract. Given a positively graded commutative coherent ring $A = \oplus_{j \geqslant 0} A_j$, finitely generated as an $A_0$-algebra, a bijection between the tensor Serre subcategories of $\mathrm{qgr}\, A$ and the set of all subsets $Y \subseteq \mathsf{Proj}\, A$ of the form $Y = \bigcup_{i \in \Omega} Y_i$ with quasi-compact open complement $\mathsf{Proj}\, A \setminus Y_i$ for all $i \in \Omega$ is established. To construct this correspondence, properties of the Ziegler and Zariski topologies on the set of isomorphism classes of indecomposable injective graded modules are used in an essential way. Also, there is constructed an isomorphism of ringed spaces

$$(\mathsf{Proj}\, A, \mathcal{O}_{\mathsf{Proj}\, A}) \xrightarrow{\sim} (\mathsf{Spec}(\mathrm{qgr}\, A), \mathcal{O}_{\mathrm{qgr}\, A}),$$

where $(\mathsf{Spec}(\mathrm{qgr}\, A), \mathcal{O}_{\mathrm{qgr}\, A})$ is a ringed space associated to the lattice $L_{\mathrm{Serre}}(\mathrm{qgr}\, A)$ of tensor Serre subcategories of $\mathrm{qgr}\, A$.

## 1. Introduction

In his celebrated work on abelian categories P. Gabriel [5] proved that any noetherian scheme $X$ can be reconstructed uniquely up to isomorphism from the category, $\mathrm{Qcoh}\, X$, of quasi-coherent sheaves over $X$. This reconstruction result has been generalized to arbitrary schemes by A. Rosenberg in [15].

Another result of Gabriel [5, VI.2, Prop. 4] states that for any noetherian scheme $X$ the assignments

$$\mathrm{coh}\, X \supseteq \mathcal{D} \mapsto \bigcup_{x \in \mathcal{D}} \mathsf{supp}_X(x) \quad \text{and} \quad X \supseteq U \mapsto \{x \in \mathrm{coh}\, X \mid \mathsf{supp}_X(x) \subseteq U\} \tag{1.1}$$

induce bijections between

1. the set of all Serre subcategories of $\mathrm{coh}\, X$, and
2. the set of all subsets $U \subseteq X$ of the form $U = \bigcup_{i \in \Omega} Y_i$ where, for all $i \in \Omega$, $Y_i$ has quasi-compact open complement $X \setminus Y_i$.

As a consequence of this result, $X$ can be reconstructed from its abelian category, $\mathrm{coh}\, X$, of coherent sheaves (see Buan-Krause-Solberg [4, Sec. 8]).

Given a quasi-compact, quasi-separated scheme $X$, let $\mathcal{D}_{\mathrm{per}}(X)$ denote the derived category of perfect complexes. It comes equipped with a tensor product $\otimes := \otimes^L_{\mathcal{O}_X}$. A thick triangulated subcategory $\mathcal{T}$ of $\mathcal{D}_{\mathrm{per}}(X)$ is said to be a tensor subcategory if for every $E \in \mathcal{D}_{\mathrm{per}}(X)$ and every object $A \in \mathcal{T}$, the tensor product $E \otimes A$ also is in $\mathcal{T}$. Thomason [19] establishes a classification similar to (1.1) for

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tensor thick subcategories of $\mathcal{D}_{\mathrm{per}}(X)$ in terms of the topology of $X$. Hopkins and Neeman (see [9, 13]) did the case where $X$ is affine and noetherian.

Based on Thomason's classification theorem, Balmer [2] reconstructs the noetherian scheme $X$ from the tensor thick triangulated subcategories of $\mathcal{D}_{\mathrm{per}}(X)$. This result has been generalized to quasi-compact, quasi-separated schemes by Buan-Krause-Solberg [4].

In his fundamental paper [16], Serre proved a theorem which describes the quasi-coherent sheaves on a projective scheme in terms of graded modules as follows. Let $A$ be a finitely generated graded algebra over a field $k$, and let $X = \mathsf{Proj}\, A$ be the associated projective scheme. Let $\operatorname{coh} X$ denote the category of coherent sheaves on $X$, and let $\mathcal{O}_X(n)$ denote the $n$th power of the twisting sheaf on $X$. Define a functor $\Gamma_* : \operatorname{coh} X \to \operatorname{qgr} A$, where $\operatorname{qgr} A$ is the category of finitely presented graded $A$-modules modulo the shifts of $A_0$, by

$$\Gamma_*(\mathcal{F}) = \bigoplus_{d=-\infty}^{\infty} H^0(X, \mathcal{F} \otimes \mathcal{O}_X(d)).$$

Serre's theorem asserts that if $A$ is generated over $k$ by finitely many elements of degree 1 then $\Gamma_*$ defines an equivalence of categories $\operatorname{coh} X \to \operatorname{qgr} A$.

Let $A$ be a coherent graded commutative ring $A = \oplus_{j \geqslant 0} A_j$ which is finitely generated as an $A_0$-algebra. The purpose of this paper is to establish a classification similar to (1.1) for tensor Serre subcategories of $\operatorname{qgr} A$ in terms of $\mathsf{Proj}\, A$. More precisely, we demonstrate the following.

**Theorem** (Classification)**.** *Let $A$ be a coherent graded commutative ring which is finitely generated as an $A_0$-algebra. The assignments*

$$\operatorname{qgr} A \supseteq \mathcal{S} \mapsto \bigcup_{M \in \mathcal{S}} \mathsf{supp}_A(M) \ \textit{and}\ \mathsf{Proj}\, A \supseteq U \mapsto \{M \in \operatorname{qgr} A \mid \mathsf{supp}_A(M) \subseteq U\}$$

*induce bijections between*

(1) *the set of all tensor Serre subcategories of* $\operatorname{qgr} A$*, and*
(2) *the set of all subsets $U \subseteq \mathsf{Proj}\, A$ of the form $U = \bigcup_{i \in \Omega} Y_i$ with quasi-compact open complement $X \setminus Y_i$ for all $i \in \Omega$,*

*where* $\mathsf{supp}_A(M) = \{P \in \mathsf{Proj}\, A \mid M_P \neq 0\}$.

Following Buan-Krause-Solberg [4] we consider the lattice $L_{\mathrm{Serre}}(\operatorname{qgr} A)$ of tensor Serre subcategories of $\operatorname{qgr} A$ and its prime ideal spectrum $\mathsf{Spec}(\operatorname{qgr} A)$. This space comes naturally equipped with a sheaf of rings $\mathcal{O}_{\operatorname{qgr} A}$. The following result says that the scheme $(\mathsf{Proj}\, A, \mathcal{O}_{\mathsf{Proj}\, A})$ is isomorphic to $(\mathsf{Spec}(\operatorname{qgr} A), \mathcal{O}_{\operatorname{qgr} A})$.

**Theorem** (Reconstruction)**.** *Let $A$ be a coherent graded ring which is finitely generated as an $A_0$-algebra. There is a natural isomorphism*

$$f : (\mathsf{Proj}\, A, \mathcal{O}_{\mathsf{Proj}\, A}) \xrightarrow{\sim} (\mathsf{Spec}(\operatorname{qgr} A), \mathcal{O}_{\operatorname{qgr} A})$$

*of ringed spaces.*

This theorem says that the abelian category $\operatorname{qgr} A$ contains all the necessary information to reconstruct the projective scheme $(\mathsf{Proj}\, A, \mathcal{O}_{\mathsf{Proj}\, A})$.

Our approach, similar to that used in [6], makes use of results on the Ziegler topology and its dual which had their origins in the model theory of modules [21, 14].

Throughout this paper we fix a positively graded commutative ring $A = \oplus_{j \geqslant 0} A_j$ with unit.

## 2. Preliminaries

In this section we recall some basic facts about graded rings and modules.

**Definition.** A *(positively) graded ring* is a ring $A$ together with a direct sum decomposition $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ as abelian groups, such that $A_i A_j \subset A_{i+j}$ for $i, j \geqslant 0$. A *homogeneous element* of $A$ is simply an element of one of the groups $A_j$, and a *homogeneous ideal* of $A$ is an ideal that is generated by homogeneous elements. A *graded $A$-module* is an $A$-module $M$ together with a direct sum decomposition $M = \oplus_{j\in\mathbb{Z}} M_j$ as abelian groups, such that $A_i M_j \subset M_{i+j}$ for $i \geqslant 0, j \in \mathbb{Z}$. One calls $M_j$ the $j$th *homogeneous component of* $M$. The elements $x \in M_j$ are called *homogeneous (of degree $j$)*.

Note that $A_0$ is a commutative ring with $1 \in A_0$, that all summands $M_j$ are $A_0$-modules, and that $M = \oplus_{j\in\mathbb{Z}} M_j$ is a direct sum decomposition of $M$ as an $A_0$-module.

Let $A$ be a graded ring. The *category of graded $A$-modules*, denoted by $\operatorname{Gr} A$, has as objects the graded $A$-modules. A *morphism* of graded $A$-modules $f : M \to N$ is an $A$-module homomorphism satisfying $f(M_j) \subset N_j$ for all $j \in \mathbb{Z}$. An $A$-module homomorphism which is a morphism in $\operatorname{Gr} A$ will be called *homogeneous.* In what follows, the set of homogeneous homomorphisms between $M$ and $N$ will be denoted by $\operatorname{Gr} A(M, N)$.

Let $M$ be a graded $A$-module and let $N$ be a submodule of $M$. $N$ is called a *graded submodule* if it is a graded module such that the inclusion map is a morphism in $\operatorname{Gr} A$. The graded submodules of $A$ are called *graded ideals.* If $d$ is an integer the *tail* $M_{\geqslant d}$ is the graded submodule of $M$ having the same homogeneous components $(M_{\geqslant d})_j$ as $M$ in degrees $j \geqslant d$ and zero for $j < d$. We also denote the ideal $A_{\geqslant 1}$ by $A_+$.

