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# Homotopy theory of associative rings

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**Abstract**

A kind of unstable homotopy theory on the category of associative rings (without unit) is developed. There are the notions of fibrations, homotopy (in the sense of Karoubi), path spaces, Puppe sequences, etc. One introduces the notion of a quasi-isomorphism (or weak equivalence) for rings and shows that—similar to spaces—the derived category obtained by inverting the quasi-isomorphisms is naturally left triangulated. Also, homology theories on rings are studied. These must be homotopy invariant in the algebraic sense, meet the Mayer–Vietoris property and plus some minor natural axioms. To any functor $\mathcal{X}$ from rings to pointed simplicial sets a homology theory is associated in a natural way. If $\mathcal{X} = GL$ and fibrations are the $GL$-fibrations, one recovers Karoubi–Villamayor's functors $KV_i$, $i > 0$. If $\mathcal{X}$ is Quillen's $K$-theory functor and fibrations are the surjective homomorphisms, one recovers the (non-negative) homotopy $K$-theory in the sense of Weibel. Technical tools we use are the homotopy information for the category of simplicial functors on rings and the Bousfield localization theory for model categories. The machinery developed in the paper also allows to give another definition for the triangulated category $kk$ constructed by Cortiñas and Thom [G. Cortiñas, A. Thom, Bivariant algebraic K-theory, preprint, math.KT/0603531]. The latter category is an algebraic analog for triangulated structures on operator algebras used in Kasparov's $KK$-theory.


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**Contents**



## 1. Introduction

In the sixties mathematicians invented lower algebraic $K$-groups of a ring and proved various exact sequences involving $K_0$ and $K_1$ (see Bass [1]). For instance, given a cartesian square of rings

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ C & \xrightarrow{g} & D \end{array} \tag{1}$$

with $f$ or $g$ surjective, Milnor [1] proved a Mayer–Vietoris sequence involving $K_0$ and $K_1$: the induced sequence of abelian groups

$$K_1(A) \to K_1(C) \oplus K_1(B) \to K_1(D) \xrightarrow{\partial} K_0(A) \to K_0(C) \oplus K_0(B) \to K_0(D) \tag{2}$$

is exact.

After Quillen [18] the higher algebraic $K$-groups of a ring $R$ are defined by producing a space $K(R)$ and setting $K_n(R) = \pi_n K(R)$. $K$ can be defined so that it actually gives a functor (*Rings*) $\to$ (*Spaces*), and so the groups $K_n(R)$ start to look like a homology theory on rings. However, there are negative results which limit any search for extending the exact sequence (2) to the left involving higher $K$-groups. For example, Swan [19] has shown that there is no satisfactory $K$-theory, extending $K_0$ and $K_1$ and yielding Mayer–Vietoris sequences, even if both $f$ and $g$ are surjective. Moreover, the algebraic $K$-theory is not homotopy invariant in the algebraic sense. These remarks show that $K$ is not a homology theory in the usual sense.

Given any admissible category $\mathfrak{R}$ of rings with or without unit (defined in Section 2) Gersten [9] considers group valued functors $G$ on $\mathfrak{R}$ which preserve zero object, cartesian squares, and kernels of surjective ring homomorphisms. He calls such a functor a left exact MV-functor. It leads naturally to a homology theory $\{k_i^G,\ i \geqslant 1\}$ of group valued functors on $\mathfrak{R}$. We require a homology theory to be homotopy invariant in the algebraic sense, to meet the Mayer–Vietoris property, and some other minor natural properties given in Section 4. If $G = GL$ one recovers the functors $KV_i$ of Karoubi and Villamayor [15]. The groups $KV_i(A)$ coincide with $K_i(A)$ for any regular ring $A$.

