

# Torsion classes of finite type and spectra

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**Abstract.** Given a commutative ring  $R$  (respectively a positively graded commutative ring  $A = \bigoplus_{j \geq 0} A_j$  which is finitely generated as an  $A_0$ -algebra), a bijection between the torsion classes of finite type in  $\text{Mod } R$  (respectively tensor torsion classes of finite type in  $\text{QGr } A$ ) and the set of all subsets  $Y \subseteq \text{Spec } R$  (respectively  $Y \subseteq \text{Proj } A$ ) of the form  $Y = \bigcup_{i \in \Omega} Y_i$ , with  $\text{Spec } R \setminus Y_i$  (respectively  $\text{Proj } A \setminus Y_i$ ) quasi-compact and open for all  $i \in \Omega$ , is established. Using these bijections, there are constructed isomorphisms of ringed spaces

$$(\text{Spec } R, \mathcal{O}_R) \xrightarrow{\sim} (\text{Spec}(\text{Mod } R), \mathcal{O}_{\text{Mod } R})$$

and

$$(\text{Proj } A, \mathcal{O}_{\text{Proj } A}) \xrightarrow{\sim} (\text{Spec}(\text{QGr } A), \mathcal{O}_{\text{QGr } A}),$$

where  $(\text{Spec}(\text{Mod } R), \mathcal{O}_{\text{Mod } R})$  and  $(\text{Spec}(\text{QGr } A), \mathcal{O}_{\text{QGr } A})$  are ringed spaces associated to the lattices  $L_{\text{tor}}(\text{Mod } R)$  and  $L_{\text{tor}}(\text{QGr } A)$  of torsion classes of finite type. Also, a bijective correspondence between the thick subcategories of perfect complexes  $\mathcal{D}_{\text{per}}(R)$  and the torsion classes of finite type in  $\text{Mod } R$  is established.

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## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Torsion classes of finite type</b>	<b>3</b>
<b>3</b>	<b>The fg-topology</b>	<b>5</b>
<b>4</b>	<b>Torsion classes and thick subcategories</b>	<b>8</b>
<b>5</b>	<b>Graded rings and modules</b>	<b>11</b>
<b>6</b>	<b>Torsion modules and the category <math>\text{QGr } A</math></b>	<b>13</b>

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<b>7</b>	<b>The prime spectrum of an ideal lattice</b>	<b>16</b>
<b>8</b>	<b>Reconstructing affine and projective schemes</b>	<b>18</b>
<b>9</b>	<b>Appendix</b>	<b>19</b>

## 1. Introduction

Non-commutative geometry comes in various flavours. One is based on abelian and triangulated categories, the latter being replacements of classical schemes. This is based on classical results of Gabriel and later extensions, in particular by Thomason. Precisely, Gabriel [6] proved that any noetherian scheme  $X$  can be reconstructed uniquely up to isomorphism from the abelian category,  $\text{Qcoh } X$ , of quasi-coherent sheaves over  $X$ . This reconstruction result has been generalized to quasi-compact schemes by Rosenberg in [15]. Based on Thomason's classification theorem, Balmer [3] reconstructs a noetherian scheme  $X$  from the triangulated category of perfect complexes  $\mathcal{D}_{\text{per}}(X)$ . This result has been generalized to quasi-compact, quasi-separated schemes by Buan-Krause-Solberg [5].

In this paper we reconstruct affine and projective schemes from appropriate abelian categories. Our approach, similar to that used in [8, 9], is different from Rosenberg's [15] and less abstract. Moreover, some results of the paper are of independent interest.

Let  $\text{Mod } R$  (respectively  $\text{QGr } A$ ) denote the category of  $R$ -modules (respectively graded  $A$ -modules modulo torsion modules) with  $R$  (respectively  $A = \bigoplus_{n \geq 0} A_n$ ) a commutative ring (respectively a commutative graded ring). We first demonstrate the following result (cf. [8, 9]).

**Theorem** (Classification). *Let  $R$  (respectively  $A$ ) be a commutative ring (respectively commutative graded ring which is finitely generated as an  $A_0$ -algebra). Then the maps*

$$V \mapsto \mathcal{S} = \{M \in \text{Mod } R \mid \text{supp}_R(M) \subseteq V\}, \quad \mathcal{S} \mapsto V = \bigcup_{M \in \mathcal{S}} \text{supp}_R(M)$$

and

$$V \mapsto \mathcal{S} = \{M \in \text{QGr } A \mid \text{supp}_A(M) \subseteq V\}, \quad \mathcal{S} \mapsto V = \bigcup_{M \in \mathcal{S}} \text{supp}_A(M)$$

induce bijections between

1. the set of all subsets  $V \subseteq \text{Spec } R$  (respectively  $V \subseteq \text{Proj } A$ ) of the form  $V = \bigcup_{i \in \Omega} Y_i$  with  $\text{Spec } R \setminus Y_i$  (respectively  $\text{Proj } A \setminus Y_i$ ) quasi-compact and open for all  $i \in \Omega$ ,
2. the set of all torsion classes of finite type in  $\text{Mod } R$  (respectively tensor torsion classes of finite type in  $\text{QGr } A$ ).

This theorem says that  $\mathrm{Spec} R$  and  $\mathrm{Proj} A$  contain all the information about finite localizations in  $\mathrm{Mod} R$  and  $\mathrm{QGr} A$  respectively. The next result says that there is a 1-1 correspondence between the finite localizations in  $\mathrm{Mod} R$  and the triangulated localizations in  $\mathcal{D}_{\mathrm{per}}(R)$  (cf. [11, 8]).

**Theorem.** *Let  $R$  be a commutative ring. The map*

$$\mathcal{S} \mapsto \mathcal{T} = \{X \in \mathcal{D}_{\mathrm{per}}(R) \mid H_n(X) \in \mathcal{S} \text{ for all } n \in \mathbf{Z}\}$$

*induces a bijection between*

1. *the set of all torsion classes of finite type in  $\mathrm{Mod} R$ ,*
2. *the set of all thick subcategories of  $\mathcal{D}_{\mathrm{per}}(R)$ .*

Following Buan-Krause-Solberg [5] we consider the lattices  $L_{\mathrm{tor}}(\mathrm{Mod} R)$  and  $L_{\mathrm{tor}}(\mathrm{QGr} A)$  of (tensor) torsion classes of finite type in  $\mathrm{Mod} R$  and  $\mathrm{QGr} A$ , as well as their prime ideal spectra  $\mathrm{Spec}(\mathrm{Mod} R)$  and  $\mathrm{Spec}(\mathrm{QGr} A)$ . These spaces come naturally equipped with sheaves of rings  $\mathcal{O}_{\mathrm{Mod} R}$  and  $\mathcal{O}_{\mathrm{QGr} A}$ . The following result says that the schemes  $(\mathrm{Spec} R, \mathcal{O}_R)$  and  $(\mathrm{Proj} A, \mathcal{O}_{\mathrm{Proj} A})$  are isomorphic to  $(\mathrm{Spec}(\mathrm{Mod} R), \mathcal{O}_{\mathrm{Mod} R})$  and  $(\mathrm{Spec}(\mathrm{QGr} A), \mathcal{O}_{\mathrm{QGr} A})$  respectively.

**Theorem** (Reconstruction). *Let  $R$  (respectively  $A$ ) be a commutative ring (respectively commutative graded ring which is finitely generated as an  $A_0$ -algebra). Then there are natural isomorphisms of ringed spaces*

$$(\mathrm{Spec} R, \mathcal{O}_R) \xrightarrow{\sim} (\mathrm{Spec}(\mathrm{Mod} R), \mathcal{O}_{\mathrm{Mod} R})$$

*and*

$$(\mathrm{Proj} A, \mathcal{O}_{\mathrm{Proj} A}) \xrightarrow{\sim} (\mathrm{Spec}(\mathrm{QGr} A), \mathcal{O}_{\mathrm{QGr} A}).$$

## 2. Torsion classes of finite type

We refer the reader to the Appendix for necessary facts about localization and torsion classes in Grothendieck categories.

**Proposition 2.1.** *Assume that  $\mathfrak{B}$  is a set of finitely generated ideals of a commutative ring  $R$ . The set of those ideals which contain a finite products of ideals belonging to  $\mathfrak{B}$  is a Gabriel filter of finite type.*

*Proof.* See [17, VI.6.10]. □

Given a module  $M$ , we denote by  $\mathrm{supp}_R(M) = \{P \in \mathrm{Spec} R \mid M_P \neq 0\}$ . Here  $M_P$  denotes the localization of  $M$  at  $P$ , that is, the module of fractions  $M[(R \setminus P)^{-1}]$ . Note that  $V(I) = \{P \in \mathrm{Spec} R \mid I \leq P\}$  is equal to  $\mathrm{supp}_R(R/I)$  for every ideal  $I$  and

$$\mathrm{supp}_R(M) = \bigcup_{x \in M} V(\mathrm{ann}_R(x)), \quad M \in \mathrm{Mod} R.$$

Recall from [10] that a topological space is *spectral* if it is  $T_0$ , quasi-compact, if the quasi-compact open subsets are closed under finite intersections and form an open basis, and if every non-empty irreducible closed subset has a generic point. Given a spectral topological space,  $X$ , Hochster [10] endows the underlying set with a new, “dual”, topology, denoted  $X^*$ , by taking as open sets those of the form  $Y = \bigcup_{i \in \Omega} Y_i$  where  $Y_i$  has quasi-compact open complement  $X \setminus Y_i$  for all  $i \in \Omega$ . Then  $X^*$  is spectral and  $(X^*)^* = X$  (see [10, Prop. 8]). The spaces,  $X$ , which we shall consider are not in general spectral; nevertheless we make the same definition and denote the space so obtained by  $X^*$ .

Given a commutative ring  $R$ , every closed subset of  $\mathbf{Spec} R$  with quasi-compact complement has the form  $V(I)$  for some finitely generated ideal,  $I$ , of  $R$  (see [2, Chpt. 1, Ex. 17(vii)]). Therefore a subset of  $\mathbf{Spec}^* R$  is open if and only if it is of the form  $\bigcup_{\lambda} V(I_{\lambda})$  with each  $I_{\lambda}$  finitely generated. Notice that  $V(I)$  with  $I$  a non-finitely generated ideal is not open in  $\mathbf{Spec}^* R$  in general. For instance (see [18, 3.16.2]), let  $R = \mathbb{C}[x_1, x_2, \dots]$  and  $m = (x_1, x_2, \dots)$ . It is clear that  $V(m) = \{m\}$  is not open in  $\mathbf{Spec}^* \mathbb{C}[x_1, x_2, \dots]$ .