For $n \in \mathbb{Z}$, $\operatorname{Gr} A$ comes equipped with a shift functor $M \mapsto M(n)$ where $M(n)$ is defined by $M(n)_j = M_{n+j}$. It is a Grothendieck category with the generating family $\{A(n)\}_{n\in\mathbb{Z}}$. The tensor product for the category of all $A$-modules induces a tensor product on $\operatorname{Gr} A$: given two graded $A$-modules $M, N$ and homogeneous elements $x \in M_i, y \in N_j$, set $\deg(x \otimes y) := i + j$. We define the *homomorphism $A$-module* $\mathcal{H}om_A(M, N)$ as follows. In dimension $n \in \mathbb{Z}$, the group $\mathcal{H}om_A(M, N)_n$ is the group of graded $A$-module homomorphisms of degree $n$, i.e.,

$$\mathcal{H}om_A(M, N)_n = \operatorname{Gr} A(M, N(n)).$$

We refer to a graded $A$-module $M$ as *finitely generated* if it is a quotient of a free graded module of finite rank $\bigoplus_{s=1}^{n} A(d_s)$ where $d_1, \ldots, d_s \in \mathbb{Z}$. $M$ is *finitely presented* if there is an exact sequence

$$\bigoplus_{t=1}^{m} A(c_t) \to \bigoplus_{s=1}^{n} A(d_s) \to M \to 0.$$

The full subcategory of graded finitely presented modules will be denoted by $\operatorname{gr} A$. Note that any graded $A$-module is a direct limit of finitely presented graded $A$-modules.

The graded ring $A$ is said to be *coherent* if every finitely generated graded ideal of $A$ is finitely presented. It is easy to see that $A$ is coherent if and only if the category $\operatorname{gr} A$ is abelian.

Let $E$ be any indecomposable injective graded $A$-module (we remind the reader that the corresponding ungraded module, $\bigoplus_n E_n$, need not be injective in the category of ungraded $A$-modules). Set $P = P(E)$ to be the sum of annihilator ideals $\mathsf{ann}_A(x)$ of non-zero homogeneous elements $x \in E$. Observe that each ideal $\mathsf{ann}_A(x)$ is homogeneous. Since $E$ is uniform the set of annihilator ideals of non-zero homogeneous elements of $E$ is closed under finite sum so the only issue is whether the sum, $P(E)$, of them all is itself one of these annihilator ideals.

Given a prime homogeneous ideal $P$, we use the notation $E_P$ to denote the injective hull, $E(A/P)$, of $A/P$. Notice that $E_P$ is indecomposable. We also denote the set of isomorphism classes of indecomposable injective graded $A$-modules by $\mathsf{Inj}\, A$.

**Lemma 2.1.** *If $E \in \mathsf{Inj}\, A$ then $P(E)$ is a homogeneous prime ideal. If the module $E$ has the form $E_P(n)$ for some prime homogeneous ideal $P$ and integer $n$, then $P = P(E)$.*

*Proof.* The proof is similar to that of [14, 9.2]. □

It follows from the preceding lemma that the map

$$P \subset A \mapsto E_P \in \mathsf{Inj}\, A$$

from the set of homogeneous prime ideals to $\mathsf{Inj}\, A$ is injective.

## 3. The Zariski and Ziegler topologies

Given any graded ring $A$ and any homogeneous ideal $I$ of $A$, let us set $D^{\mathrm{m}}(I) = \{E \in \mathsf{Inj}\, A \mid \mathcal{H}om_A(A/I, E) = 0\}$ ("m" for "morphism"). Since $D^{\mathrm{m}}(I) \cap D^{\mathrm{m}}(J) = D^{\mathrm{m}}(I \cap J)$ (for the non-immediate inclusion, note that any morphism from $A/(I \cap J)$ to $E$ extends, by injectivity of $E$, to one from $A/I \oplus A/J$) these form a basis for a topology on $\mathsf{Inj}\, A$.

**Lemma 3.1.** *If $A$ is any graded ring and $I$ is a finitely generated homogeneous ideal of $A$ and $I = \sum_1^n I_i$, then $D^{\mathrm{m}}(I) = \bigcup_1^n D^{\mathrm{m}}(I_i)$.*

*Proof.* Suppose that $E \notin \bigcup_1^n D^{\mathrm{m}}(I_i)$. Then for each $i$ there is a non-zero morphism $f_i : R/I_i(d_j) \to E$. The intersection of the images of these morphisms is non-zero and, since $A$ is commutative, any element in this intersection is annihilated by each $I_i$, hence by $I$, that is, $\mathcal{H}om_A(R/I, E) \neq 0$ so $E \notin D^{\mathrm{m}}(I)$, as required. □

There is another topology on $\mathsf{Inj}\, A$. The collection of subsets

$$[M] = \{E \in \mathsf{Inj}\, A \mid \mathcal{H}om_A(M, E) = 0\}$$

with $M \in \operatorname{gr} A$ forms a basis of open subsets for the *Zariski topology* on $\mathsf{Inj}\, A$. This topological space will be denoted by $\mathsf{Inj}_{\mathrm{zar}}\, A$. Observe that $[M] = [M(d)]$ for any integer $d$.

For a coherent graded ring $A$ the sets $D^{\mathrm{m}}(I)$ with $I$ running over finitely generated homogeneous ideals form a basis for topology on $\mathsf{Inj}\, A$ which we call the *fg-ideals topology*. We use the fact that the intersection, $I \cap J$, of two finitely generated ideals $I, J$ is finitely generated in a coherent graded ring. Indeed, since $I \cap J = \operatorname{Ker}(A \to A/I \oplus A/J)$ and $\operatorname{gr} A$ is abelian, then $I \cap J$ is in $\operatorname{gr} A$.

By definition, $D^{\mathrm{m}}(I) = [A/I]$ for $I$ a finitely generated homogeneous ideal. Let $M$ be a finitely presented graded $A$-module. It is finitely generated by $b_1, ..., b_n$ say, $\deg(b_j) = d_j$. Set $M_k = \sum_{j \leqslant k} b_j A$, $M_0 = 0$. Each factor $C_j = M_j/M_{j-1}$ is cyclic and, we claim, $[M] = [C_1] \cap ... \cap [C_n]$. For, if there is a non-zero morphism from $C_j$ to $E(n)$, $n \in \mathbb{Z}$, then, by injectivity of $E(n)$, this extends to a morphism from $M/M_{j-1}$ to $E(n)$ and hence there is induced a non-zero morphism from $M$ to $E(n)$. Conversely, if $f : M \to E(n)$ is non-zero let $j$ be minimal such that the restriction of $f$ to $M_j$ is non-zero. Then $f$ induces a non-zero morphism from $C_j$ to $E(n)$. Since each $C_j$ is cyclic and finitely presented there are finitely generated ideals $I_j$, $1 \leqslant j \leqslant n$, such that $C_j \cong A/I_j(d_j)$. It follows that each $[C_j]$ coincides with $D^{\mathrm{m}}(I_j)$, and hence $[M] = D^{\mathrm{m}}(I)$ with $I = \bigcap_{1 \leqslant j \leqslant n} I_j$ finitely generated. Thus we have shown the following

**Proposition 3.2.** *Given a coherent graded ring $A$, the Zariski topology on $\mathsf{Inj}\, A$ coincides with the fg-ideals topology.*

Let us consider the collection of sets

$$(M) = \mathsf{Inj}_{\mathrm{zar}}\, A \setminus [M] = \{E \in \mathsf{Inj}\, A \mid \mathcal{H}om_A(M, E) \neq 0\}$$

with $M \in \mathrm{gr}\, A$ and set

$$O(M) = \{E \in \mathsf{Inj}\, A \mid \mathrm{Gr}\, A(M, E) \neq 0\}.$$

Obviously, $(M) = \bigcup_{n \in \mathbb{Z}} O(M(n))$.

**Proposition 3.3.** *Let $A$ be a coherent graded ring. The collection of sets $(M)$ with $M \in \mathrm{gr}\, A$ forms a basis of quasi-compact open sets for a topology, called the Ziegler topology, on $\mathsf{Inj}\, A$. This topological space will be denoted by $\mathsf{Inj}_{\mathrm{zg}}\, A$.*

*Proof.* Since $A$ is coherent by assumption, so $\mathrm{Gr}\, A$ is a locally coherent category, the collection of sets $O(M)$ with $M \in \mathrm{gr}\, A$ forms a basis of quasi-compact open sets for a topology on $\mathsf{Inj}\, A$ (see [7, 12]). Given $L, M \in \mathrm{gr}\, A$, it follows that $O(L) \cap O(M) = \bigcup_\alpha O(K_\alpha)$ for some $K_\alpha \in \mathrm{gr}\, A$. We have

$$(L) \cap (M) = [\bigcup_{n \in \mathbb{Z}} O(L(n))] \cap [\bigcup_{n \in \mathbb{Z}} O(M(n))] =$$

$$= \bigcup_{n \in \mathbb{Z}} [O(L(n)) \cap O(M(n))] \text{ which clearly equals } \bigcup_{n, \alpha} O(K_\alpha(n)) = \bigcup_\alpha (K_\alpha).$$

Since each $O(M)$ is quasi-compact and the translates of any cover of $O(M)$ cover $(M)$, then so is $(M)$ for the Ziegler topology on $\mathsf{Inj}\, A$. □

If $\mathcal{A}$ is an abelian category then a Serre subcategory is a full subcategory $\mathcal{S}$ such that if $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathcal{A}$ then $B \in \mathcal{S}$ if and only if $A, C \in \mathcal{S}$. Given a subcategory $\mathcal{X}$ in $\mathrm{gr}\, A$ with $A$ graded coherent, we may consider the smallest Serre subcategory of $\mathrm{gr}\, A$ containing $\mathcal{X}$. This Serre subcategory we denote, following Herzog [7], by

$$\sqrt{\mathcal{X}} = \bigcap \{\mathcal{S} \subseteq \mathrm{gr}\, A \mid \mathcal{S} \supseteq \mathcal{X} \text{ is Serre}\}.$$

There is an explicit description of $\sqrt{\mathcal{X}}$.