After developing the general localization theory for model categories in the 90s (see the monograph by Hirschhorn [13]) we now have new devices for producing homology theories on rings. More precisely, we fix an admissible category of rings $\mathfrak{R}$ and a family of fibrations $\mathfrak{F}$ on it like, for example, the $GL$-fibrations or the surjective homomorphisms. Then any simplicial functor on $\mathfrak{R}$ gives rise to a homology theory:

**Theorem.** *To any functor $\mathscr{X}$ from $\mathfrak{R}$ to pointed simplicial sets a homology theory $\{k_i^{\mathscr{X}}, i \geqslant 0\}$ is associated. Such a homology theory is defined by means of an explicitly constructed functor $Ex_{I,J}(\mathscr{X})$ from $\mathfrak{R}$ to pointed simplicial sets and, by definition,*

$$k_i^{\mathscr{X}}(A) := \pi_i\big(Ex_{I,J}(\mathscr{X})(A)\big)$$

*for any $A \in \mathfrak{R}$ and $i \geqslant 0$. Moreover, there is a natural transformation $\theta_{\mathscr{X}} : \mathscr{X} \to Ex_{I,J}(\mathscr{X})$, functorial in $\mathscr{X}$.*

Roughly speaking, we turn any pointed simplicial functor into a homology theory. If $\mathscr{X} = G$ one recovers the functors $k_i^G$ of Gersten. In this way, the important simplicial functors $GL$ and $K$ give rise to the homology theories $\{KV_i \mid \mathfrak{F} = GL\text{-fibrations}\}$ and $\{KH_i \mid \mathfrak{F} = \text{surjective maps}\}$ respectively. Here $KH$ stands for the (non-negative) homotopy $K$-theory in the sense of Weibel [25].

Next we present another part, developing a sort of unstable homotopy theory on an admissible category of associative rings $\mathfrak{R}$. We are based on the feeling that if rings are in a certain sense similar to spaces then there should exist a homotopy theory where the homomorphism $A \to A[x]$ is a homotopy equivalence, the Puppe sequence, constructed by Gersten in [10], leads to various long exact sequences, the loop ring $\Omega A = (x^2 - x)A[x]$ is interpreted as the loop space, etc.

For this we give definitions of quasi-isomorphisms for rings and left derived categories $D^-(\mathfrak{R}, \mathfrak{F})$ associated to any family of fibrations $\mathfrak{F}$ on $\mathfrak{R}$. We show how to construct $D^-(\mathfrak{R}, \mathfrak{F})$, mimicking the passage from spaces or chain complexes to the homotopy category and the localization from this homotopy category to the derived category.

In this way, the left derived category $D^-(\mathfrak{R}, \mathfrak{F})$ is obtained from the admissible category of rings $\mathfrak{R}$ in two stages. First one constructs a quotient $\mathscr{H}\mathfrak{R}$ of $\mathfrak{R}$ by equating homotopy equivalent (in the sense of Karoubi) homomorphisms between rings. Then one localizes $\mathscr{H}\mathfrak{R}$ by inverting quasi-isomorphisms via a calculus of fractions. These steps are explained in Section 5. If $\mathfrak{F}$ is saturated, which is always the case in practice, then $D^-(\mathfrak{R}, \mathfrak{F})$ is naturally left triangulated. The left triangulated structure as such is a tool for producing homology theories on rings.

**Theorem.** *Let $\mathfrak{F}$ be a saturated family of fibrations in $\mathfrak{R}$. One can define the category of left triangles $\mathscr{L}tr(\mathfrak{R}, \mathfrak{F})$ in $D^-(\mathfrak{R}, \mathfrak{F})$ having the usual set of morphisms from $\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$ to $\Omega C' \xrightarrow{f'} A' \xrightarrow{g'} B' \xrightarrow{h'} C'$. Then $\mathscr{L}tr(\mathfrak{R}, \mathfrak{F})$ is a left triangulation of $D^-(\mathfrak{R}, \mathfrak{F})$, i.e. it is closed* *under isomorphisms and enjoys the axioms which are versions of Vierdier's axioms for triangulated categories. Stabilization of the loop functor $\Omega$ produces a triangulated category $D(\mathfrak{R}, \mathfrak{F})$ out of the left triangulated category $D^-(\mathfrak{R}, \mathfrak{F})$.*

Motivated by ideas and work of J. Cuntz on bivariant $K$-theory of locally convex algebras (see [6,7]), Cortiñas and Thom [5] construct a bivariant homology theory $kk_*(A, B)$ on the category $\mathrm{Alg}_H$ of algebras over a unital ground ring $H$. It is Morita invariant, homotopy invariant, excisive $K$-theory of algebras, which is universal in the sense that it maps uniquely to any other such theory. This bivariant $K$-theory is defined in a triangulated category $kk$ whose objects are the $H$-algebras without unit and $kk_n(A, B) = kk(A, \Omega^n B)$, $n \in \mathbb{Z}$. We make use of our machinery to study various triangulated structures on admissible categories of rings which are not necessarily small. As an application, we give another, but equivalent, description of the triangulated category $kk$.