For definitions of terms used in the next result see the Appendix to this paper.

**Theorem 2.2** (Classification). *Let  $R$  be a commutative ring. There are bijections between*

1. *the set of all open subsets  $V \subseteq \mathbf{Spec}^* R$ ,*
2. *the set of all Gabriel filters  $\mathfrak{F}$  of finite type,*
3. *the set of all torsion classes  $\mathcal{S}$  of finite type in  $\mathbf{Mod} R$ .*

*These bijections are defined as follows:*

$$\begin{aligned} V &\mapsto \begin{cases} \mathfrak{F}_V &= \{I \subset R \mid V(I) \subseteq V\} \\ \mathcal{S}_V &= \{M \in \mathbf{Mod} R \mid \text{supp}_R(M) \subseteq V\} \end{cases} \\ \mathfrak{F} &\mapsto \begin{cases} V_{\mathfrak{F}} &= \bigcup_{I \in \mathfrak{F}} V(I) \\ \mathcal{S}_{\mathfrak{F}} &= \{M \in \mathbf{Mod} R \mid \text{ann}_R(x) \in \mathfrak{F} \text{ for every } x \in M\} \end{cases} \\ \mathcal{S} &\mapsto \begin{cases} \mathfrak{F}_{\mathcal{S}} &= \{I \subset R \mid R/I \in \mathcal{S}\} \\ V_{\mathcal{S}} &= \bigcup_{M \in \mathcal{S}} \text{supp}_R(M) \end{cases} \end{aligned}$$

*Proof.* The bijection between Gabriel filters of finite type and torsion classes of finite type is a consequence of a theorem of Gabriel (see, e.g., [7, 5.8]).

Let  $\mathfrak{F}$  be a Gabriel filter of finite type. Then the set  $\Lambda_{\mathfrak{F}}$  of finitely generated ideals  $I$  belonging to  $\mathfrak{F}$  is a filter basis for  $\mathfrak{F}$ . Therefore  $V_{\mathfrak{F}} = \bigcup_{I \in \Lambda_{\mathfrak{F}}} V(I)$  is open in  $\mathbf{Spec}^* R$ .

Now let  $V$  be an open subset of  $\mathbf{Spec}^* R$ . Let  $\Lambda$  denote the set of finitely generated ideals  $I$  such that  $V(I) \subseteq V$ . By definition of the topology  $V = \bigcup_{I \in \Lambda} V(I)$  and  $I_1 \cdots I_n \in \Lambda$  for any  $I_1, \dots, I_n \in \Lambda$ . We denote by  $\mathfrak{F}'_V$  the set of ideals  $I \subset R$  such that  $I \supseteq J$  for some  $J \in \Lambda$ . By Proposition 2.1  $\mathfrak{F}'_V$  is a Gabriel filter of finite type. Clearly,  $\mathfrak{F}'_V \subset \mathfrak{F}_V = \{I \subset R \mid V(I) \subseteq V\}$ . Suppose  $I \in \mathfrak{F}_V \setminus \mathfrak{F}'_V$ ; by [17, VI.6.13-15] (cf. the proof of Theorem 6.4) there exists a prime ideal  $P \in V(I)$  such

that  $P \notin \mathfrak{F}'_V$ . But  $V(I) \subseteq V$  and therefore  $P \supset J$  for some  $J \in \Lambda$ , so  $P \in \mathfrak{F}'_V$ , a contradiction. Thus  $\mathfrak{F}'_V = \mathfrak{F}_V$ .

Clearly,  $V = V_{\mathfrak{F}'_V}$  for every open subset  $V \subseteq \mathbf{Spec}^* R$ . Let  $\mathfrak{F}$  be a Gabriel filter of finite type and  $I \in \mathfrak{F}$ . Clearly  $\mathfrak{F} \subset \mathfrak{F}_{V_{\mathfrak{F}}}$  and, as above, there is no ideal belonging to  $\mathfrak{F}_{V_{\mathfrak{F}}} \setminus \mathfrak{F}$ . We have shown the bijection between the sets of all Gabriel filters of finite type and all open subsets in  $\mathbf{Spec}^* R$ . The description of the bijection between the set of torsion classes of finite type and the set of open subsets in  $\mathbf{Spec}^* R$  is now easily checked.  $\square$

### 3. The fg-topology

Let  $\text{Inj } R$  denote the set of isomorphism classes of indecomposable injective modules. Given a finitely generated ideal  $I$  of  $R$ , we denote by  $\mathcal{S}_I$  the torsion class of finite type corresponding to the Gabriel filter of finite type having  $\{I^n\}_{n \geq 1}$  as a basis (see Proposition 2.1 and Theorem 2.2). Note that a module  $M$  has  $\mathcal{S}_I$ -torsion if and only if every element  $x \in M$  is annihilated by some power  $I^{n(x)}$  of the ideal  $I$ . Let us set

$$D^{\text{fg}}(I) := \{E \in \text{Inj } R \mid E \text{ is } \mathcal{S}_I\text{-torsion free}\}, \quad V^{\text{fg}}(I) := \text{Inj } R \setminus D^{\text{fg}}(I)$$

(“fg” referring to this topology being defined using only finitely generated ideals).

Let  $E$  be any indecomposable injective  $R$ -module. Set  $P = P(E)$  to be the sum of annihilator ideals of non-zero elements, equivalently non-zero submodules, of  $E$ . Since  $E$  is uniform the set of annihilator ideals of non-zero elements of  $E$  is closed under finite sum. It is easy to check ([14, 9.2]) that  $P(E)$  is a prime ideal and  $P(E_P) = P$ . Here  $E_P$  stands for the injective hull of  $R/P$ . There is an embedding

$$\alpha : \mathbf{Spec} R \rightarrow \text{Inj } R, \quad P \mapsto E_P,$$

which need not be surjective. We shall identify  $\mathbf{Spec} R$  with its image in  $\text{Inj } R$ .

If  $P$  is a prime ideal of a commutative ring  $R$  its complement in  $R$  is a multiplicatively closed set  $S$ . Given a module  $M$  we denote the module of fractions  $M[S^{-1}]$  by  $M_P$ . There is a corresponding Gabriel filter

$$\mathfrak{F}^P = \{I \mid P \notin V(I)\}.$$

Clearly,  $\mathfrak{F}^P$  is of finite type. The  $\mathfrak{F}^P$ -torsion modules are characterized by the property that  $M_P = 0$  (see [17, p. 151]).

More generally, let  $\mathcal{P}$  be a subset of  $\mathbf{Spec} R$ . To  $\mathcal{P}$  we associate a Gabriel filter

$$\mathfrak{F}^{\mathcal{P}} = \bigcap_{P \in \mathcal{P}} \mathfrak{F}^P = \{I \mid \mathcal{P} \cap V(I) = \emptyset\}.$$

The corresponding torsion class consists of all modules  $M$  with  $M_P = 0$  for all  $P \in \mathcal{P}$ .

Given a family of injective  $R$ -modules  $\mathcal{E}$ , denote by  $\mathfrak{F}_{\mathcal{E}}$  the Gabriel filter determined by  $\mathcal{E}$ . By definition, this corresponds to the localizing subcategory  $\mathcal{S}_{\mathcal{E}} = \{M \in \text{Mod } R \mid \text{Hom}_R(M, E) = 0 \text{ for all } E \in \mathcal{E}\}$ .

**Proposition 3.1.** *A Gabriel filter  $\mathfrak{F}$  is of finite type if and only if it is of the form  $\mathfrak{F}^{\mathcal{P}}$  with  $\mathcal{P}$  a closed set in  $\mathrm{Spec}^* R$ . Moreover,  $\mathfrak{F}^{\mathcal{P}}$  is determined by  $\mathcal{E}_{\mathcal{P}} = \{E_P \mid P \in \mathcal{P}\}$ :  $\mathfrak{F}^{\mathcal{P}} = \{I \mid \mathrm{Hom}_R(R/I, \mathcal{E}_{\mathcal{P}}) = 0\}$ .*

*Proof.* This is a consequence of Theorem 2.2.  $\square$

**Proposition 3.2.** *Let  $\mathcal{P}$  be the closure of  $P$  in  $\mathrm{Spec}^* R$ . Then  $\mathcal{P} = \{Q \in \mathrm{Spec} R \mid Q \subseteq P\}$ . Also  $\mathfrak{F}^{\mathcal{P}} = \mathfrak{F}^P$ .*

*Proof.* This is direct from the definition of the topology.  $\square$

Recall that for any ideal  $I$  of a ring,  $R$ , and  $r \in R$  we have an isomorphism  $R/(I : r) \cong (rR + I)/I$ , where  $(I : r) = \{s \in R \mid rs \in I\}$ , induced by sending  $1 + (I : r)$  to  $r + I$ .

**Proposition 3.3.** *Let  $E$  be an indecomposable injective module and let  $P(E)$  be the prime ideal defined before. Let  $I$  be a finitely generated ideal of  $R$ . Then  $E \in V^{\mathrm{fg}}(I)$  if and only if  $E_{P(E)} \in V^{\mathrm{fg}}(I)$ .*

*Proof.* Let  $I$  be such that  $E = E(R/I)$ . For each  $r \in R \setminus I$  we have, by the remark just above, that the annihilator of  $r + I \in E$  is  $(I : r)$  and so, by definition of  $P(E)$ , we have  $(I : r) \leq P(E)$ . The natural projection  $(rR + I)/I \cong R/(I : r) \longrightarrow R/P(E)$  extends to a morphism from  $E$  to  $E_{P(E)}$  which is non-zero on  $r + I$ . Forming the product of these morphisms as  $r$  varies over  $R \setminus I$ , we obtain a morphism from  $E$  to a product of copies of  $E_{P(E)}$  which is monic on  $R/I$  and hence is monic. Therefore  $E$  is a direct summand of a product of copies of  $E_{P(E)}$  and so  $E \in V^{\mathrm{fg}}(J)$  implies  $E_{P(E)} \in V^{\mathrm{fg}}(J)$ , where  $J$  is a finitely generated ideal.