**Proposition 3.4.** [7, 3.1] *Let $A$ be a graded coherent ring and let $\mathcal{X}$ be a subcategory of $\mathrm{gr}\, A$. A graded finitely presented module $M$ is in $\sqrt{\mathcal{X}}$ if and only if there is a finite filtration of $M$ by graded finitely presented submodules*

$$M = M_0 \geqslant M_1 \geqslant \cdots \geqslant M_n = 0$$

*and, for each $i < n$, there is $N_i \in \mathcal{X}$ and there are graded finitely presented submodules*

$$N_i \geqslant N_{i_1} \geqslant N_{i_2}$$

*such that* $M_i/M_{i+1} \cong N_{i_1}/N_{i_2}$.

Given a subcategory $\mathcal{X}$ of $\operatorname{gr} A$ denote by

$$[\mathcal{X}] = \{E \in \mathsf{Inj}\, A \mid \mathcal{H}om_A(M, E) = 0 \text{ for all } M \in \mathcal{X}\}.$$

We shall also write $(\mathcal{X})$ to denote $\mathsf{Inj}\, A \setminus [\mathcal{X}]$.

**Corollary 3.5.** [7, 3.3] *Given a graded coherent ring $A$ and $\mathcal{X} \subseteq \operatorname{gr} A$, we have* $[\mathcal{X}] = [\sqrt{\mathcal{X}}]$ *and* $(\mathcal{X}) = (\sqrt{\mathcal{X}})$.

*Proof.* This immediately follows from Proposition 3.4 and the fact that the functor $\operatorname{Gr} A(-, E)$ with $E$ graded injective preserves exact sequences. □

Let $A$ be a graded coherent ring. A *tensor Serre subcategory* of $\operatorname{gr} A$ (or $\operatorname{Gr} A$) is a Serre subcategory $\mathcal{S} \subset \operatorname{gr} A$ (or $\operatorname{Gr} A$) such that for any $X \in \mathcal{S}$ and any $Y \in \operatorname{gr} A$ the tensor product $X \otimes Y$ is in $\mathcal{S}$.

**Lemma 3.6.** *Let $A$ be a graded coherent ring. $\mathcal{S}$ is a tensor Serre subcategory of* $\operatorname{gr} A$ *if and only if it is closed under shifts of objects, i.e. $X \in \mathcal{S}$ implies $X(n) \in \mathcal{S}$ for any $n \in \mathbb{Z}$.*

*Proof.* Suppose that $\mathcal{S}$ is a tensor Serre subcategory of $\operatorname{gr} A$. Then it is closed under shifts of objects, because $X(n) \cong X \otimes A(n)$.

Assume the converse. Let $X \in \mathcal{S}$ and $Y \in \operatorname{gr} A$. Then there is a surjection $\oplus_1^n A(n_i) \overset{f}{\twoheadrightarrow} Y$. It follows that $1_X \otimes f : \oplus_1^n X(n_i) \to X \otimes Y$ is a surjection. Since each $X(n_i)$ belongs to $\mathcal{S}$ then so does $X \otimes Y$. □

**Proposition 3.7.** *Let $A$ be a graded coherent ring. The maps*

$$U \overset{\varphi}{\mapsto} \mathcal{S}_U = \{M \in \operatorname{gr} A \mid (M) \subset U\}$$

*and*

$$\mathcal{S} \overset{\psi}{\mapsto} U_{\mathcal{S}} = \bigcup_{M \in \mathcal{S}} (M)$$

*induce a 1-1 correspondence between the lattices of open sets of* $\mathsf{Inj}_{\mathrm{zg}}\, A$ *and tensor Serre subcategories of* $\operatorname{gr} A$.

*Proof.* By [7, 3.8] and [12, 4.2] the maps

$$\operatorname{gr} A \supseteq \mathcal{S} \mapsto O = \bigcup_{M \in \mathcal{S}} O(M)$$

and

$$O \mapsto \mathcal{S}_O = \{M \in \operatorname{gr} A \mid O(M) \subset O\}$$

induce a 1-1 correspondence between the Serre subcategories of $\operatorname{gr} A$ and the open sets $O$ of $\mathsf{Inj}\, A$ for the topology defined by the sets $O(M), M \in \operatorname{gr} A$ (see above). Our assertion now follows from the fact that $(M) = \cup_{n \in \mathbb{Z}} O(M(n))$ and Lemma 3.6. □

Recall that for any homogeneous ideal $I$ of a graded ring, $A$, and homogeneous $r \in A$ we have an isomorphism $A/(I : r) \cong (rA + I)/I(\deg(r))$, where $(I : r)$ is the homogeneous ideal $\{s \in A \mid rs \in I\}$, induced by sending $1 + (I : r)$ to $r + I$.

The next result is the "nerve" in our analysis.

**Theorem 3.8.** (cf. Prest [14, 9.6]) *Let $A$ be a commutative coherent graded ring, let $E$ be an indecomposable injective graded $A$-module and let $P(E)$ be the prime ideal defined before. Then $E$ and $E_{P(E)}$ are topologically indistinguishable in $\mathsf{Inj}_{\mathrm{zg}} R$ and hence also in $\mathsf{Inj}_{\mathrm{zar}} R$.*

*Proof.* Let $I$ be such that $E = E(A/I)(d)$. For each homogeneous $r \in (A \setminus I)_n$ we have, by the remark just above, that the annihilator of $r + I \in E$ is $(I : r)$ and so, by definition of $P(E)$, we have $(I : r) \leqslant P(E)$. The natural projection $(rA + I)/I(d) \cong (A/(I : r))(d - n) \to (A/P(E))(d - n)$ extends to a morphism from $E$ to $E_{P(E)}(d - n)$ which is non-zero on $r + I$. Forming the product of these morphisms as $r$ varies over homogeneous elements in $A \setminus I$, we obtain a morphism from $E$ to a product of appropriately shifted copies of $E_{P(E)}$ which is monic on $A/I$ and hence is monic. Therefore $E$ is a direct summand of a product of appropriately shifted copies of $E_{P(E)}$ and so $E \in (M)$ implies $E_{P(E)} \in (M)$, where $M \in \operatorname{gr} A$.

For the converse, take a basic Ziegler-open neighbourhood of $E_{P(E)}$. This has the form $(M)$ for a finitely presented graded module $M$. Now, $E_{P(E)} \in (M)$ means that there is a non-zero morphism $f : M(d) \longrightarrow E_{P(E)}$ for some integer $d$. We can construct a pullback diagram

$$\begin{tikzcd} L \arrow[r] \arrow[d, tail] & A/P(E) \arrow[d, tail] \\ M(d) \arrow[r, "f"] & E_{P(E)} \end{tikzcd}$$

in which the vertical arrows are monic. Let $0 \neq K \subset L$ be a finitely generated submodule of $L$ such that the restriction, $f'$, of $f$ to $K$ is non-zero. Notice that the image of $f'$ is contained in $A/P(E)$. $K$ is in $\operatorname{gr} A$ because it is a finitely generated submodule of $M(d) \in \operatorname{gr} A$ and $A$ is coherent by assumption. Since $A/P(E) = \varinjlim A/I_\lambda$, where $I_\lambda$ ranges over the annihilators of non-zero homogeneous elements of $E$, and $K$ is finitely presented, $f'$ factorises through one of the maps $A/I_{\lambda_0} \to A/P(E)$ ($I_{\lambda_0} = \mathsf{ann}_A(x)$ for some homogeneous $x \in E$). In particular, there is a non-zero morphism $K \to E(\deg(x))$ and hence, by injectivity of $E$, an extension to a morphism $M(d - \deg(x)) \to E$, showing that $E \in (M)$, as required. $\square$

## 4. Torsion modules and the category $\mathrm{QGr}\, A$

Throughout this section the graded ring $A$ is supposed to be coherent and the homogeneous ideal $A_+ \subset A$ is supposed to be finitely generated. This is equivalent to saying that $A$ is coherent and a finitely generated $A_0$-algebra. In this section we introduce the category $\mathrm{QGr}\, A$ ($\mathrm{qgr}\, A$) which is analogous to the category of quasi-coherent (coherent) sheaves on a projective variety. The non-commutative analog of the category $\mathrm{QGr}\, A$ plays a prominent role in "non-commutative projective geometry" (see, e.g., [1, 17, 20]). We use the general theory of locally coherent Grothendieck categories and their localizations (see [7, 12] for details).

Recall that a Serre subcategory $\mathcal{S}$ of $\mathrm{Gr}\, A$ is *localizing* if it is closed under taking direct limits. A localizing subcategory $\mathcal{S}$ is said to be *of finite type* if the canonical functor from the quotient category $\mathrm{Gr}\, A/\mathcal{S} \to \mathrm{Gr}\, A$ respects direct limits, equivalently if $\mathcal{S}$ is the closure under direct limits of a Serre subcategory of $\operatorname{gr} A$.

Let us consider the commutative ring $A_0$ as a graded $A$-module. It is isomorphic to $A/A_+$ and belongs to $\operatorname{gr} A$, because $A_+$ is finitely generated by assumption.

Put $\operatorname{tors} A = \sqrt{\{A_0(d)\}_{d\in\mathbb{Z}}}$, the tensor Serre subcategory in $\operatorname{gr} A$ generated by the shifts of $A_0$, and set $\operatorname{Tors} A = \{\varinjlim T_i \mid T_i \in \operatorname{tors} A\}$. Then $\operatorname{Tors} A$ is a localizing subcategory of finite type in $\operatorname{Gr} A$ and $\operatorname{tors} A = \operatorname{Tors} A \cap \operatorname{gr} A$ (see [7, 2.8], [12, 2.8]). We refer to the objects of $\operatorname{Tors} A$ as *torsion graded modules*.

Recall that $A_{\geqslant n}$, $n \in \mathbb{Z}$, is the graded ideal of $A$ having the same homogeneous components $(A_{\geqslant n})_j$ as $A$ in degrees $j \geqslant n$ and zero for $j < n$. Notice that $A_+ = A_{\geqslant 1}$. The following lemma says that all shifts of the quotient module $A/A_{\geqslant n}$ are torsion.