**Theorem.** *Let $\mathfrak{R}$ be an arbitrary admissible category of rings and let $\mathfrak{W}$ be any subcategory of homomorphisms containing $A \to A[x]$ such that the triple $(\mathfrak{R}, \mathfrak{W}, \mathfrak{F} = \{\text{surjective maps}\})$ is a Brown category. There is a triangulated category $D(\mathfrak{R}, \mathfrak{W})$ whose objects and morphisms are defined similar to those of $D(\mathfrak{R}, \mathfrak{F})$. If $\mathfrak{R} = \mathrm{Alg}_H$ and $\mathfrak{W}_{CT}$ is the class of weak equivalences generated by Morita invariant, homotopy invariant, excisive homology theories, then there is a natural triangulated equivalence of the triangulated categories $D(\mathrm{Alg}_H, \mathfrak{W}_{CT})$ and kk.*

The main tools of the paper are coming from modern homotopical algebra (as exposed for instance in the work of Hovey [14], Hirschhorn [13], Dugger [8], Goerss and Jardine [11]). To develop homotopy theory of rings we consider the model category $U\mathfrak{R}$ of simplicial functors on $\mathfrak{R}$, i.e. simplicial presheaves on $\mathfrak{R}^{\mathrm{op}}$ instead of simplicial presheaves on $\mathfrak{R}$. The model structure is given by injective maps (cofibrations) and objectwise weak equivalences of simplicial sets (Quillen equivalences). There is a contravariant embedding $r$ of $\mathfrak{R}$ into $U\mathfrak{R}$ as representable functors. We need to localize this model structure to take into account the pullback squares (1) with $f$ a fibration in $\mathfrak{F}$ and the fact that $rA[x] \to rA$ should be a Quillen equivalence. Let us remark that we require a homology theory to take such distinguished squares to the Mayer–Vietoris sequence. To do so, we define a set $\mathscr{S}$ to consist of the maps $rA[x] \to rA$ for any ring $A$ and maps $rB \coprod_{rD} rC \to rA$ for every pullback square (1) in $\mathfrak{R}$ with $f$ a fibration. Then one localizes $U\mathfrak{R}$ at $\mathscr{S}$. This procedure is a reminiscence of an unstable motivic model category. The latter model structure is obtained from simplicial presheaves $\mathscr{E}$ on smooth schemes by localizing $\mathscr{E}$ at the set $\mathscr{S}$ of the maps $X \times \mathbb{A}^1 \to X$ for any smooth scheme $X$ and maps $P \to D$ for every pullback square (1) of smooth schemes with $f$ etale, $g$ an open embedding, and $f^{-1}(D - C) \to D - C$ an isomorphism. There is then some work involving properties of the Nisnevich topology to show that this model category is equivalent to the Morel–Voevodsky motivic model category of [16].

### *1.1. Organization of the paper*

After fixing some notation and terminology in Section 2, we study the notion of $I$-homotopy for simplicial functors on an admissible category of rings $\mathfrak{R}$. It has a lot of common properties with $\mathbb{A}^1$-homotopy for simplicial (pre-)sheaves on schemes. We show there how to convert a simplicial functor into a homotopy invariant one. All this material is the content of Section 3. Then comes Section 4 in which homology theories on rings are investigated. We also construct there the simplicial functor $Ex_{I,J}(\mathscr{X})$. Derived categories on rings and their left triangulated structure are studied in Section 5. In Section 6 the stabilization procedure is described as well as the triangulated categories $D(\mathfrak{R}, \mathfrak{F})$. In Section 7 we apply the machinery developed in the preceding sections to study various triangulated structures on admissible categories of rings which are not necessarily small. We also give an equivalent definition of *kk* there. The necessary facts about Bousfield localization in model categories are given in Section 8.