Now,  $E_{P(E)} \in V^{\mathrm{fg}}(I)$ , where  $I$  is a finitely generated ideal, means that there is a non-zero morphism  $f : R/I^n \longrightarrow E_{P(E)}$  for some  $n$ . Since  $R/P(E)$  is essential in  $E_{P(E)}$  the image of  $f$  has non-zero intersection with  $R/P(E)$  so there is an ideal  $J$ , without loss of generality finitely generated, with  $I^n < J \leq R$ ,  $J/I^n$  a cyclic module, and such that the restriction,  $f'$ , of  $f$  to  $J/I^n$  is non-zero (and the image is contained in  $R/P(E)$ ). Since  $J/I^n$  is a cyclic  $\mathcal{S}_I$ -torsion module, there is an epimorphism  $g : R/I^m \rightarrow J/I^n$  for some  $m$ . By construction,  $R/P(E) = \varinjlim R/I_\lambda$ , where  $I_\lambda$  ranges over the annihilators of non-zero elements of  $E$ . Since  $R/I^m$  is finitely presented,  $0 \neq f'g$  factorises through one of the maps  $R/I_\lambda \longrightarrow R/P(E)$ . In particular, there is a non-zero morphism  $R/I^m \longrightarrow E$  showing that  $E \in V^{\mathrm{fg}}(I)$ , as required.  $\square$

Given a module  $M$ , we set

$$[M] := \{E \in \mathrm{Inj} R \mid \mathrm{Hom}_R(M, E) = 0\}, \quad (M) := \mathrm{Inj} R \setminus [M].$$

**Remark 3.4.** *For any finitely generated ideal  $I$  we have:  $D^{\mathrm{fg}}(I) \cap \mathrm{Spec} R = D(I)$  and  $V^{\mathrm{fg}}(I) \cap \mathrm{Spec} R = V(I)$ . Moreover,  $D^{\mathrm{fg}}(I) = [R/I]$ .*

If  $I, J$  are finitely generated ideals, then  $D(IJ) = D(I) \cap D(J)$ . It follows from Proposition 3.3 and Remark 3.4 that  $D^{\mathrm{fg}}(I) \cap D^{\mathrm{fg}}(J) = D^{\mathrm{fg}}(IJ)$ . Thus the sets  $D^{\mathrm{fg}}(I)$  with  $I$  running over finitely generated ideals form a basis for a topology on

$\text{Inj} R$  which we call the *fg-ideals topology*. This topological space will be denoted by  $\text{Inj}_{\text{fg}} R$ . Observe that if  $R$  is coherent then the fg-topology equals the Zariski topology on  $\text{Inj} R$  (see [14, 8]). The latter topological space is defined by taking the  $[M]$  with  $M$  finitely presented as a basis of open sets.

**Theorem 3.5.** (cf. Prest [14, 9.6]) *Let  $R$  be a commutative ring, let  $E$  be an indecomposable injective module and let  $P(E)$  be the prime ideal defined before. Then  $E$  and  $E_{P(E)}$  are topologically indistinguishable in  $\text{Inj}_{\text{fg}} R$ .*

*Proof.* This follows from Proposition 3.3 and Remark 3.4.  $\square$

**Theorem 3.6.** (cf. Garkusha-Prest [8, Thm. A]) *Let  $R$  be a commutative ring. The space  $\text{Spec} R$  is dense and a retract in  $\text{Inj}_{\text{fg}} R$ . A left inverse to the embedding  $\text{Spec} R \hookrightarrow \text{Inj}_{\text{fg}} R$  takes an indecomposable injective module  $E$  to the prime ideal  $P(E)$ . Moreover,  $\text{Inj}_{\text{fg}} R$  is quasi-compact, the basic open subsets  $D^{\text{fg}}(I)$ , with  $I$  finitely generated, are quasi-compact, the intersection of two quasi-compact open subsets is quasi-compact, and every non-empty irreducible closed subset has a generic point.*

*Proof.* For any finitely generated ideal  $I$  we have

$$D^{\text{fg}}(I) \cap \text{Spec} R = D(I)$$

(see Remark 3.4). From this relation and Theorem 3.5 it follows that  $\text{Spec} R$  is dense in  $\text{Inj}_{\text{fg}} R$  and that  $\alpha : \text{Spec} R \rightarrow \text{Inj}_{\text{fg}} R$  is a continuous map.

One may check (see [14, 9.2]) that

$$\beta : \text{Inj}_{\text{fg}} R \rightarrow \text{Spec} R, \quad E \mapsto P(E),$$

is left inverse to  $\alpha$ . Remark 3.4 implies that  $\beta$  is continuous. Thus  $\text{Spec} R$  is a retract of  $\text{Inj}_{\text{fg}} R$ .

Let us show that each basic open set  $D^{\text{fg}}(I)$  is quasi-compact (in particular  $\text{Inj}_{\text{fg}} R = D^{\text{fg}}(R)$  is quasi-compact). Let  $D^{\text{fg}}(I) = \bigcup_{i \in \Omega} D^{\text{fg}}(I_i)$  with each  $I_i$  finitely generated. It follows from Remark 3.4 that  $D(I) = \bigcup_{i \in \Omega} D(I_i)$ . Since  $I$  is finitely generated,  $D(I)$  is quasi-compact in  $\text{Spec} R$  by [2, Chpt. 1, Ex. 17(vii)]. We see that  $D(I) = \bigcup_{i \in \Omega_0} D(I_i)$  for some finite subset  $\Omega_0 \subset \Omega$ .

Assume  $E \in D^{\text{fg}}(I) \setminus \bigcup_{i \in \Omega_0} D^{\text{fg}}(I_i)$ . It follows from Theorem 3.5 that  $E_{P(E)} \in D^{\text{fg}}(I) \setminus \bigcup_{i \in \Omega_0} D^{\text{fg}}(I_i)$ . But  $E_{P(E)} \in D^{\text{fg}}(I) \cap \text{Spec} R = D(I) = \bigcup_{i \in \Omega_0} D(I_i)$ , and hence it is in  $D(I_{i_0}) = D^{\text{fg}}(I_{i_0}) \cap \text{Spec} R$  for some  $i_0 \in \Omega_0$ , a contradiction. So  $D^{\text{fg}}(I)$  is quasi-compact. It also follows that the intersection  $D^{\text{fg}}(I) \cap D^{\text{fg}}(J) = D^{\text{fg}}(IJ)$  of two quasi-compact open subsets is quasi-compact. Furthermore, every quasi-compact open subset in  $\text{Inj}_{\text{fg}} R$  must therefore have the form  $D^{\text{fg}}(I)$  with  $I$  finitely generated.

Finally, it follows from Remark 3.4 and Theorem 3.5 that a subset  $V$  of  $\text{Inj}_{\text{fg}} R$  is Zariski-closed and irreducible if and only if there is a prime ideal  $Q$  of  $R$  such that  $V = \{E \mid P(E) \geq Q\}$ . This obviously implies that the point  $E_Q \in V$  is generic.  $\square$

Notice that  $\text{Inj}_{\text{fg}} R$  is not a spectral space in general, for it is not necessarily  $T_0$ .

**Lemma 3.7.** *Let the ring  $R$  be commutative. Then the maps*

$$\text{Spec}^* R \supseteq V \xrightarrow{\phi} \mathcal{Q}_V = \{E \in \text{Inj} R \mid P(E) \in V\}$$

and

$$(\text{Inj}_{\text{fg}} R)^* \supseteq \mathcal{Q} \xrightarrow{\psi} V_{\mathcal{Q}} = \{P(E) \in \text{Spec}^* R \mid E \in \mathcal{Q}\} = \mathcal{Q} \cap \text{Spec}^* R$$

induce a 1-1 correspondence between the lattices of open sets of  $\text{Spec}^* R$  and those of  $(\text{Inj}_{\text{fg}} R)^*$ .

*Proof.* First note that  $E_P \in \mathcal{Q}_V$  for any  $P \in V$  (see [14, 9.2]). Let us check that  $\mathcal{Q}_V$  is an open set in  $(\text{Inj}_{\text{fg}} R)^*$ . Every closed subset of  $\text{Spec} R$  with quasi-compact complement has the form  $V(I)$  for some finitely generated ideal,  $I$ , of  $R$  (see [2, Chpt. 1, Ex. 17(vii)]), so there are finitely generated ideals  $I_\lambda \subseteq R$  such that  $V = \bigcup_\lambda V(I_\lambda)$ . Since the points  $E$  and  $E_{P(E)}$  are, by Theorem 3.5, indistinguishable in  $(\text{Inj}_{\text{fg}} R)^*$  we see that  $\mathcal{Q}_V = \bigcup_\lambda V^{\text{fg}}(I_\lambda)$ , hence this set is open in  $(\text{Inj}_{\text{fg}} R)^*$ .

The same arguments imply that  $V_{\mathcal{Q}}$  is open in  $\text{Spec}^* R$ . It is now easy to see that  $V_{\mathcal{Q}_V} = V$  and  $\mathcal{Q}_{V_{\mathcal{Q}}} = \mathcal{Q}$ .  $\square$

## 4. Torsion classes and thick subcategories

We shall write  $L(\text{Spec}^* R)$ ,  $L((\text{Inj}_{\text{fg}} R)^*)$ ,  $L_{\text{thick}}(\mathcal{D}_{\text{per}}(R))$ ,  $L_{\text{tor}}(\text{Mod } R)$  to denote:

- the lattice of all open subsets of  $\text{Spec}^* R$ ,
- the lattice of all open subsets of  $(\text{Inj}_{\text{fg}} R)^*$ ,
- the lattice of all thick subcategories of  $\mathcal{D}_{\text{per}}(R)$ ,
- the lattice of all torsion classes of finite type in  $\text{Mod } R$ , ordered by inclusion.

(A thick subcategory is a triangulated subcategory closed under direct summands).