**Lemma 4.1.** *Under the assumptions above the graded module $A/A_{\geqslant n}(d)$ is torsion for any $n \geqslant 1$ and $d \in \mathbb{Z}$.*

*Proof.* The graded module $A/A_{\geqslant 1}(d)$ is torsion by assumption. We proceed by induction on $n$. Suppose $A/A_{\geqslant n}(d)$ is torsion. We want to check that $A/A_{\geqslant n+1}(d)$ is torsion.

Consider the short exact sequence in $\operatorname{Gr} A$

$$0 \to A_{\geqslant n}/A_{\geqslant n+1} \to A/A_{\geqslant n+1} \to A/A_{\geqslant n} \to 0.$$

Clearly $A_{\geqslant n}/A_{\geqslant n+1}$ is an epimorphic image of the torsion graded module $\oplus_{x\in A_n} A_0(n)$, hence is torsion itself. Since $A_{\geqslant n}/A_{\geqslant n+1}$ is torsion and $\operatorname{Tors} A$ is closed under extensions, we see that $A/A_{\geqslant n+1}$ is torsion as well and therefore so is each $A/A_{\geqslant n+1}(d)$, $d \in \mathbb{Z}$. $\square$

Let $\operatorname{QGr} A = \operatorname{Gr} A/\operatorname{Tors} A$. We define $\tau$ as the functor which assigns to a graded $A$-module its maximal torsion module. By $\pi : \operatorname{Gr} A \to \operatorname{QGr} A$ we denote the quotient functor. By standard localization theory $\pi$ is exact and respects direct limits. We denote the fully faithful right adjoint to $\pi$ by $\omega$ and we denote the composition $\omega\pi$ by $Q$. Since $\pi\omega$ is the identity, it follows that $Q^2 = Q$. A graded module is said to be Tors-*closed* if it has the form $Q(M)$ for some $M \in \operatorname{Gr} A$. Note that any Tors-closed module is torsion free. We shall identify $\operatorname{QGr} A$ with the full subcategory of Tors-closed modules. For any $X \in \operatorname{Gr} A$ and $Y \in \operatorname{QGr} A$ there is an isomorphism

$$\operatorname{Gr} A(X, Y) \cong \operatorname{QGr} A(Q(X), Y)$$

natural both in $X$ and $Y$. The shift functor $M \mapsto M(n)$ defines a shift functor on $\operatorname{QGr} A$ for which we shall use the same notation. Observe that the functors $\tau, Q$ commute with the shift functor. Finally we shall write $\mathcal{O} = Q(A)$.

Put $\operatorname{qgr} A = \operatorname{gr} A/\operatorname{tors} A$. It is easy to see that the obvious functor $\operatorname{qgr} A \to \operatorname{QGr} A$ is fully faithful. We shall identify $\operatorname{qgr} A$ with its image in $\operatorname{QGr} A$. $\operatorname{qgr} A$ is an abelian category and equals $\sqrt{\{\mathcal{O}(d)\}_{d\in\mathbb{Z}}}$, the smallest Serre subcategory of $\operatorname{qgr} A$ containing $\{\mathcal{O}(d)\}_{d\in\mathbb{Z}}$. Moreover, every object of $\operatorname{QGr} A$ can be written as a direct limit $\varinjlim M_i$ of objects $M_i \in \operatorname{qgr} A$. It follows from [7, 2.16] that $M \in \operatorname{qgr} A$ if and only if it has the form $Q(L)$ for some $L \in \operatorname{gr} A$. Therefore every $M \in \operatorname{qgr} A$ has a presentation in $\operatorname{qgr} A$

$$\bigoplus_{t=1}^{m} \mathcal{O}(c_t) \to \bigoplus_{s=1}^{n} \mathcal{O}(d_s) \to M \to 0.$$

The tensor product in $\operatorname{Gr} A$ induces a tensor product in $\operatorname{QGr} A$, denoted by $\boxtimes$. More precisely, one sets

$$X \boxtimes Y := Q(X \otimes Y)$$

for any $X, Y \in \operatorname{QGr} A$.

**Lemma 4.2.** *Given $X, Y \in \mathrm{Gr}\, A$ there is a natural isomorphism in* $\mathrm{QGr}\, A$*: $Q(X) \boxtimes Q(Y) \cong Q(X \otimes Y)$. Moreover, the functor $- \boxtimes Y : \mathrm{QGr}\, A \to \mathrm{QGr}\, A$ is right exact and preserves direct limits.*

*Proof.* By standard localization theory (see [5]) there is a long exact sequence

$$0 \to T \to X \xrightarrow{\lambda_X} Q(X) \to T' \to 0,$$

where $T = \tau(X)$, the largest torsion submodule of $X$, $T' \in \mathrm{Tors}\, A$, and $Q(X)$ is the maximal essential extension of $X' := X/T$ such that $Q(X)/X' \in \mathrm{Tors}\, A$. We have an exact sequence in $\mathrm{Gr}\, A$:

$$T \otimes Y \to X \otimes Y \xrightarrow{\ell} X' \otimes Y \to 0.$$

Since Tors is a tensor Serre subcategory, we have $T \otimes Y \in \mathrm{Tors}$. Therefore $Q(\ell)$ is an isomorphism.

On the other hand, one has an exact sequence

$$\cdots \to \mathrm{Tor}_1(T', Y) \to X' \otimes Y \xrightarrow{\iota} Q(X) \otimes Y \to T' \otimes Y \to 0 \tag{4.1}$$

with $T' \otimes Y \in \mathrm{Tors}\, A$.

We claim that $\mathrm{Tor}_1(T', Y) \in \mathrm{Tors}\, A$. Indeed, choose a free resolution $F_*$

$$\cdots \to F_1 \to F_0 \to Y \to 0$$

of $Y$. Then the homology groups of the complex $T' \otimes F_*$ are $\mathrm{Tor}_{n \geqslant 0}(T', Y)$. But $T' \otimes F_*$ is a complex in $\mathrm{Tors}\, A$, hence its homology belongs to $\mathrm{Tors}\, A$.

Applying the exact functor $Q$ to sequence (4.1) we infer that $Q(\iota)$ is an isomorphism. It follows that $Q(\lambda_X \otimes 1_Y) = Q(\iota)Q(\ell)$ is an isomorphism as well. We have:

$$Q(X \otimes Y) \cong Q(Q(X) \otimes Y)$$

which, by the same reasoning, is isomorphic to

$$Q(Q(X) \otimes Q(Y)) = Q(X) \boxtimes Q(Y).$$

We leave the reader to check that the functor $- \boxtimes Y$ is right exact and preserves direct limits. $\square$

As a consequence of the preceding lemma we get an isomorphism $X(d) \cong \mathcal{O}(d) \boxtimes X$ for any $X \in \mathrm{QGr}\, A$ and $d \in \mathbb{Z}$.

The notion of a tensor Serre subcategory of $\mathrm{qgr}\, A$ (with respect to the tensor product $\boxtimes$) is defined similar to tensor Serre subcategories of $\mathrm{gr}\, A$. The proof of the next lemma is like that of Lemma 3.6 (also use Lemma 4.2).

**Lemma 4.3.** *$\mathcal{S}$ is a tensor Serre subcategory of $\mathrm{qgr}\, A$ if and only if it is closed under shifts of objects, i.e. $X \in \mathcal{S}$ implies $X(n) \in \mathcal{S}$ for any $n \in \mathbb{Z}$.*

## 5. Some properties of $\mathsf{Proj}\, A$

Recall that the projective scheme $\mathsf{Proj}\, A$ is a topological space whose points are the graded prime ideals not containing $A_+$. The topology of $\mathsf{Proj}\, A$ is defined by taking the closed sets to be the sets of the form $V(I) = \{P \in \mathsf{Proj}\, A \mid P \supseteq I\}$ for some homogeneous ideal $I$ of $A$. We set $D(I) := \mathsf{Proj}\, A \setminus V(I)$.

We also recall from [8] that a topological space is *spectral* if it is $T_0$ and quasi-compact, the quasi-compact open subsets are closed under finite intersections and form an open basis, and every non-empty irreducible closed subset has a generic point.

**Proposition 5.1.** *Let $A$ be a graded ring which is finitely generated as an $A_0$-algebra. Then the space $\mathsf{Proj}\,A$ is spectral.*

*Proof.* Let $P, Q$ be two different graded prime ideals of $A$. Without loss of generality we may assume that there is a homogeneous element $a \in P$ such that $a \notin Q$. It follows that $P \notin D(a)$ but $Q \in D(a)$. Therefore $\mathsf{Proj}\,A$ is $T_0$.

Below we shall need the following

**Sublemma.** *A graded ideal $P$ is prime if and only if for any two homogeneous elements $x, y \in A$ the condition $xy \in P$ implies that $x \in P$ or $y \in P$.*

*Proof.* Assume that for any two homogeneous elements $x, y \in A$ the condition $xy \in P$ implies $x \in P$ or $y \in P$. Let $a, b \in A$ be such that $ab \in P$. We write $a = \sum_i a_i$, $a_i \in A_i$, and $b = \sum_j b_j$, $b_j \in A_j$. Assume that $a \notin P$ and $b \notin P$. Then there exist integers $p, q$ such that $a_p \notin P$, but $a_i \in P$ for $i < p$, and $b_q \notin P$, but $b_j \in P$ for $j < q$. The $(p+q)$th homogeneous component of $ab$ is $\sum_{i+j=p+q} a_i b_j$. Thus $\sum_{i+j=p+q} a_i b_j \in P$, since $P$ is graded. All summands of this sum, except possibly $a_p b_q$, belong to $P$, and so it follows that $a_p b_q \in P$ as well. We conclude by assumption that $a_p \in P$ or $b_q \in P$, a contradiction. $\square$

Let $a$ be a non-zero homogeneous element of $A_+$. Let us show that $D(a)$ is quasi-compact. For this, we consider a cover $D(a) = \bigcup_\Lambda D(I_\lambda)$ of $D(a)$ by open sets $D(I_\lambda)$. Assume $D(a) \neq D(I_{\lambda_1}) \cup \cdots \cup D(I_{\lambda_n})$ for any $\lambda_1, \ldots, \lambda_n \in \Lambda$. Set $I := \sum_\Lambda I_\lambda$. Then $a \notin I$ because otherwise $a \in I_{\lambda_1} + \cdots + I_{\lambda_n}$ for some $\lambda_1, \ldots, \lambda_n \in \Lambda$ and then $D(a) = D(I_{\lambda_1}) \cup \cdots \cup D(I_{\lambda_n})$. It also follows that $a^t \notin I$ for any $t$.