## 2. Preliminaries

We shall work in the category $\mathcal{R}ing$ of associative rings (with or without unit) and ring homomorphisms. Following Gersten [9] a category of rings $\mathfrak{R}$ is *admissible* if it is a full subcategory of $\mathcal{R}ing$ and

(1) if $R$ is in $\mathfrak{R}$, $I$ is a (two-sided) ideal of $R$ then $I$ and $R/I$ are in $\mathfrak{R}$;
(2) if $R$ is in $\mathfrak{R}$, then so is $R[x]$, the polynomial ring in one variable;
(3) given a cartesian square

$$\begin{array}{ccc} D & \xrightarrow{\rho} & A \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{g} & C \end{array}$$

in $\mathcal{R}ing$ with $A, B, C$ in $\mathfrak{R}$, then $D$ is in $\mathfrak{R}$.

One may abbreviate (1), (2), and (3) by saying that $\mathfrak{R}$ is closed under operations of taking ideals, homomorphic images, polynomial extensions in a finite number of variables, and fibre products. If otherwise stated we shall always work in a fixed (skeletally) small admissible category $\mathfrak{R}$.

**Remark.** Given a ring homomorphism $f : R \to R'$ in $\mathcal{R}ing$ between two rings with unit, $f(1)$ need not be equal to 1. We only assume that $f(r_1 r_2) = f(r_1) f(r_2)$ and $f(r_1 + r_2) = f(r_1) + f(r_2)$ for any two elements $r_1, r_2 \in R$. It follows that the trivial ring 0 is a zero object in $\mathcal{R}ing$.

If $R$ is a ring then the polynomial ring $R[x]$ admits two homomorphisms onto $R$

$$R[x] \underset{\partial_x^1}{\overset{\partial_x^0}{\rightrightarrows}} R$$

where

$$\partial_x^i\big|_R = 1_R, \quad \partial_x^i(x) = i, \quad i = 0, 1.$$

Of course, $\partial_x^1(x) = 1$ has to be understood in the sense that $\Sigma r_n x^n \mapsto \Sigma r_n$.

**Definition.** Two ring homomorphisms $f_0, f_1 : S \to R$ are *elementary homotopic*, written $f_0 \sim f_1$, if there exists a ring homomorphism

$$f : S \to R[x]$$

such that $\partial_x^0 f = f_0$ and $\partial_x^1 f = f_1$. A map $f : S \to R$ is called an *elementary homotopy equivalence* if there is a map $g : R \to S$ such that $fg$ and $gf$ are elementary homotopic to $\mathrm{id}_R$ and $\mathrm{id}_S$ respectively.

For example, let $A$ be a $\mathbb{N}$-graded ring, then the inclusion $A_0 \to A$ is an elementary homotopy equivalence. The homotopy inverse is given by the projection $A \to A_0$. Indeed, the map $A \to A[x]$ sending a homogeneous element $a_n \in A_n$ to $a_n t^n$ is a homotopy between the composite $A \to A_0 \to A$ and the identity $\mathrm{id}_A$.

The relation "elementary homotopic" is reflexive and symmetric [9, p. 62]. One may take the transitive closure of this relation to get an equivalence relation (denoted by the symbol "$\simeq$"). The set of equivalence classes of morphisms $R \to S$ is written $[R, S]$.

**Lemma 2.1.** *(Gersten [10]) Given morphisms in* $\mathcal{R}ing$

$$R \xrightarrow{f} S \underset{g'}{\overset{g}{\rightrightarrows}} T \xrightarrow{h} U$$

*such that* $g \simeq g'$, *then* $gf \simeq g'f$ *and* $hg \simeq hg'$.

Thus homotopy behaves well with respect to composition and we have category *Hotring*, the *homotopy category of rings*, whose objects are rings and such that $Hotring(R, S) = [R, S]$. The homotopy category of an admissible category of rings $\mathfrak{R}$ will be denoted by $\mathscr{H}(\mathfrak{R})$.

The diagram in $\mathcal{R}ing$

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is a short exact sequence if $f$ is injective ($\equiv \operatorname{Ker} f = 0$), $g$ is surjective, and the image of $f$ is equal to the kernel of $g$. Thus $f$ is a monomorphism in $\mathfrak{R}$ and $f = \ker g$.