Given a perfect complex  $X \in \mathcal{D}_{\text{per}}(R)$  denote by  $\text{supp}(X) = \{P \in \text{Spec } R \mid X \otimes_R^L R_P \neq 0\}$ . It is easy to see that

$$\text{supp}(X) = \bigcup_{n \in \mathbb{Z}} \text{supp}_R(H_n(X)),$$

where  $H_n(X)$  is the  $n$ th homology group of  $X$ .

**Theorem 4.1** (Thomason [18]). *Let  $R$  be a commutative ring. The assignments*

$$\mathcal{T} \in L_{\text{thick}}(\mathcal{D}_{\text{per}}(R)) \xrightarrow{\mu} \bigcup_{X \in \mathcal{T}} \text{supp}(X)$$

and

$$V \in L(\text{Spec}^* R) \xrightarrow{\nu} \{X \in \mathcal{D}_{\text{per}}(R) \mid \text{supp}(X) \subseteq V\}$$

are mutually inverse lattice isomorphisms.



Given a subcategory  $\mathcal{X}$  in  $\text{Mod } R$ , we may consider the smallest torsion class of finite type in  $\text{Mod } R$  containing  $\mathcal{X}$ . This torsion class we denote by

$$\sqrt{\mathcal{X}} = \bigcap \{ \mathcal{S} \subseteq \text{Mod } R \mid \mathcal{S} \supseteq \mathcal{X} \text{ is a torsion class of finite type} \}.$$

**Theorem 4.2.** (cf. Garkusha-Prest [8, Thm. C]) *Let  $R$  be a commutative ring. There are bijections between*

- the set of all open subsets  $Y \subseteq (\text{Inj}_{\text{fg}} R)^*$ ,
- the set of all torsion classes of finite type in  $\text{Mod } R$ ,
- the set of all thick subcategories of  $\mathcal{D}_{\text{per}}(R)$ .

These bijections are defined as follows:

$$\begin{aligned} Y &\mapsto \begin{cases} \mathcal{S} &= \{M \mid (M) \subseteq Y\} \\ \mathcal{T} &= \{X \in \mathcal{D}_{\text{per}}(R) \mid (H_n(X)) \subseteq Y \text{ for all } n \in \mathbb{Z}\} \end{cases} \\ \mathcal{S} &\mapsto \begin{cases} Y &= \bigcup_{M \in \mathcal{S}} (M) \\ \mathcal{T} &= \{X \in \mathcal{D}_{\text{per}}(R) \mid H_n(X) \in \mathcal{S} \text{ for all } n \in \mathbb{Z}\} \end{cases} \\ \mathcal{T} &\mapsto \begin{cases} Y &= \bigcup_{X \in \mathcal{T}, n \in \mathbb{Z}} (H_n(X)) \\ \mathcal{S} &= \sqrt{\{H_n(X) \mid X \in \mathcal{T}, n \in \mathbb{Z}\}} \end{cases} \end{aligned}$$

*Proof.* That  $\mathcal{S}_Y = \{M \mid (M) \subseteq Y\}$  is a torsion class follows because it is defined as the class of modules having no non-zero morphism to a family of injective modules,  $\mathcal{E} := \text{Inj } R \setminus Y$ . By Lemma 3.7,  $\mathcal{E} \cap \text{Spec}^* R = U$  is a closed set in  $\text{Spec}^* R$ , that is  $P(E) \in U$  for all  $E \in \mathcal{E}$ .  $\mathcal{S}_Y$  is also determined by the family of injective modules  $\{E_P\}_{P \in U}$ . Indeed, any  $E \in \mathcal{E}$  is a direct summand of some power of  $E_{P(E)}$  by the proof of Proposition 3.3. Therefore  $\text{Hom}_R(M, E_{P(E)}) = 0$  implies  $\text{Hom}_R(M, E) = 0$ . By Proposition 3.1  $\mathcal{S}_Y$  is of finite type. Conversely, given a torsion class of finite type  $\mathcal{S}$ , the set  $Y_{\mathcal{S}} = \bigcup_{M \in \mathcal{S}} (M)$  is plainly open in  $(\text{Inj}_{\text{fg}} R)^*$ . Moreover,  $\mathcal{S}_{Y_{\mathcal{S}}} = \mathcal{S}$  and  $Y = Y_{\mathcal{S}_Y}$ .

Consider the following diagram:

$$\begin{array}{ccc} L(\text{Spec}^* R) & \begin{array}{c} \xrightarrow{\nu} \\ \xleftarrow{\mu} \end{array} & L_{\text{thick}}(\mathcal{D}_{\text{per}}(R)) \\ \begin{array}{c} \psi \uparrow \\ \downarrow \phi \end{array} & & \begin{array}{c} \sigma \uparrow \\ \downarrow \rho \end{array} \\ L((\text{Inj}_{\text{fg}} R)^*) & \begin{array}{c} \xrightarrow{\zeta} \\ \xleftarrow{\delta} \end{array} & L_{\text{tor}}(\text{Mod } R), \end{array}$$

where  $\phi, \psi$  are as in Lemma 3.7,  $\mu, \nu$  are as in Theorem 4.1 and the remaining maps are the corresponding maps indicated in the formulation of the theorem. We have  $\nu = \mu^{-1}$  by Theorem 4.1,  $\phi = \psi^{-1}$  by Lemma 3.7, and  $\zeta = \delta^{-1}$  by the above.

By construction,

$$\sigma \zeta \phi(V) = \{X \mid \bigcup_{n \in \mathbb{Z}} \text{supp}_R(H_n(X)) \subseteq V\} = \{X \mid \text{supp}(X) \subseteq V\}$$

for all  $V \in L(\mathbf{Spec}^* R)$ . Thus  $\sigma\zeta\phi = \nu$ . Since  $\zeta, \phi, \nu$  are bijections so is  $\sigma$ .

On the other hand,

$$\psi\delta\rho(\mathcal{T}) = \bigcup_{X \in \mathcal{T}, n \in \mathbb{Z}} \text{supp}_R(H_n(X)) = \bigcup_{X \in \mathcal{T}} \text{supp}(X)$$

for any  $\mathcal{T} \in L_{\text{thick}}(\mathcal{D}_{\text{per}}(R))$ . We have used here the relation

$$\bigcup_{M \in \rho(\mathcal{T})} \text{supp}_R(M) = \bigcup_{X \in \mathcal{T}, n \in \mathbb{Z}} \text{supp}_R(H_n(X)).$$

One sees that  $\psi\delta\rho = \mu$ . Since  $\delta, \psi, \mu$  are bijections so is  $\rho$ . Obviously,  $\sigma = \rho^{-1}$  and the diagram above yields the desired bijective correspondences. The theorem is proved.  $\square$

To conclude this section, we should mention the relation between torsion classes of finite type in  $\text{Mod } R$  and the Ziegler subspace topology on  $\text{Inj } R$  (we denote this space by  $\text{Inj}_{\text{zg}} R$ ). The latter topology arises from Ziegler's work on the model theory of modules [20]. The points of the Ziegler spectrum of  $R$  are the isomorphism classes of indecomposable pure-injective  $R$ -modules and the closed subsets correspond to complete theories of modules. It is well known (see [14, 9.12]) that for every coherent ring  $R$  there is a 1-1 correspondence between the open (equivalently closed) subsets of  $\text{Inj}_{\text{zg}} R$  and torsion classes of finite type in  $\text{Mod } R$ . However, this is not the case for general commutative rings.

The topology on  $\text{Inj}_{\text{zg}} R$  can be defined as follows. Let  $\mathcal{M}$  be the set of those modules  $M$  which are kernels of homomorphisms between finitely presented modules; that is  $M = \text{Ker}(K \xrightarrow{f} L)$  with  $K, L$  finitely presented. The sets  $(M)$  with  $M \in \mathcal{M}$  form a basis of open sets for  $\text{Inj}_{\text{zg}} R$ . We claim that there is a ring  $R$  and a module  $M \in \mathcal{M}$  such that the intersection  $(M) \cap \mathbf{Spec}^* R$  is not open in  $\mathbf{Spec}^* R$ , and hence such that the open subset  $(M)$  cannot correspond to any torsion class of finite type on  $\text{Mod } R$ . Such a ring has been pointed out by G. Puninski.

Let  $V$  be a commutative valuation domain with value group isomorphic to  $\Gamma = \bigoplus_{n \in \mathbb{Z}} \mathbb{Z}$ , a  $\mathbb{Z}$ -indexed direct sum of copies of  $\mathbb{Z}$ . The order on  $\Gamma$  is defined as follows.  $(a_n)_{n \in \mathbb{Z}} > (b_n)_{n \in \mathbb{Z}}$  if  $a_i > b_i$  for some  $i$  and  $a_k = b_k$  for every  $k < i$ . Then  $J^2 = J$  where  $J$  is the Jacobson radical of  $V$ . Let  $r$  be an element with value  $v(r) = (a_n)_{n \in \mathbb{Z}}$  where  $a_0 = 1$  and  $a_n = 0$  for all  $n \neq 0$ . Consider the ring  $R = V/rJ$ . Again  $J(R)^2 = J(R)$ . Denoting the image of  $r$  in  $R$  by  $r'$ , note that  $\text{ann}_R(r') = J(R)$  which is not finitely generated, and so  $R$  is not coherent by Chase's Theorem (see [17, 1.13.3]). Note that  $R$  is a local ring and, as already observed, the simple module  $R/J(R)$  is isomorphic to  $r'R$ . Therefore  $R/J(R) = \text{Ker}(R \rightarrow R/r'R)$ . Thus  $R/J(R) \in \mathcal{M}$ . We have

$$(R/J(R)) \cap \mathbf{Spec}^* R = V(J(R)) = \{J(R)\}.$$

Suppose  $V(J(R))$  is open in  $\mathbf{Spec}^* R$ ; then  $V(J(R)) = \bigcup_{\lambda} V(I_{\lambda})$  with each  $I_{\lambda}$  finitely generated. Since  $J(R)$  is the largest proper ideal each  $V(I_{\lambda})$ , if non-empty, equals  $\{J(R)\}$ . Therefore  $J(R) = \sqrt{I_{\lambda}}$  for some  $\lambda$ . But the prime radical of every

finitely generated ideal in  $R$  is prime (since  $R$  is a valuation ring) and different from  $J(R)$ . To see the latter, we have, since  $I_\lambda$  is finitely generated, that all elements of  $I_\lambda$  have value  $> (a'_n)_n$  for some  $(a'_n)_n$  with  $a'_n = 0$  for all  $n \leq N$  for some fixed  $N$ . (Recall that the valuation  $v$  on  $R$  satisfies  $v(r+s) \geq \min\{v(r), v(s)\}$  and  $v(rs) = v(r) + v(s)$ .) It follows that there is a prime ideal properly between  $I_\lambda$  and  $J(R)$ . This gives a contradiction, as required.