**Sublemma.** *Let $Q$ be a graded ideal with $a^t \notin Q$ for all $t$, $Q \geqslant I$, and let $Q$ be maximal such. Then $Q$ is prime.*

*Proof.* By the sublemma above it is enough to check that for any two homogeneous elements $b, c \in A$ the condition $bc \in Q$ implies $b \in Q$ or $c \in Q$. Let $b, c \in A$ be such that $bc \in Q$ and $b, c \notin Q$. Then $a^t = q + br \in Q + bA$ and $a^s = q' + cr' \in Q + cA$ for some $s, t \in \mathbb{N}$ and $q, q' \in Q$. We see that $a^{t+s} = q'' + bcr'' \in Q$, a contradiction. $\square$

Choose $Q$ as in the sublemma. Note that $Q \not\supseteq A_+$ so $Q \in \mathsf{Proj}\,A$. Since $a \notin Q$, $Q \in D(a)$ and so $Q \in D(I_{\lambda_0})$ for some $\lambda_0 \in \Lambda$. It follows that $Q \not\supseteq I_{\lambda_0}$, a contradiction. So $D(a)$ is quasi-compact. It follows that every subset $D(J)$ with $J$ a finitely generated graded ideal is quasi-compact and every quasi-compact open subset is of this form.

Let $a_1, \ldots, a_n$ be generators for $A_+$. It follows that $\mathsf{Proj}\,A = D(a_1) \cup \cdots \cup D(a_n)$ is quasi-compact. Also the quasi-compact open subsets form an open basis since the set of them is closed under finite intersections: given any finitely generated graded ideals $I, J$ with generators $b_1, \ldots, b_l$ and $c_1, \ldots, c_m$ respectively, we have $D(I) \cap D(J) = \bigcup_{i,j} D(b_i c_j)$ and each $D(b_i c_j)$ is quasi-compact.

It remains to verify that every non-empty irreducible closed subset $V$ has a generic point.

There exists a graded ideal $I$ such that $V = V(I)$. Without loss of generality we may assume that $I = \sqrt{I}$, where $\sqrt{I}$ denotes the graded ideal generated by the homogeneous elements $b$ such that $b^t \in I$ for some $t$. If $I$ is prime then it is a generic point of $V$. If $I$ is not prime there are homogeneous $a, b \in A \setminus I$ such that $ab \in I$ (see the first sublemma above). Then $V = V(aA + I) \cup V(bA + I)$, hence $V = V(aA + I)$ for $V$ is irreducible. Therefore every graded prime ideal $P$ containing $I$ must contain $a$.

Let $Q$ be a graded ideal with $a^t \notin Q$ for any $t$, $Q \geqslant I$, and let $Q$ be maximal such. The proof of the second sublemma above shows that $Q$ is prime. It follows that $Q \in V \setminus V(aA + I) = \emptyset$, contradiction. $\square$

In the remainder of this section $A$ is coherent and a finitely generated $A_0$-algebra. We denote by $\mathsf{QInj}\, A$ the subset of $\mathsf{Inj}\, A$ consisting of torsion-free indecomposable injectives.

**Lemma 5.2.** *If $A$ is coherent and a finitely generated $A_0$-algebra then $E \in \mathsf{QInj}\, A$ if and only if the graded prime ideal $P(E)$ is in $\mathsf{Proj}\, A$.*

*Proof.* By assumption $A_0 \in \operatorname{gr} A$. Let $E \in \mathsf{QInj}\, A$. Then $\mathcal{H}om(A_0, E) = 0$ and hence $\mathcal{H}om(A_0, E_{P(E)}) = 0$ by Theorem 3.8. Therefore $P(E) \not\supseteq A_+$ because otherwise the composite map $A_0 \cong A/A_+ \twoheadrightarrow A/P(E) \rightarrowtail E_{P(E)}$ would be non-zero. Thus $P(E) \in \mathsf{Proj}\, A$.

Now assume the converse. Suppose that there is a non-zero map $f : A_0 \to E_{P(E)}(d)$ for some integer $d$. It follows that $A_+ \subset P(E)$, a contradiction. By Theorem 3.8 $\mathcal{H}om(A_0, E) = 0$, hence $E$ has no torsion. $\square$

**Corollary 5.3.** *For any power $A_+^t$ of $A_+$ the module $A/A_+^t(d), d \in \mathbb{Z}$, is torsion.*

*Proof.* Let $E \in \mathsf{QInj}\, A$. Suppose $\mathcal{H}om(A/A_+^t, E) \neq 0$. Since $A_+$ is finitely generated, then so is $A_+^t$ and hence $A/A_+^t \in \operatorname{qgr} A$. By Theorem 3.8 $\mathcal{H}om(A/A_+^t, E_{P(E)}) \neq 0$, and so $P(E) \supset A_+^t$. It follows that $P(E) \supset A_+$ what is impossible by the preceding lemma. $\square$

It is useful to have the following characterization of torsion modules.

**Proposition 5.4.** *A graded module $T$ is torsion if and only if every element $x \in T$ is annihilated by some power $A_+^t$ of $A_+$, that is, $\mathsf{ann}_A(x) \supseteq A_+^t$.*

*Proof.* Suppose first that every element $x \in T$ is annihilated by some power $A_+^{t_x}$ of $A_+$. Then there is an epimorphism

$$\bigoplus_{x \in T} A/A_+^{t_x}(\deg(x)) \to T.$$

The module on the left is torsion by Corollary 5.3, and hence so is $T$.

Now observe that the full subcategory of $\operatorname{QGr} A$ consisting of those modules whose elements are annihilated by powers of $A_+$ is localizing. Moreover this subcategory contains $A/A_+$, hence all of $\operatorname{tors} A$. Therefore it contains, hence, by the first paragraph, coincides with, $\operatorname{Tors} A$. $\square$

Given a prime ideal $P \in \mathsf{Proj}\, A$ and a graded module $M$, denote by $M_P$ the homogeneous localization of $M$ at $P$. If $f$ is a homogeneous element of $A$, by $M_f$ we denote the localization of $M$ at the multiplicative set $S_f = \{f^n\}_{n \geqslant 0}$.

**Corollary 5.5.** *If $T$ is a torsion module then $T_P = 0$ and $T_f = 0$ for any $P \in \mathsf{Proj}\, A$ and $f \in A_+$. As a consequence, $M_P \cong Q(M)_P$ and $M_f \cong Q(M)_f$ for any $M \in \operatorname{Gr} A$.*

*Proof.* Every homogeneous element of $T_P$ is of the form $x/s$, where $x \in T, s \in A \setminus P$ are homogeneous elements. Since $P \in \mathsf{Proj}\, A$ there exist a homogeneous element $f \in (A \setminus P) \cap A_+$. By Proposition 5.4 $x$ is annihilated by some power $A_+^t$ of $A_+$. It follows that $f^t s \cdot x/s = 0$, and hence $T_P = 0$. The fact that $T_f = 0$ is proved in a similar fashion.

Now let $X$ be a graded $A$-module. There is a long exact sequence

$$0 \to T \to X \xrightarrow{\lambda_X} Q(X) \to T' \to 0,$$

where $T, T' \in \operatorname{Tors} A$. We conclude that $(\lambda_X)_P$ and $(\lambda_X)_f$ are isomorphisms. □

## 6. The Zariski and Ziegler topologies on $\mathsf{QInj}\, A$

Throughout this section the graded ring $A$ is supposed to be coherent and a finitely generated $A_0$-algebra. The *Zariski topology* on $\mathsf{QInj}\, A$ is given by the collection of sets

$$[M]_q := \{E \in \mathsf{QInj}\, A \mid \mathcal{H}om_A(M, E) = 0\}$$

with $M \in \operatorname{qgr} A$ forming a basis of open subsets. This topological space will be denoted by $\mathsf{QInj}_{\mathrm{zar}}\, A$. Observe that $[M]_q = [M(d)]_q$ for any integer $d$. The Zariski topology on $\mathsf{QInj}\, A$ coincides with the subspace topology induced by the Zariski topology on $\mathsf{Inj}\, A$, because

$$\operatorname{Gr} A(L, E) \cong \operatorname{QGr} A(Q(L), E)$$

for any $L \in \operatorname{gr} A$ and hence $[L] \cap \mathsf{QInj}\, A = [Q(L)]_q$.

The Ziegler topology on $\mathsf{QInj}\, A$ is defined in a similar way. Namely, set

$$O_q(M) := \{E \in \mathsf{QInj}\, A \mid \operatorname{QGr} A(M, E) \neq 0\}$$

and

$$(M)_q := \bigcup_{d \in \mathbb{Z}} O_q(M(d)) = \{E \in \mathsf{QInj}\, A \mid \mathcal{H}om_A(M, E) \neq 0\}$$

with $M \in \operatorname{qgr} A$. The proof of Proposition 3.3 shows that the collection of sets $(M)_q$ with $M \in \operatorname{qgr} A$ forms a basis of quasi-compact open sets for the *Ziegler topology* on $\mathsf{QInj}\, A$. This topological space will be denoted by $\mathsf{QInj}_{\mathrm{zg}}\, A$. Moreover, the Ziegler topology on $\mathsf{QInj}\, A$ is compatible with the Ziegler topology on $\mathsf{Inj}\, A$, because $O(L) \cap \mathsf{QInj}\, A = O_q(Q(L))$ and $(L) \cap \mathsf{QInj}\, A = (Q(L))_q$ for any $L \in \operatorname{gr} A$.