## 5. Graded rings and modules

In this section we recall some basic facts about graded rings and modules.

**Definition.** A *(positively) graded ring* is a ring  $A$  together with a direct sum decomposition  $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$  as abelian groups, such that  $A_i A_j \subset A_{i+j}$  for  $i, j \geq 0$ . A *homogeneous element* of  $A$  is simply an element of one of the groups  $A_j$ , and a *homogeneous ideal* of  $A$  is an ideal that is generated by homogeneous elements. A *graded  $A$ -module* is an  $A$ -module  $M$  together with a direct sum decomposition  $M = \bigoplus_{j \in \mathbb{Z}} M_j$  as abelian groups, such that  $A_i M_j \subset M_{i+j}$  for  $i \geq 0, j \in \mathbb{Z}$ . One calls  $M_j$  the  *$j$ th homogeneous component of  $M$* . The elements  $x \in M_j$  are said to be *homogeneous (of degree  $j$ )*.

Note that  $A_0$  is a commutative ring with  $1 \in A_0$ , that all summands  $M_j$  are  $A_0$ -modules, and that  $M = \bigoplus_{j \in \mathbb{Z}} M_j$  is a direct sum decomposition of  $M$  as an  $A_0$ -module.

Let  $A$  be a graded ring. The *category of graded  $A$ -modules*, denoted by  $\text{Gr } A$ , has as objects the graded  $A$ -modules. A *morphism* of graded  $A$ -modules  $f : M \rightarrow N$  is an  $A$ -module homomorphism satisfying  $f(M_j) \subset N_j$  for all  $j \in \mathbb{Z}$ . An  $A$ -module homomorphism which is a morphism in  $\text{Gr } A$  will be called *homogeneous*.

Let  $M$  be a graded  $A$ -module and let  $N$  be a submodule of  $M$ . Say that  $N$  is a *graded submodule* if it is a graded module such that the inclusion map is a morphism in  $\text{Gr } A$ . The graded submodules of  $A$  are called *graded ideals*. If  $d$  is an integer the *tail*  $M_{\geq d}$  is the graded submodule of  $M$  having the same homogeneous components  $(M_{\geq d})_j$  as  $M$  in degrees  $j \geq d$  and zero for  $j < d$ . We also denote the ideal  $A_{\geq 1}$  by  $A_+$ .

For  $n \in \mathbb{Z}$ ,  $\text{Gr } A$  comes equipped with a shift functor  $M \mapsto M(n)$  where  $M(n)$  is defined by  $M(n)_j = M_{n+j}$ . Then  $\text{Gr } A$  is a Grothendieck category with generating family  $\{A(n)\}_{n \in \mathbb{Z}}$ . The tensor product for the category of all  $A$ -modules induces a tensor product on  $\text{Gr } A$ : given two graded  $A$ -modules  $M, N$  and homogeneous elements  $x \in M_i, y \in N_j$ , set  $\deg(x \otimes y) := i + j$ . We define the *homomorphism  $A$ -module*  $\mathcal{H}om_A(M, N)$  to be the graded  $A$ -module which is, in dimension  $n \in \mathbb{Z}$ , the group  $\mathcal{H}om_A(M, N)_n$  of graded  $A$ -module homomorphisms of degree  $n$ , i.e.,

$$\mathcal{H}om_A(M, N)_n = \text{Gr } A(M, N(n)).$$

We say that a graded  $A$ -module  $M$  is *finitely generated* if it is a quotient of a free graded module of finite rank  $\bigoplus_{s=1}^n A(d_s)$  where  $d_1, \dots, d_s \in \mathbb{Z}$ . Say that  $M$

is *finitely presented* if there is an exact sequence

$$\bigoplus_{t=1}^m A(c_t) \rightarrow \bigoplus_{s=1}^n A(d_s) \rightarrow M \rightarrow 0.$$

The full subcategory of finitely presented graded modules will be denoted by  $\text{gr } A$ . Note that any graded  $A$ -module is a direct limit of finitely presented graded  $A$ -modules, and therefore  $\text{Gr } A$  is a locally finitely presented Grothendieck category.

Let  $E$  be any indecomposable injective graded  $A$ -module (we remind the reader that the corresponding ungraded module,  $\bigoplus_n E_n$ , need not be injective in the category of ungraded  $A$ -modules). Set  $P = P(E)$  to be the sum of the annihilator ideals  $\text{ann}_A(x)$  of non-zero homogeneous elements  $x \in E$ . Observe that each ideal  $\text{ann}_A(x)$  is homogeneous. Since  $E$  is uniform the set of annihilator ideals of non-zero homogeneous elements of  $E$  is upwards closed so the only issue is whether the sum,  $P(E)$ , of them all is itself one of these annihilator ideals.

Given a prime homogeneous ideal  $P$ , we use the notation  $E_P$  to denote the injective hull,  $E(A/P)$ , of  $A/P$ . Notice that  $E_P$  is indecomposable. We also denote the set of isomorphism classes of indecomposable injective graded  $A$ -modules by  $\text{Inj } A$ .

**Lemma 5.1.** *If  $E \in \text{Inj } A$  then  $P(E)$  is a homogeneous prime ideal. If the module  $E$  has the form  $E_P(n)$  for some prime homogeneous ideal  $P$  and integer  $n$ , then  $P = P(E)$ .*

*Proof.* The proof is similar to that of [14, 9.2]. □

It follows from the preceding lemma that the map

$$P \subset A \mapsto E_P \in \text{Inj } A$$

from the set of homogeneous prime ideals to  $\text{Inj } A$  is injective.

A *tensor torsion class* in  $\text{Gr } A$  is a torsion class with torsion class  $\mathcal{S} \subset \text{Gr } A$  such that for any  $X \in \mathcal{S}$  and any  $Y \in \text{Gr } A$  the tensor product  $X \otimes Y$  is in  $\mathcal{S}$ .

**Lemma 5.2.** *Let  $A$  be a graded ring. Then a torsion class  $\mathcal{S}$  is a tensor torsion class of  $\text{Gr } A$  if and only if it is closed under shifts of objects, i.e.  $X \in \mathcal{S}$  implies  $X(n) \in \mathcal{S}$  for any  $n \in \mathbb{Z}$ .*

*Proof.* Suppose that  $\mathcal{S}$  is a tensor torsion class of  $\text{Gr } A$ . Then it is closed under shifts of objects, because  $X(n) \cong X \otimes A(n)$ .

Assume the converse. Let  $X \in \mathcal{S}$  and  $Y \in \text{Gr } A$ . Then there is a surjection  $\bigoplus_{i \in I} A(i) \xrightarrow{f} Y$ . It follows that  $1_X \otimes f : \bigoplus_{i \in I} X(i) \rightarrow X \otimes Y$  is a surjection. Since each  $X(i)$  belongs to  $\mathcal{S}$  then so does  $X \otimes Y$ . □

**Lemma 5.3.** *The map*

$$\mathcal{S} \mapsto \mathfrak{F}(\mathcal{S}) = \{\mathfrak{a} \subseteq A \mid A/\mathfrak{a} \in \mathcal{S}\}$$

*establishes a bijection between the tensor torsion classes in  $\text{Gr } A$  and the sets  $\mathfrak{F}$  of homogeneous ideals satisfying the following axioms:*

- T1.  $A \in \mathfrak{F}$ ;
- T2. if  $\mathfrak{a} \in \mathfrak{F}$  and  $a$  is a homogeneous element of  $A$  then  $(\mathfrak{a} : a) = \{x \in A \mid xa \in \mathfrak{a}\} \in \mathfrak{F}$ ;
- T3. if  $\mathfrak{a}$  and  $\mathfrak{b}$  are homogeneous ideals of  $A$  such that  $\mathfrak{a} \in \mathfrak{F}$  and  $(\mathfrak{b} : a) \in \mathfrak{F}$  for every homogeneous element  $a \in \mathfrak{a}$  then  $\mathfrak{b} \in \mathfrak{F}$ .

We shall refer to such filters as  $t$ -filters. Moreover,  $\mathcal{S}$  is of finite type if and only if  $\mathfrak{F}(\mathcal{S})$  has a basis of finitely generated ideals, that is every ideal in  $\mathfrak{F}(\mathcal{S})$  contains a finitely generated ideal belonging to  $\mathfrak{F}(\mathcal{S})$ . In this case  $\mathfrak{F}(\mathcal{S})$  will be referred to as a  $t$ -filter of finite type.

*Proof.* It is enough to observe that there is a bijection between the Gabriel filters on the family  $\{A(n)\}_{n \in \mathbb{Z}}$  of generators closed under the shift functor (i.e., if  $\mathfrak{a}$  belongs to the Gabriel filter then so does  $\mathfrak{a}(n)$  for all  $n \in \mathbb{Z}$ ) and the  $t$ -filters.  $\square$

**Proposition 5.4.** *The following statements are true:*

1. Let  $\mathfrak{F}$  be a  $t$ -filter. If  $I, J$  belong to  $\mathfrak{F}$ , then  $IJ \in \mathfrak{F}$ .
2. Assume that  $\mathfrak{B}$  is a set of homogeneous finitely generated ideals. The set  $\mathfrak{B}'$  of finite products of ideals belonging to  $\mathfrak{B}$  is a basis for a  $t$ -filter of finite type.

*Proof.* (1). For any homogeneous element  $a \in I$  we have  $(IJ : a) \supset J$ , so  $IJ \in \mathfrak{F}$  by T3 and the fact that every homogeneous ideal containing an ideal from  $\mathfrak{F}$  must belong to  $\mathfrak{F}$ .

(2). We follow [17, VI.6.10]. We must check that the set  $\mathfrak{F}$  of homogeneous ideals containing ideals in  $\mathfrak{B}'$  is a  $t$ -filter of finite type. T1 is plainly satisfied. Let  $a$  be a homogeneous element in  $A$  and  $I \in \mathfrak{F}$ . There is an ideal  $I' \in \mathfrak{B}'$  contained in  $I$ . Then  $(I : a) \supset I'$  and therefore  $(I : a) \in \mathfrak{F}$ , hence T2 is satisfied as well.

Next we verify that  $\mathfrak{F}$  satisfies T3. Suppose that  $I$  is a homogeneous ideal and there exists  $J \in \mathfrak{F}$  such that  $(I : a) \in \mathfrak{F}$  for every homogeneous element  $a \in J$ . We may assume that  $J \in \mathfrak{B}'$ . Let  $a_1, \dots, a_n$  be generators of  $J$ . Then  $(I : a_i) \in \mathfrak{F}$ ,  $i \leq n$ , and  $(I : a_i) \supset J_i$  for some  $J_i \in \mathfrak{B}'$ . It follows that  $a_i J_i \subset I$  for each  $i$ , and hence  $JJ_1 \cdots J_n \subset J(J_1 \cap \dots \cap J_n) \subset I$ , so  $I \in \mathfrak{F}$ .  $\square$

## 6. Torsion modules and the category $\text{QGr } A$

Let  $A$  be a commutative graded ring. In this section we introduce the category  $\text{QGr } A$ , which is analogous to the category of quasi-coherent sheaves on a projective variety. The non-commutative analog of the category  $\text{QGr } A$  plays a prominent role in “non-commutative projective geometry” (see, e.g., [1, 16, 19]).

Recall that the projective scheme  $\text{Proj } A$  is a topological space whose points are the homogeneous prime ideals not containing  $A_+$ . The topology of  $\text{Proj } A$  is defined by taking the closed sets to be the sets of the form  $V(I) = \{P \in \text{Proj } A \mid P \supseteq I\}$  for  $I$  a homogeneous ideal of  $A$ . We set  $D(I) := \text{Proj } A \setminus V(I)$ .

**In the remainder of this section the homogeneous ideal  $A_+ \subset A$  is assumed to be finitely generated.** This is equivalent to assuming that  $A$  is a finitely generated  $A_0$ -algebra. The space  $\text{Proj } A$  is spectral and the quasi-compact open sets are those of the form  $D(I)$  with  $I$  finitely generated (see, e.g., [9, 5.1]). Let  $\text{Tors } A$  denote the tensor torsion class of finite type corresponding to the family of homogeneous finitely generated ideals  $\{A_+^n\}_{n \geq 1}$  (see Proposition 5.4). We refer to the objects of  $\text{Tors } A$  as *torsion graded modules*.

Let  $\text{QGr } A = \text{Gr } A / \text{Tors } A$ . Let  $Q$  denote the quotient functor  $\text{Gr } A \rightarrow \text{QGr } A$ . We shall identify  $\text{QGr } A$  with the full subcategory of Tors-closed modules. The shift functor  $M \mapsto M(n)$  defines a shift functor on  $\text{QGr } A$  for which we shall use the same notation. Observe that  $Q$  commutes with the shift functor. Finally we shall write  $\mathcal{O} = Q(A)$ . Note that  $\text{QGr } A$  is a locally finitely generated Grothendieck category with the family,  $\{Q(M)\}_{M \in \text{Gr } A}$ , of finitely generated generators (see [7, 5.8]).

The tensor product in  $\text{Gr } A$  induces a tensor product in  $\text{QGr } A$ , denoted by  $\boxtimes$ . More precisely, one sets

$$X \boxtimes Y := Q(X \otimes Y)$$

for any  $X, Y \in \text{QGr } A$ .

**Lemma 6.1.** *Given  $X, Y \in \text{Gr } A$  there is a natural isomorphism in  $\text{QGr } A$ :  $Q(X) \boxtimes Q(Y) \cong Q(X \otimes Y)$ . Moreover, the functor  $- \boxtimes Y : \text{QGr } A \rightarrow \text{QGr } A$  is right exact and preserves direct limits.*

*Proof.* See [9, 4.2]. □

As a consequence of this lemma we get an isomorphism  $X(d) \cong \mathcal{O}(d) \boxtimes X$  for any  $X \in \text{QGr } A$  and  $d \in \mathbb{Z}$ .

The notion of a tensor torsion class of  $\text{QGr } A$  (with respect to the tensor product  $\boxtimes$ ) is defined analogously to that in  $\text{Gr } A$ . The proof of the next lemma is like that of Lemma 5.2 (also use Lemma 6.1).

**Lemma 6.2.** *A torsion class  $\mathcal{S}$  is a tensor torsion class of  $\text{QGr } A$  if and only if it is closed under shifts of objects, i.e.  $X \in \mathcal{S}$  implies  $X(n) \in \mathcal{S}$  for any  $n \in \mathbb{Z}$ .*

Given a prime ideal  $P \in \text{Proj } A$  and a graded module  $M$ , denote by  $M_P$  the homogeneous localization of  $M$  at  $P$ . If  $f$  is a homogeneous element of  $A$ , by  $M_f$  we denote the localization of  $M$  at the multiplicative set  $S_f = \{f^n\}_{n \geq 0}$ .

**Lemma 6.3.** *If  $T$  is a torsion module then  $T_P = 0$  and  $T_f = 0$  for any  $P \in \text{Proj } A$  and  $f \in A_+$ . As a consequence,  $M_P \cong Q(M)_P$  and  $M_f \cong Q(M)_f$  for any  $M \in \text{Gr } A$ .*

*Proof.* See [9, 5.5]. □

Denote by  $L_{\text{tor}}(\text{Gr } A, \text{Tors } A)$  (respectively  $L_{\text{tor}}(\text{QGr } A)$ ) the lattice of the tensor torsion classes of finite type in  $\text{Gr } A$  with torsion classes containing  $\text{Tors } A$

(respectively the tensor torsion classes of finite type in  $\text{QGr } A$ ) ordered by inclusion. The map

$$\ell : L_{\text{tor}}(\text{Gr } A, \text{Tors } A) \longrightarrow L_{\text{tor}}(\text{QGr } A), \quad \mathcal{S} \longmapsto \mathcal{S}/\text{Tors } A$$

is a lattice isomorphism, where  $\mathcal{S}/\text{Tors } A = \{Q(M) \mid M \in \mathcal{S}\}$  (see, e.g., [7, 1.7]). We shall consider the map  $\ell$  as an identification.

**Theorem 6.4** (Classification). *Let  $A$  be a commutative graded ring which is finitely generated as an  $A_0$ -algebra. Then the maps*

$$V \mapsto \mathcal{S} = \{M \in \text{QGr } A \mid \text{supp}_A(M) \subseteq V\} \quad \text{and} \quad \mathcal{S} \mapsto V = \bigcup_{M \in \mathcal{S}} \text{supp}_A(M)$$

induce bijections between

1. the set of all open subsets  $V \subseteq \text{Proj}^* A$ ,
2. the set of all tensor torsion classes of finite type in  $\text{QGr } A$ .

*Proof.* By Lemma 5.3 it is enough to show that the maps

$$V \mapsto \mathfrak{F}_V = \{I \in A \mid V(I) \subseteq V\} \quad \text{and} \quad \mathfrak{F} \mapsto V_{\mathfrak{F}} = \bigcup_{I \in \mathfrak{F}} V(I)$$

induce bijections between the set of all open subsets  $V \subseteq \text{Proj}^* A$  and the set of all  $t$ -filters of finite type containing  $\{A_+^n\}_{n \geq 1}$ .

Let  $\mathfrak{F}$  be such a  $t$ -filter. Then the set  $\Lambda_{\mathfrak{F}}$  of finitely generated graded ideals  $I$  belonging to  $\mathfrak{F}$  is a basis for  $\mathfrak{F}$ . Clearly  $V_{\mathfrak{F}} = \bigcup_{I \in \Lambda_{\mathfrak{F}}} V(I)$ , so  $V_{\mathfrak{F}}$  is open in  $\text{Proj}^* A$ .

Now let  $V$  be an open subset of  $\text{Proj}^* A$ . Let  $\Lambda$  be the set of finitely generated homogeneous ideals  $I$  such that  $V(I) \subseteq V$ . Then  $V = \bigcup_{I \in \Lambda} V(I)$  and  $I_1 \cdots I_n \in \Lambda$  for any  $I_1, \dots, I_n \in \Lambda$ . We denote by  $\mathfrak{F}'_V$  the set of homogeneous ideals  $I \subset A$  such that  $I \supseteq J$  for some  $J \in \Lambda$ . By Proposition 5.4(2)  $\mathfrak{F}'_V$  is a  $t$ -filter of finite type. Clearly,  $\mathfrak{F}'_V \subset \mathfrak{F}_V$ . Suppose  $I \in \mathfrak{F}_V \setminus \mathfrak{F}'_V$ .

We can use Zorn's lemma to find an ideal  $J \supset I$  which is maximal with respect to  $J \notin \mathfrak{F}'_V$  (we use the fact that  $\mathfrak{F}'_V$  has a basis of finitely generated ideals). We claim that  $J$  is prime. Indeed, suppose  $a, b \in A$  are two homogeneous elements not belonging to  $J$ . Then  $J + aA$  and  $J + bA$  must be members of  $\mathfrak{F}'_V$ , and also  $(J + aA)(J + bA) \in \mathfrak{F}'_V$  by Proposition 5.4(1). But  $(J + aA)(J + bA) \subset J + abA$ , and therefore  $ab \notin J$ . We see that  $J \in V(I) \subset V$ , and hence  $J \in V(I')$  for some  $I' \in \Lambda$ . But this implies  $J \in \mathfrak{F}'_V$ , a contradiction. Thus  $\mathfrak{F}'_V = \mathfrak{F}_V$ . Clearly,  $V = V_{\mathfrak{F}'_V}$  for every open subset  $V \subseteq \text{Proj}^* A$ . Let  $\mathfrak{F}$  be a  $t$ -filter of finite type and  $I \in \mathfrak{F}$ . Then  $I \supset J$  for some  $J \in \Lambda_{\mathfrak{F}}$ , and hence  $V(I) \subset V(J) \subset V_{\mathfrak{F}}$ . It follows that  $\mathfrak{F} \subset \mathfrak{F}_{V_{\mathfrak{F}}}$ . As above, there is no ideal belonging to  $\mathfrak{F}_{V_{\mathfrak{F}}} \setminus \mathfrak{F}$ . We have shown the desired bijection between the sets of all  $t$ -filters of finite type and all open subsets in  $\text{Proj}^* A$ .  $\square$

## 7. The prime spectrum of an ideal lattice

Inspired by recent work of Balmer [4], Buan, Krause, and Solberg [5] introduce the notion of an ideal lattice and study its prime ideal spectrum. Applications arise from abelian or triangulated tensor categories.