Given any homogeneous ideal $I$ of $A$, let us set $D^{\mathrm{m,q}}(I) = \{E \in \mathsf{QInj}\, A \mid \mathcal{H}om_A(A/I, E) = 0\}$. The *fg-topology* on $\mathsf{QInj}\, A$ is defined by the collection of (basic open) subsets $D^{\mathrm{m,q}}(I) = [Q(A/I)]_q$ with $I$ a finitely generated homogeneous ideal. We have $D^{\mathrm{m}}(I) \cap \mathsf{QInj}\, A = D^{\mathrm{m,q}}(I)$. Moreover, the Zariski topology and the fg-ideals topology for $\mathsf{QInj}\, A$ coincide.

Using Lemmas 2.1 and 5.2, there is an embedding

$$\alpha : \mathsf{Proj}\, A \to \mathsf{QInj}\, A, \quad P \mapsto E_P.$$

We shall identify $\mathsf{Proj}\, A$ with its image in $\mathsf{QInj}\, A$.

**Theorem 6.1.** *The space $\mathsf{Proj}\, A$ is dense and a retract in $\mathsf{QInj}_{\mathrm{zar}}\, A$. A left inverse to the embedding $\mathsf{Proj}\, A \hookrightarrow \mathsf{QInj}_{\mathrm{zar}}\, A$ takes an indecomposable injective torsion-free graded module $E$ to the prime ideal $P(E)$ which is the sum of annihilator ideals of non-zero homogeneous elements of $E$. Moreover, $\mathsf{QInj}_{\mathrm{zar}}\, A$ is quasi-compact, the basic open subsets $[M]_q, M \in \operatorname{qgr} A$, are quasi-compact, the intersection of two quasi-compact open subsets is quasi-compact, and every non-empty irreducible closed subset has a generic point.*

*Proof.* For any ideal $I$ of $A$ we have

$$D^{\mathrm{m,q}}(I) \cap \mathsf{Proj}\, A = D(I). \tag{6.1}$$

From this relation, the fact that the Zariski topology and the fg-ideals topology for $\mathsf{QInj}\, A$ coincide, and Theorem 3.8 it follows that $\mathsf{Proj}\, A$ is dense in $\mathsf{QInj}_{\mathrm{zar}}\, A$ and that $\alpha : \mathsf{Proj}\, A \to \mathsf{QInj}_{\mathrm{zar}}\, A$ is a continuous map.

It follows from Lemma 2.1 that

$$\beta : \mathsf{QInj}_{\mathrm{zar}}\, A \to \mathsf{Proj}\, A, \quad E \mapsto P(E),$$

is left inverse to $\alpha$. The relation (6.1) implies that $\beta$ is continuous as well. Thus $\mathsf{Proj}\, A$ is a retract of $\mathsf{QInj}_{\mathrm{zar}}\, A$.

Let us show that each basic open set $[M]_q$, $M \in \operatorname{qgr} A$, is quasi-compact (in particular $\mathsf{QInj}_{\mathrm{zar}}\, A = [0]_q$ is quasi-compact). $[M]_q = D^{\mathrm{m,q}}(I)$ for some finitely generated graded ideal $I$ of $A$.

Let $D^{\mathrm{m,q}}(I) = \bigcup_{i\in\Omega} D^{\mathrm{m,q}}(I_i)$ with each $I_i$ graded finitely generated. It follows from (6.1) that $D(I) = \bigcup_{i\in\Omega} D(I_i)$. Since $I$ is finitely generated, $D(I)$ is quasi-compact in $\mathsf{Proj}\, A$ by the proof of Proposition 5.1. We see that $D(I) = \bigcup_{i\in\Omega_0} D(I_i)$ for some finite subset $\Omega_0 \subset \Omega$.

Assume $E \in D^{\mathrm{m,q}}(I)\setminus\bigcup_{i\in\Omega_0} D^{\mathrm{m,q}}(I_i)$. It follows from Theorem 3.8 that $E_{P(E)} \in D^{\mathrm{m,q}}(I) \setminus \bigcup_{i\in\Omega_0} D^{\mathrm{m,q}}(I_i)$. But $E_{P(E)} \in D^{\mathrm{m,q}}(I) \cap \mathsf{Proj}\, A = D(I) = \bigcup_{i\in\Omega_0} D(I_i)$, and hence it is in $D(I_{i_0}) = D^{\mathrm{m,q}}(I_{i_0}) \cap \mathsf{Proj}\, A$ for some $i_0 \in \Omega_0$, a contradiction. So $[M]_q$ is quasi-compact.

It follows from above that every quasi-compact open subset in $\mathsf{QInj}_{\mathrm{zar}}\, A$ is of the form $[M]_q$ with $M \in \operatorname{qgr} A$. Therefore the intersection $[M]_q \cap [N]_q = [M \oplus N]_q$ of two quasi-compact open subsets is quasi-compact.

By Theorem 3.8, the proof of Proposition 5.1, and relation (6.1) we see that a subset $V$ of $\mathsf{QInj}_{\mathrm{zar}}\, A$ is Zariski-closed and irreducible if and only if there is a graded prime ideal $Q$ of $A$ such that $V = \{E \mid P(E) \geqslant Q\}$. Theorem 3.8, Lemma 2.1, and (6.1) obviously imply that the point $E_Q \in V$ is generic. $\square$

**Remark.** A similar result for affine schemes $\mathsf{Spec}\, R$ with $R$ a commutative coherent ring has been shown in [6, Theorem A].

Given a spectral topological space, $X$, Hochster [8] endows the underlying set with a new, "dual", topology, denoted $X^*$, by taking as open sets those of the form $Y = \bigcup_{i\in\Omega} Y_i$ where $Y_i$ has quasi-compact open complement $X \setminus Y_i$ for all $i \in \Omega$. Then $X^*$ is spectral and $(X^*)^* = X$ (see [8, Prop. 8]). The spaces, $X$, which we consider here are not in general spectral; nevertheless we make the same definition and denote the space so obtained by $X^*$. We also write $\mathsf{Proj}^*\, A$ for $(\mathsf{Proj}\, A)^*$.

**Corollary 6.2.** *The following relations hold:*

$$\mathsf{QInj}_{\mathrm{zar}}\, A = (\mathsf{QInj}_{\mathrm{zg}}\, A)^* \ \textit{and} \ \mathsf{QInj}_{\mathrm{zg}}\, A = (\mathsf{QInj}_{\mathrm{zar}}\, A)^*.$$

*Proof.* This follows from Theorem 6.1 and the fact that a Ziegler-open subset $U$ is quasi-compact if and only if it is one of the basic open subsets $(M)_q, M \in \operatorname{qgr} A$. $\square$

The proof of the following statement literally repeats that of Proposition 3.7.

**Proposition 6.3.** *The maps*

$$U \overset{\varphi}{\mapsto} \mathcal{S}_U = \{M \in \operatorname{qgr} A \mid (M)_q \subset U\}$$

*and*

$$\mathcal{S} \overset{\psi}{\mapsto} U_{\mathcal{S}} = \bigcup_{M \in \mathcal{S}} (M)_q$$

*induce a 1-1 correspondence between the lattices of open sets of* $\mathsf{QInj}_{\mathrm{zg}}\,A$ *and tensor Serre subcategories of* $\operatorname{qgr} A$.

**Lemma 6.4.** *The maps*

$$\mathsf{Proj}^*\,A \supseteq U \overset{\varphi}{\mapsto} \mathcal{Q}_U = \{E \in \mathsf{QInj}_{\mathrm{zg}}\,A \mid P(E) \in U\}$$

*and*

$$\mathsf{QInj}_{\mathrm{zg}}\,A \supseteq \mathcal{Q} \overset{\psi}{\mapsto} U_{\mathcal{Q}} = \{P(E) \in \mathsf{Proj}^*\,A \mid E \in \mathcal{Q}\} = \mathcal{Q} \cap \mathsf{Proj}^*\,A$$

*induce a 1-1 correspondence between the lattices of open sets of* $\mathsf{Proj}^*\,A$ *and those of* $\mathsf{QInj}_{\mathrm{zg}}\,A$.

*Proof.* First note that $E_P \in \mathcal{Q}_U$ for any $P \in U$ (see Lemmas 2.1 and 5.2). Let us check that $\mathcal{Q}_U$ is an open set in $\mathsf{QInj}_{\mathrm{zg}}\,A$. Given a graded ideal $I$ of $A$, denote by $V(I) := \mathsf{Proj}\,A \setminus D(I)$ and $V^{\mathrm{m,q}}(I) := \mathsf{QInj}_{\mathrm{zg}}\,A \setminus D^{\mathrm{m,q}}(I)$. By definition, each $V(I)$ with $I$ a finitely generated graded ideal of $A$ is a basic open set in $\mathsf{Proj}^*\,A$. It follows from (6.1) that

$$V(I) = V^{\mathrm{m,q}}(I) \cap \mathsf{Proj}^*\,A. \tag{6.2}$$

Every closed subset of $\mathsf{Proj}\,A$ with quasi-compact complement has the form $V(I)$ for some finitely generated graded ideal, $I$, of $A$ (see Proposition 5.1), so there are finitely generated graded ideals $I_\lambda \subseteq A$ such that $U = \bigcup_\lambda V(I_\lambda)$. Since the points $E$ and $E_{P(E)}$ are, by Theorem 3.8, indistinguishable in $\mathsf{QInj}_{\mathrm{zg}}\,A$ we see that $\mathcal{Q}_U = \bigcup_\lambda V^{\mathrm{m,q}}(I_\lambda)$, hence this set is open in $\mathsf{QInj}_{\mathrm{zg}}\,A = (\mathsf{QInj}_{\mathrm{zar}}\,A)^*$.

The same arguments imply that $U_{\mathcal{Q}}$ is open in $\mathsf{Proj}^*\,A$. It is now easy to see that $U_{\mathcal{Q}_U} = U$ and $\mathcal{Q}_{U_{\mathcal{Q}}} = \mathcal{Q}$. □

## 7. Classifying Serre subcategories of $\operatorname{qgr} A$

Throughout this section the graded ring $A$ is supposed to be coherent and a finitely generated $A_0$-algebra.