**Definition** (Buan, Krause, Solberg [5]). An *ideal lattice* is by definition a partially ordered set  $L = (L, \leq)$ , together with an associative multiplication  $L \times L \rightarrow L$ , such that the following holds.

(L1) The poset  $L$  is a *complete lattice*, that is,

$$\sup A = \bigvee_{a \in A} a \quad \text{and} \quad \inf A = \bigwedge_{a \in A} a$$

exist in  $L$  for every subset  $A \subseteq L$ .

(L2) The lattice  $L$  is *compactly generated*, that is, every element in  $L$  is the supremum of a set of compact elements. (An element  $a \in L$  is *compact*, if for all  $A \subseteq L$  with  $a \leq \sup A$  there exists some finite  $A' \subseteq A$  with  $a \leq \sup A'$ .)

(L3) We have for all  $a, b, c \in L$

$$a(b \vee c) = ab \vee ac \quad \text{and} \quad (a \vee b)c = ac \vee bc.$$

(L4) The element  $1 = \sup L$  is compact, and  $1a = a = a1$  for all  $a \in L$ .

(L5) The product of two compact elements is again compact.

A *morphism*  $\phi: L \rightarrow L'$  of ideal lattices is a map satisfying

$$\begin{aligned} \phi\left(\bigvee_{a \in A} a\right) &= \bigvee_{a \in A} \phi(a) \quad \text{for } A \subseteq L, \\ \phi(1) &= 1 \quad \text{and} \quad \phi(ab) = \phi(a)\phi(b) \quad \text{for } a, b \in L. \end{aligned}$$

Let  $L$  be an ideal lattice. Following [5] we define the spectrum of prime elements in  $L$ . An element  $p \neq 1$  in  $L$  is *prime* if  $ab \leq p$  implies  $a \leq p$  or  $b \leq p$  for all  $a, b \in L$ . We denote by  $\mathbf{Spec} L$  the set of prime elements in  $L$  and define for each  $a \in L$

$$V(a) = \{p \in \mathbf{Spec} L \mid a \leq p\} \quad \text{and} \quad D(a) = \{p \in \mathbf{Spec} L \mid a \not\leq p\}.$$

The subsets of  $\mathbf{Spec} L$  of the form  $V(a)$  are closed under forming arbitrary intersections and finite unions. More precisely,

$$V\left(\bigvee_{i \in \Omega} a_i\right) = \bigcap_{i \in \Omega} V(a_i) \quad \text{and} \quad V(ab) = V(a) \cup V(b).$$

Thus we obtain the *Zariski topology* on  $\mathbf{Spec} L$  by declaring a subset of  $\mathbf{Spec} L$  to be *closed* if it is of the form  $V(a)$  for some  $a \in L$ . The set  $\mathbf{Spec} L$  endowed with



this topology is called the *prime spectrum* of  $L$ . Note that the sets of the form  $D(a)$  with compact  $a \in L$  form a basis of open sets. The prime spectrum  $\text{Spec } L$  of an ideal lattice  $L$  is spectral [5, 2.5].

There is a close relation between spectral spaces and ideal lattices. Given a topological space  $X$ , we denote by  $L_{\text{open}}(X)$  the lattice of open subsets of  $X$  and consider the multiplication map

$$L_{\text{open}}(X) \times L_{\text{open}}(X) \rightarrow L_{\text{open}}(X), \quad (U, V) \mapsto UV = U \cap V.$$

The lattice  $L_{\text{open}}(X)$  is complete.

The following result, which appears in [5], is part of the Stone Duality Theorem (see, for instance, [12]).

**Proposition 7.1.** *Let  $X$  be a spectral space. Then  $L_{\text{open}}(X)$  is an ideal lattice. Moreover, the map*

$$X \rightarrow \text{Spec } L_{\text{open}}(X), \quad x \mapsto X \setminus \overline{\{x\}},$$

*is a homeomorphism.*

We deduce from the classification of torsion classes of finite type (Theorems 2.2 and 6.4) the following.

**Proposition 7.2.** *Let  $R$  (respectively  $A$ ) be a commutative ring (respectively graded commutative ring which is finitely generated as an  $A_0$ -algebra). Then  $L_{\text{tor}}(\text{Mod } R)$  and  $L_{\text{tor}}(\text{QGr } A)$  are ideal lattices.*

*Proof.* The spaces  $\text{Spec } R$  and  $\text{Proj } A$  are spectral. Thus  $\text{Spec}^* R$  and  $\text{Proj}^* A$  are spectral, also  $L_{\text{open}}(\text{Spec}^* R)$  and  $L_{\text{open}}(\text{Proj}^* A)$  are ideal lattices by Proposition 7.1. By Theorems 2.2 and 6.4 we have isomorphisms  $L_{\text{open}}(\text{Spec}^* R) \cong L_{\text{tor}}(\text{Mod } R)$  and  $L_{\text{open}}(\text{Proj}^* A) \cong L_{\text{tor}}(\text{QGr } A)$ . Therefore  $L_{\text{tor}}(\text{Mod } R)$  and  $L_{\text{tor}}(\text{QGr } A)$  are ideal lattices.  $\square$

**Corollary 7.3.** *Let  $R$  (respectively  $A$ ) be a commutative ring (respectively graded commutative ring which is finitely generated as an  $A_0$ -algebra). The points of  $\text{Spec } L_{\text{tor}}(\text{Mod } R)$  (respectively  $\text{Spec } L_{\text{tor}}(\text{QGr } A)$ ) are the  $\cap$ -irreducible torsion classes of finite type in  $\text{Mod } R$  (respectively tensor torsion classes of finite type in  $\text{QGr } A$ ) and the maps*

$$\begin{aligned} f : \text{Spec}^* R &\longrightarrow \text{Spec } L_{\text{tor}}(\text{Mod } R), & P &\longmapsto \mathcal{S}_P = \{M \in \text{Mod } R \mid M_P = 0\} \\ f : \text{Proj}^* A &\longrightarrow \text{Spec } L_{\text{tor}}(\text{QGr } A), & P &\longmapsto \mathcal{S}_P = \{M \in \text{QGr } A \mid M_P = 0\} \end{aligned}$$

*are homeomorphisms of spaces.*

*Proof.* This is a consequence of Theorems 2.2, 6.4 and Propositions 7.1, 7.2.  $\square$

## 8. Reconstructing affine and projective schemes

Let  $R$  (respectively  $A$ ) be a commutative ring (respectively graded commutative ring which is finitely generated as an  $A_0$ -algebra). We shall write  $\mathrm{Spec}(\mathrm{Mod} R) := \mathrm{Spec}^* L_{\mathrm{tor}}(\mathrm{Mod} R)$  (respectively  $\mathrm{Spec}(\mathrm{QGr} A) := \mathrm{Spec}^* L_{\mathrm{tor}}(\mathrm{QGr} A)$ ) and  $\mathrm{supp}(M) := \{\mathcal{P} \in \mathrm{Spec}(\mathrm{Mod} R) \mid M \notin \mathcal{P}\}$  (respectively  $\mathrm{supp}(M) := \{\mathcal{P} \in \mathrm{Spec}(\mathrm{QGr} A) \mid M \notin \mathcal{P}\}$ ) for  $M \in \mathrm{Mod} R$  (respectively  $M \in \mathrm{QGr} A$ ). It follows from Corollary 7.3 that

$$\mathrm{supp}_R(M) = f^{-1}(\mathrm{supp}(M)) \quad (\text{respectively } \mathrm{supp}_A(M) = f^{-1}(\mathrm{supp}(M))).$$

Following [4, 5], we define a structure sheaf on  $\mathrm{Spec}(\mathrm{Mod} R)$  ( $\mathrm{Spec}(\mathrm{QGr} A)$ ) as follows. For an open subset  $U \subseteq \mathrm{Spec}(\mathrm{Mod} R)$  ( $U \subseteq \mathrm{Spec}(\mathrm{QGr} A)$ ), let

$$\mathcal{S}_U = \{M \in \mathrm{Mod} R \text{ (QGr } A) \mid \mathrm{supp}(M) \cap U = \emptyset\}$$

and observe that  $\mathcal{S}_U = \{M \mid M_{\mathcal{P}} = 0 \text{ for all } \mathcal{P} \in f^{-1}(U)\}$  is a (tensor) torsion class. We obtain a presheaf of rings on  $\mathrm{Spec}(\mathrm{Mod} R)$  ( $\mathrm{Spec}(\mathrm{QGr} A)$ ) by

$$U \mapsto \mathrm{End}_{\mathrm{Mod} R/\mathcal{S}_U}(R) \quad (\mathrm{End}_{\mathrm{QGr} A/\mathcal{S}_U}(\mathcal{O})).$$

If  $V \subseteq U$  are open subsets, then the restriction map

$$\mathrm{End}_{\mathrm{Mod} R/\mathcal{S}_U}(R) \rightarrow \mathrm{End}_{\mathrm{Mod} R/\mathcal{S}_V}(R) \quad (\mathrm{End}_{\mathrm{QGr} A/\mathcal{S}_U}(\mathcal{O}) \rightarrow \mathrm{End}_{\mathrm{QGr} A/\mathcal{S}_V}(\mathcal{O}))$$

is induced by the quotient functor  $\mathrm{Mod} R/\mathcal{S}_U \rightarrow \mathrm{Mod} R/\mathcal{S}_V$  ( $\mathrm{QGr} A/\mathcal{S}_U \rightarrow \mathrm{QGr} A/\mathcal{S}_V$ ). The sheafification is called the *structure sheaf* of  $\mathrm{Mod} R$  ( $\mathrm{QGr} A$ ) and is denoted by  $\mathcal{O}_{\mathrm{Mod} R}$  ( $\mathcal{O}_{\mathrm{QGr} A}$ ). This is a sheaf of commutative rings by [13, XI.2.4]. Next let  $\mathcal{P} \in \mathrm{Spec}(\mathrm{Mod} R)$  and  $P := f^{-1}(\mathcal{P})$ . We have

$$\mathcal{O}_{\mathrm{Mod} R, \mathcal{P}} = \varinjlim_{\mathcal{P} \in U} \mathrm{End}_{\mathrm{Mod} R/\mathcal{S}_U}(R) = \varinjlim_{f \notin P} \mathrm{End}_{\mathrm{Mod} R/\mathcal{S}_{D(f)}}(R) \cong \varinjlim_{f \notin P} R_f = \mathcal{O}_{R, P}.$$

Similarly, for  $\mathcal{P} \in \mathrm{Spec}(\mathrm{QGr} A)$  and  $P := f^{-1}(\mathcal{P})$  we have

$$\mathcal{O}_{\mathrm{QGr} A, \mathcal{P}} \cong \mathcal{O}_{\mathrm{Proj} A, P}.$$

The next theorem says that the abelian category  $\mathrm{Mod} R$  ( $\mathrm{QGr} A$ ) contains all the necessary information to reconstruct the affine (projective) scheme  $(\mathrm{Spec} R, \mathcal{O}_R)$  (respectively  $(\mathrm{Proj} A, \mathcal{O}_{\mathrm{Proj} A})$ ).

**Theorem 8.1** (Reconstruction). *Let  $R$  (respectively  $A$ ) be a ring (respectively graded ring which is finitely generated as an  $A_0$ -algebra). The maps of Corollary 7.3 induce isomorphisms of ringed spaces*

$$f : (\mathrm{Spec} R, \mathcal{O}_R) \xrightarrow{\sim} (\mathrm{Spec}(\mathrm{Mod} R), \mathcal{O}_{\mathrm{Mod} R})$$

and

$$f : (\mathrm{Proj} A, \mathcal{O}_{\mathrm{Proj} A}) \xrightarrow{\sim} (\mathrm{Spec}(\mathrm{QGr} A), \mathcal{O}_{\mathrm{QGr} A}).$$

*Proof.* The proof is similar to that of [5, 9.4]. Fix an open subset  $U \subseteq \mathbf{Spec}(\mathrm{Mod} R)$  and consider the composition of the functors

$$F : \mathrm{Mod} R \xrightarrow{\widetilde{(-)}} \mathrm{Qcoh} \mathrm{Spec} R \xrightarrow{(-)|_{f^{-1}(U)}} \mathrm{Qcoh} f^{-1}(U).$$

Here, for any  $R$ -module  $M$ , we denote by  $\widetilde{M}$  its associated sheaf. By definition, the stalk of  $\widetilde{M}$  at a prime  $P$  equals the localized module  $M_P$ . We claim that  $F$  annihilates  $\mathcal{S}_U$ . In fact,  $M \in \mathcal{S}_U$  implies  $f^{-1}(\mathrm{supp}(M)) \cap f^{-1}(U) = \emptyset$  and therefore  $\mathrm{supp}_R(M) \cap f^{-1}(U) = \emptyset$ . Thus  $M_P = 0$  for all  $P \in f^{-1}(U)$  and therefore  $F(M) = 0$ . It follows that  $F$  factors through  $\mathrm{Mod} R/\mathcal{S}_U$  and induces a map  $\mathrm{End}_{\mathrm{Mod} R/\mathcal{S}_U}(R) \rightarrow \mathcal{O}_R(f^{-1}(U))$  which extends to a map  $\mathcal{O}_{\mathrm{Mod} R}(U) \rightarrow \mathcal{O}_R(f^{-1}(U))$ . This yields the morphism of sheaves  $f^\# : \mathcal{O}_{\mathrm{Mod} R} \rightarrow f_*\mathcal{O}_R$ .

By the above  $f^\#$  induces an isomorphism  $f_P^\# : \mathcal{O}_{\mathrm{Mod} R, f(P)} \rightarrow \mathcal{O}_{R, P}$  at each point  $P \in \mathrm{Spec} R$ . We conclude that  $f_P^\#$  is an isomorphism. It follows that  $f$  is an isomorphism of ringed spaces if the map  $f : \mathrm{Spec} R \rightarrow \mathrm{Spec}(\mathrm{Mod} R)$  is a homeomorphism. This last condition is a consequence of Propositions 7.1 and 7.2. The same arguments apply to show that

$$f : (\mathrm{Proj} A, \mathcal{O}_{\mathrm{Proj} A}) \xrightarrow{\sim} (\mathrm{Spec}(\mathrm{QGr} A), \mathcal{O}_{\mathrm{QGr} A})$$

is an isomorphism of ringed spaces.  $\square$

## 9. Appendix

A subcategory  $\mathcal{S}$  of a Grothendieck category  $\mathcal{C}$  is said to be *Serre* if for any short exact sequence

$$0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$$

$X', X'' \in \mathcal{S}$  if and only if  $X \in \mathcal{S}$ . A Serre subcategory  $\mathcal{S}$  of  $\mathcal{C}$  is said to be a *torsion class* if  $\mathcal{S}$  is closed under taking coproducts. An object  $C$  of  $\mathcal{C}$  is said to be *torsionfree* if  $(\mathcal{S}, C) = 0$ . The pair consisting of a torsion class and the corresponding class of torsionfree objects is referred to as a *torsion theory*. Given a torsion class  $\mathcal{S}$  in  $\mathcal{C}$  the *quotient category*  $\mathcal{C}/\mathcal{S}$  is the full subcategory with objects those torsionfree objects  $C \in \mathcal{C}$  satisfying  $\mathrm{Ext}^1(T, C) = 0$  for every  $T \in \mathcal{S}$ . The inclusion functor  $i : \mathcal{S} \rightarrow \mathcal{C}$  admits the right adjoint  $t : \mathcal{C} \rightarrow \mathcal{S}$  which takes every object  $X \in \mathcal{C}$  to the maximal subobject  $t(X)$  of  $X$  belonging to  $\mathcal{S}$ . The functor  $t$  we call the *torsion functor*. Moreover, the inclusion functor  $i : \mathcal{C}/\mathcal{S} \rightarrow \mathcal{C}$  has a left adjoint, the *localization functor*  $(-)_\mathcal{S} : \mathcal{C} \rightarrow \mathcal{C}/\mathcal{S}$ , which is also exact. Then,

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \cong \mathrm{Hom}_{\mathcal{C}/\mathcal{S}}(X_\mathcal{S}, Y)$$

for all  $X \in \mathcal{C}$  and  $Y \in \mathcal{C}/\mathcal{S}$ . A torsion class  $\mathcal{S}$  is *of finite type* if the functor  $i : \mathcal{C}/\mathcal{S} \rightarrow \mathcal{C}$  preserves directed sums. If  $\mathcal{C}$  is a locally coherent Grothendieck category then  $\mathcal{S}$  is of finite type if and only if  $i : \mathcal{C}/\mathcal{S} \rightarrow \mathcal{C}$  preserves direct limits (see, e.g., [7]).

Let  $\mathcal{C}$  be a Grothendieck category having a family of finitely generated projective generators  $\mathcal{A} = \{P_i\}_{i \in I}$ . Let  $\mathfrak{F} = \bigcup_{i \in I} \mathfrak{F}^i$  be a family of subobjects, where each  $\mathfrak{F}^i$  is a family of subobjects of  $P_i$ . We refer to  $\mathfrak{F}$  as a *Gabriel filter* if the following axioms are satisfied:

- T1.  $P_i \in \mathfrak{F}^i$  for every  $i \in I$ ;
- T2. if  $\mathfrak{a} \in \mathfrak{F}^i$  and  $\mu : P_j \rightarrow P_i$  then  $\{\mathfrak{a} : \mu\} = \mu^{-1}(\mathfrak{a}) \in \mathfrak{F}^j$ ;
- T3. if  $\mathfrak{a}$  and  $\mathfrak{b}$  are subobjects of  $P_i$  such that  $\mathfrak{a} \in \mathfrak{F}^i$  and  $\{\mathfrak{b} : \mu\} \in \mathfrak{F}^j$  for any  $\mu : P_j \rightarrow P_i$  with  $\text{Im } \mu \subset \mathfrak{a}$  then  $\mathfrak{b} \in \mathfrak{F}^j$ .

In particular each  $\mathfrak{F}^i$  is a filter of subobjects of  $P_i$ . A Gabriel filter is *of finite type* if each of these filters has a cofinal set of finitely generated objects (that is, if for each  $i$  and each  $\mathfrak{a} \in \mathfrak{F}_i$  there is a finitely generated  $\mathfrak{b} \in \mathfrak{F}_i$  with  $\mathfrak{a} \supseteq \mathfrak{b}$ ).

Note that if  $\mathcal{A} = \{A\}$  is a ring and  $\mathfrak{a}$  is a right ideal of  $A$ , then for every endomorphism  $\mu : A \rightarrow A$

$$\mu^{-1}(\mathfrak{a}) = \{\mathfrak{a} : \mu(1)\} = \{a \in A \mid \mu(1)a \in \mathfrak{a}\}.$$

On the other hand, if  $x \in A$ , then  $\{\mathfrak{a} : x\} = \mu^{-1}(\mathfrak{a})$ , where  $\mu \in \text{End } A$  is such that  $\mu(1) = x$ .

It is well-known (see, e.g., [7]) that the map

$$\mathcal{S} \longmapsto \mathfrak{F}(\mathcal{S}) = \{\mathfrak{a} \subseteq P_i \mid i \in I, P_i/\mathfrak{a} \in \mathcal{S}\}$$

establishes a bijection between the Gabriel filters (respectively Gabriel filters of finite type) and the torsion classes on  $\mathcal{C}$  (respectively torsion classes of finite type on  $\mathcal{C}$ ).

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