If $M \in \operatorname{qgr} A$ we write

$$\mathsf{supp}_A(M) = \{P \in \mathsf{Proj}^*\,A \mid M_P \neq 0\},$$

where $M_P$ stands for the homogeneous localization of the graded module $M$ at graded prime $P$.

**Lemma 7.1.** *Let* $M \in \operatorname{qgr} A$ *and* $E \in \mathsf{QInj}\,A$. *Then* $E \in (M)_q$ *if and only if* $M_{P(E)} \neq 0$ *(or equivalently* $P(E) \in \mathsf{supp}_A(M)$*).*

*Proof.* By Theorem 3.8, $E \in (M)_q$ if and only if $E_{P(E)} \in (M)_q$. The assertion now follows from the easy fact that $M_P = 0$ if and only if $\mathcal{H}om_A(M, E_P) = 0$. □

Given a subcategory $\mathcal{X}$ of $\operatorname{qgr} A$ denote by

$$[\mathcal{X}]_q = \{E \in \mathsf{QInj}\,A \mid \mathcal{H}om_A(M, E) = 0 \text{ for all } M \in \mathcal{X}\}.$$

We shall also write $(\mathcal{X})_q$ to denote $\mathsf{QInj}\,A \setminus [\mathcal{X}]_q$.

It follows from the preceding lemma that

$$\mathsf{supp}_A(M) = (M)_q \cap \mathsf{Proj}^*\,A. \tag{7.1}$$

Hence $\mathsf{supp}_A(M)$ is an open set of $\mathsf{Proj}^* A$ by Lemma 6.4. More generally we have for any $\mathcal{X} \subseteq \operatorname{qgr} A$:

$$\mathsf{supp}_A(\mathcal{X}) := \bigcup_{M \in \mathcal{X}} \mathsf{supp}_A(M) = (\mathcal{X})_q \cap \mathsf{Proj}^* A.$$

The proof of Corollary 3.5 shows that $(\mathcal{X})_q = (\sqrt{\mathcal{X}})_q$. Therefore,

$$\mathsf{supp}_A(\mathcal{X}) = \mathsf{supp}_A(\sqrt{\mathcal{X}}). \tag{7.2}$$

We shall write $L_{\text{open}}(\mathsf{Proj}^* A)$, $L_{\text{Serre}}(\operatorname{qgr} A)$ to denote:

- ⋄ the lattice of all open subsets of $\mathsf{Proj}^* A$,
- ⋄ the lattice of all tensor Serre subcategories of $\operatorname{qgr} A$.

We are now in a position to demonstrate the main result of this section classifying tensor Serre subcategories of $\operatorname{qgr} A$.

**Theorem 7.2** (Classification)**.** *The assignments*

$$\operatorname{qgr} A \supseteq \mathcal{S} \mapsto \bigcup_{M \in \mathcal{S}} \mathsf{supp}_A(M) \ \textit{and}\ \mathsf{Proj}\, A \supseteq U \mapsto \{M \in \operatorname{qgr} A \mid \mathsf{supp}_A(M) \subseteq U\}$$

*induce a lattice isomorphism between $L_{\text{open}}(\mathsf{Proj}^* A)$ and $L_{\text{Serre}}(\operatorname{qgr} A)$.*

*Proof.* The map

$$\tau : \operatorname{qgr} A \supseteq \mathcal{S} \mapsto \bigcup_{M \in \mathcal{S}} \mathsf{supp}_A(M)$$

factors as

$$\operatorname{qgr} A \supseteq \mathcal{S} \overset{\delta}{\mapsto} \mathcal{Q} = \bigcup_{M \in \mathcal{S}} (M)_q \overset{\psi}{\mapsto} \bigcup_{M \in \mathcal{S}} \mathsf{supp}_A(M),$$

where $\psi$ is the map of Lemma 6.4 (we have used here relation (7.1)). By Proposition 6.3 the map $\delta$ induces a 1-1 correspondence between the tensor Serre subcategories of $\operatorname{qgr} A$ and the open sets $\mathcal{Q}$ of $\mathsf{QInj}_{\mathrm{zg}} A$. By Lemma 6.4 the map $\psi$ induces a 1-1 correspondence between the lattices of open sets of $\mathsf{QInj}_{\mathrm{zg}} A$ and those of $\mathsf{Proj}^* A$. Therefore the map $\tau$ induces the desired 1-1 correspondence between the tensor Serre subcategories of $\operatorname{qgr} A$ and the open sets of $\mathsf{Proj}^* A$. The inverse map to this correspondence is induced by the composite

$$\mathsf{Proj}^* A \supseteq U \overset{\varphi}{\mapsto} \mathcal{Q}_U \overset{\zeta}{\mapsto} \mathcal{S}(\mathcal{Q}_U) := \{M \in \operatorname{qgr} A \mid (M)_q \subseteq \mathcal{Q}_U\}$$

where $\zeta$ yields the inverse to the correspondence induced by $\tau$ and $\psi$ yields the inverse to the correspondence induced by $\varphi$ (see Lemma 6.4). □

To conclude the section, we give a classification of the tensor localizing subcategories of finite type in $\operatorname{QGr} A$ (=tensor Serre subcategories $\mathcal{L}$ closed under direct limits such that $\operatorname{QGr} A/\mathcal{L} \to \operatorname{QGr} A$ respects direct limits) in terms of open sets of $\mathsf{Proj}^* A$. For commutative (ungraded) noetherian rings, $R$, a similar classification of the localizing subcategories in $\operatorname{Mod} R$ in terms of open sets of $\mathsf{Spec}^* R$ has been shown by Hovey [10] and generalized to commutative (ungraded) coherent rings by Garkusha and Prest [6, 2.4].

**Corollary 7.3.** *The assignments*

$$\operatorname{QGr} A \supseteq \mathcal{L} \mapsto \bigcup_{M \in \mathcal{L}} \mathsf{supp}_A(M)$$

*and*

$$\mathsf{Proj}^* A \supseteq U \mapsto \{\varinjlim_{\lambda} M_\lambda \mid M_\lambda \in \operatorname{qgr} A, \mathsf{supp}_A(M_\lambda) \subseteq U\}$$

*induce bijections between*

- ⋄ *the set of all tensor localizing subcategories of finite type in* QGr $A$,
- ⋄ *the set of all open subsets* $U \subseteq \mathsf{Proj}^* A$.

*Proof.* By [7, 2.8] and [12, 2.10] there is a 1-1 correspondence between the tensor Serre subcategories of qgr $A$ and the localizing subcategories of finite type in QGr $A$. This correspondence is given by

$$\mathcal{S} \mapsto \vec{\mathcal{S}} := \{\varinjlim_{\lambda} M_\lambda \mid M_\lambda \in \mathcal{S}\} \quad \text{and} \quad \mathcal{L} \mapsto \mathcal{L} \cap \operatorname{qgr} A.$$

Since the functor of $P$-localization with $P \in \mathsf{Proj}\, A$ commutes with direct limits, we see that

$$\bigcup_{M \in \mathcal{L}} \mathsf{supp}_A(M) = \bigcup_{M \in \mathcal{L} \cap \operatorname{qgr} A} \mathsf{supp}_A(M)$$

Now our assertion follows from Theorem 7.2. □

## 8. The prime spectrum of an ideal lattice

Inspired by recent work of Balmer [3], Buan, Krause, and Solberg [4] introduce the notion of an ideal lattice and study its prime ideal spectrum. Applications arise from studying abelian or triangulated tensor categories.

**Definition** (Buan, Krause, Solberg [4])**.** An *ideal lattice* is by definition a partially ordered set $L = (L, \leqslant)$, together with an associative multiplication $L \times L \to L$, such that the following holds.

(L1) The poset $L$ is a *complete lattice*, that is,

$$\sup A = \bigvee_{a \in A} a \quad \text{and} \quad \inf A = \bigwedge_{a \in A} a$$

exist in $L$ for every subset $A \subseteq L$.

(L2) The lattice $L$ is *compactly generated*, that is, every element in $L$ is the supremum of compact elements. (An element $a \in L$ is *compact*, if for all $A \subseteq L$ with $a \leqslant \sup A$ there exists some finite $A' \subseteq A$ with $a \leqslant \sup A'$.)

(L3) We have for all $a, b, c \in L$

$$a(b \vee c) = ab \vee ac \quad \text{and} \quad (a \vee b)c = ac \vee bc.$$

(L4) The element $1 = \sup L$ is compact, and $1a = a = a1$ for all $a \in L$.

(L5) The product of two compact elements is again compact.

A *morphism* $\varphi \colon L \to L'$ of ideal lattices is a map satisfying

$$\varphi(\bigvee_{a \in A} a) = \bigvee_{a \in A} \varphi(a) \quad \text{for} \quad A \subseteq L,$$

$$\varphi(1) = 1 \quad \text{and} \quad \varphi(ab) = \varphi(a)\varphi(b) \quad \text{for} \quad a, b \in L.$$

Let $L$ be an ideal lattice. Following [4] we define the spectrum of prime elements in $L$. An element $p \neq 1$ in $L$ is *prime* if $ab \leqslant p$ implies $a \leqslant p$ or $b \leqslant p$ for all $a, b \in L$. We denote by $\mathsf{Spec}\, L$ the set of prime elements in $L$ and define for each $a \in L$

$$V(a) = \{p \in \mathsf{Spec}\, L \mid a \leqslant p\} \quad \text{and} \quad D(a) = \{p \in \mathsf{Spec}\, L \mid a \nleqslant p\}.$$

The subsets of $\mathsf{Spec}\, L$ of the form $V(a)$ are closed under forming arbitrary intersections and finite unions. More precisely,

$$V(\bigvee_{i\in\Omega} a_i) = \bigcap_{i\in\Omega} V(a_i) \quad \text{and} \quad V(ab) = V(a) \cup V(b).$$

Thus we obtain the *Zariski topology* on $\mathsf{Spec}\, L$ by declaring a subset of $\mathsf{Spec}\, L$ to be *closed* if it is of the form $V(a)$ for some $a \in L$. The set $\mathsf{Spec}\, L$ endowed with this topology is called the *prime spectrum* of $L$. Note that the sets of the form $D(a)$ with compact $a \in L$ form a basis of open sets. The prime spectrum $\mathsf{Spec}\, L$ of an ideal lattice $L$ is spectral [4, 2.5].

There is a close relation between spectral spaces and ideal lattices. Given a topological space $X$, we denote by $L_{\text{open}}(X)$ the lattice of open subsets of $X$ and consider the multiplication map

$$L_{\text{open}}(X) \times L_{\text{open}}(X) \to L_{\text{open}}(X), \quad (U, V) \mapsto UV = U \cap V.$$

The lattice $L_{\text{open}}(X)$ is complete.

**Proposition 8.1** (Buan, Krause, Solberg [4])**.** *Let $X$ be a spectral space. Then $L_{\text{open}}(X)$ is an ideal lattice. Moreover, the map*

$$X \to \mathsf{Spec}\, L_{\text{open}}(X), \quad x \mapsto X \setminus \overline{\{x\}},$$

*is a homeomorphism.*

This construction is referred to as the soberification of $X$ and the above result can be seen as part of the Stone Duality Theorem (see, for instance, [11]).

We deduce from the classification of tensor Serre subcategories of $\operatorname{qgr} A$ (Theorem 7.2) the following.

**Proposition 8.2.** *Let $A$ be a coherent graded ring which is finitely generated as an $A_0$-algebra. Then $L_{\text{Serre}}(\operatorname{qgr} A)$ is an ideal lattice.*

*Proof.* The space $\mathsf{Proj}\, A$ is spectral by Proposition 5.1. Thus $\mathsf{Proj}^* A$ is spectral and $L_{\text{open}}(\mathsf{Proj}^* A)$ is an ideal lattice by Proposition 8.1. By Theorem 7.2 we have an isomorphism $L_{\text{open}}(\mathsf{Proj}^* A) \cong L_{\text{Serre}}(\operatorname{qgr} A)$, and therefore $L_{\text{Serre}}(\operatorname{qgr} A)$ is an ideal lattice. □

**Corollary 8.3.** *The points of $\mathsf{Spec}\, L_{\text{Serre}}(\operatorname{qgr} A)$ are the $\cap$-irreducible tensor Serre subcategories of $\operatorname{qgr} A$ and the map*

$$f : \mathsf{Proj}^* A \longrightarrow \mathsf{Spec}\, L_{\text{Serre}}(\operatorname{qgr} A), \quad P \longmapsto \mathcal{S}_P = \{M \in \operatorname{qgr} A \mid M_P = 0\} \tag{8.1}$$

*is a homeomorphism of spaces.*

*Proof.* This is a consequence of Theorem 7.2 and Propositions 8.1 and 8.2. □

We write $\mathsf{Spec}(\operatorname{qgr} A) := \mathsf{Spec}^* L_{\text{Serre}}(\operatorname{qgr} A)$ and $\mathsf{supp}(M) := \{\mathcal{P} \in \mathsf{Spec}(\operatorname{qgr} A) \mid M \notin \mathcal{P}\}$ for $M \in \operatorname{qgr} A$. It follows from Corollary 8.3 that

$$\mathsf{supp}_A(M) = f^{-1}(\mathsf{supp}(M)).$$

Following [3, 4], we define a structure sheaf on $\mathsf{Spec}(\operatorname{qgr} A)$ as follows. For an open subset $U \subseteq \mathsf{Spec}(\operatorname{qgr} A)$, let

$$\mathcal{S}_U = \{M \in \operatorname{qgr} A \mid \mathsf{supp}(M) \cap U = \emptyset\}$$

and observe that $\mathcal{S}_U$ is a tensor Serre subcategory. We obtain a presheaf of rings on $\mathsf{Spec}(\operatorname{qgr} A)$ by

$$U \mapsto \operatorname{End}_{\operatorname{qgr} A/\mathcal{S}_U}(\mathcal{O}).$$

If $V \subseteq U$ are open subsets, then the restriction map

$$\operatorname{End}_{\operatorname{qgr} A/\mathcal{S}_U}(\mathcal{O}) \to \operatorname{End}_{\operatorname{qgr} A/\mathcal{S}_V}(\mathcal{O})$$

is induced by the quotient functor $\operatorname{qgr} A/\mathcal{S}_U \to \operatorname{qgr} A/\mathcal{S}_V$. The sheafification is called the *structure sheaf* of $\operatorname{qgr} A$ and is denoted by $\mathcal{O}_{\operatorname{qgr} A}$. This is a sheaf of commutative rings by [18, 1.7]. Next we have by [4, 7.1]

$$\mathcal{O}_{\operatorname{qgr} A,\mathcal{P}} \cong \operatorname{End}_{\operatorname{qgr} A/\mathcal{P}}(\mathcal{O}) \quad \text{for each} \quad \mathcal{P} \in \mathsf{Spec}(\operatorname{qgr} A).$$

The next theorem says that the abelian category $\operatorname{qgr} A$ contains all the necessary information to reconstruct the projective scheme $(\mathsf{Proj}\, A, \mathcal{O}_{\mathsf{Proj}\, A})$.

**Theorem 8.4** (Reconstruction)**.** *Let $A$ be a coherent graded ring which is finitely generated as an $A_0$-algebra. The map in* (8.1) *induces an isomorphism*

$$f : (\mathsf{Proj}\, A, \mathcal{O}_{\mathsf{Proj}\, A}) \xrightarrow{\sim} (\mathsf{Spec}(\operatorname{qgr} A), \mathcal{O}_{\operatorname{qgr} A})$$

*of ringed spaces.*

*Proof.* Fix an open subset $U \subseteq \mathsf{Spec}(\operatorname{qgr} A)$ and consider the composition of the functors

$$F : \operatorname{qgr} A \to \operatorname{Gr} A \xrightarrow{\widetilde{(-)}} \operatorname{Qcoh} \mathsf{Proj}\, A \xrightarrow{(-)|_{f^{-1}(U)}} \operatorname{Qcoh} f^{-1}(U).$$

Here we denote, for any graded $A$-module $M$, by $\widetilde{M}$ its associated sheaf. By definition, the stalk of $\widetilde{M}$ at a homogeneous prime $P$ equals the degree zero part $M_{(P)}$ of the localized module $M_P$. We claim that $F$ annihilates $\mathcal{S}_U$. In fact, $M \in \mathcal{S}_U$ implies $f^{-1}(\mathsf{supp}(M)) \cap f^{-1}(U) = \emptyset$ and therefore $\mathsf{supp}_A(M) \cap f^{-1}(U) = \emptyset$. Thus $M_{(P)} = 0$ for all $P \in f^{-1}(U)$ and therefore $F(M) = 0$. It follows that $F$ factors through $\operatorname{qgr} A/\mathcal{S}_U$ and induces a map $\operatorname{End}_{\operatorname{qgr} A/\mathcal{S}_U}(\mathcal{O}) \to \mathcal{O}_{\mathsf{Proj}\, A}(f^{-1}(U))$ which extends to a map $\mathcal{O}_{\operatorname{qgr} A}(U) \to \mathcal{O}_{\mathsf{Proj}\, A}(f^{-1}(U))$. This yields the morphism of sheaves $f^\sharp : \mathcal{O}_{\operatorname{qgr} A} \to f_*\mathcal{O}_{\mathsf{Proj}\, A}$.

Now fix a point $P \in \mathsf{Proj}\, A$. Then $f^\sharp$ induces a map $f^\sharp_P : \mathcal{O}_{\operatorname{qgr} A, f(P)} \to \mathcal{O}_{\mathsf{Proj}\, A,P}$. We have an isomorphism $\mathcal{O}_{\operatorname{qgr} A, f(P)} \cong \operatorname{End}_{\operatorname{qgr} A/f(P)}(\mathcal{O})$. By construction,

$$f(P) = \{M \in \operatorname{qgr} A \mid M_P = 0\}.$$

One has an isomorphism

$$\operatorname{End}_{\operatorname{qgr} A/f(P)}(\mathcal{O}) \cong \mathcal{H}om_A(\mathcal{O}, \mathcal{O})_{(P)} \cong \mathcal{H}om_A(A, A)_{(P)} = \mathcal{O}_{\mathsf{Proj}\, A,P}.$$

We have used here Corollary 5.5. We conclude that $f^\sharp_P$ is an isomorphism. It follows that $f$ is an isomorphism of ringed spaces if the map $f : \mathsf{Proj}\, A \to \mathsf{Spec}(\operatorname{qgr} A)$ is a homeomorphism. This last condition is a consequence of Theorem 7.2 and Propositions 8.1 and 8.2. $\square$

To conclude the paper observe that one can similarly reconstruct the affine scheme $(\mathsf{Spec}\, R, \mathcal{O}_R)$ with $R$ a commutative coherent ring out of Serre subcategories of the abelian category $\operatorname{mod} R$ of finitely presented modules. Theorem B in [6] implies that $L_{\mathrm{Serre}}(\operatorname{mod} R)$ is an ideal lattice and that the map

$$f : \mathsf{Spec}^*\, R \longrightarrow \mathsf{Spec}\, L_{\mathrm{Serre}}(\operatorname{mod} R), \quad P \longmapsto \mathcal{S}_P = \{M \in \operatorname{mod} R \mid M_P = 0\}$$

is a homeomorphism of spaces. Next, one proves, similarly to Theorem 8.4, that the map $f$ induces an isomorphism

$$(\mathsf{Spec}\, R, \mathcal{O}_R) \xrightarrow{\sim} (\mathsf{Spec}(\operatorname{mod} R), \mathcal{O}_{\operatorname{mod} R})$$

of ringed spaces, where the definition of $(\mathsf{Spec}(\operatorname{mod} R), \mathcal{O}_{\operatorname{mod} R})$ is like that of $(\mathsf{Spec}(\operatorname{qgr} A), \mathcal{O}_{\operatorname{qgr} A})$.

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