**Definition.** A ring $R$ is *contractible* if $0 \sim 1$; that is, if there is a ring homomorphism $f : R \to R[x]$ such that $\partial_x^0 f = 0$ and $\partial_x^1 f = 1_R$.

Following Karoubi and Villamayor [15] we define *ER*, the *path ring* on $R$, as the kernel of $\partial_x^0 : R[x] \to R$, so $ER \to R[x] \xrightarrow{\partial_x^0} R$ is a short exact sequence in $\mathcal{R}ing$. Also $\partial_x^1 : R[x] \to R$ induces a surjection

$$\partial_x^1 : ER \to R$$

and we define the *loop ring* $\Omega R$ of $R$ to be its kernel, so we have a short exact sequence in $\mathcal{R}ing$

$$\Omega R \to ER \xrightarrow{\partial_x^1} R.$$

Clearly, $\Omega R$ is the intersection of the kernels of $\partial_x^0$ and $\partial_x^1$. By [9, 3.3] $ER$ is contractible for any ring $R$.

## 3. The functor $Sing_*$

In this section we introduce and study the important notion of $I$-homotopy for simplicial functors on an admissible category of rings $\Re$. It is similar to $\mathbb{A}^1$-homotopy in the sense of Morel and Voevodsky [16].

### 3.1. Homotopization

Recall that a simplicial set map $f : X \to Y$ is a weak equivalence if all maps

(1) $\pi_0 X \to \pi_0 Y$, and
(2) $\pi_i(X, x) \to \pi_i(Y, fx)$, $x \in X_0$, $i \geqslant 1$

are bijections. Here $\pi_i(X, x) = \pi_i(|X|, x)$, in general, but

$$\pi_i(X, x) = \big[(S^i, *), (X, x)\big] = \pi\big((S^i, *), (X, x)\big)$$

if $X$ is fibrant (recall that $S^i = \Delta^i/\partial\Delta^i$ is the simplicial $i$-sphere).

Following Gersten, we say that a functor $F$ from rings to sets is *homotopy invariant* if $F(R) \cong F(R[t])$ for every $R$. Similarly, a functor $F$ from rings to simplicial sets is *homotopy invariant* if for every ring $R$ the natural map $R \to R[t]$ induces a weak equivalence of simplicial sets $F(R) \simeq F(R[t])$. Note that each homotopy group $\pi_n(F(R))$ also forms a homotopy invariant functor.

We shall introduce the simplicial ring $R[\Delta]$, and use it to define the homotopization functor $Sing_*$.

For each ring $R$ one defines a simplicial ring $R[\Delta]$,

$$R[\Delta]_n := R\big[\Delta^n\big] = R[t_0, \dots, t_n]\Big/\Big(\sum t_i - 1\Big)R \quad \big(\cong R[t_1, \dots, t_n]\big).$$

The face and degeneracy operators $\partial_i : R[\Delta^n] \to R[\Delta^{n-1}]$ and $s_i : R[\Delta^n] \to R[\Delta^{n+1}]$ are given by

$$\partial_i(t_j)\ \big(\text{respectively } s_i(t_j)\big) = \begin{cases} t_j \text{ (respectively } t_j), & j < i, \\ 0 \text{ (respectively } t_j + t_{j+1}), & j = i, \\ t_{j-1} \text{ (respectively } t_{j+1}), & i < j. \end{cases}$$

Note that the face maps $\partial_{0;1} : R[\Delta^1] \to R[\Delta^0]$ are isomorphic to $\partial_t^{0;1} : R[t] \to R$ in the sense that the diagram

$$\begin{array}{ccc} R[t] & \xrightarrow{\partial_t^\varepsilon} & R \\ {\scriptstyle t\mapsto t_0}\downarrow & & \downarrow \\ R[\Delta^1] & \xrightarrow{\partial_\varepsilon} & R[\Delta^0] \end{array}$$

is commutative and the vertical maps are isomorphisms.

**Lemma 3.1.** *The inclusion of simplicial rings* $R[\Delta] \subset R[x][\Delta]$ *is a homotopy equivalence, split by evaluation at* $x = 0$.

**Proof.** A simplicial homotopy from $R[x][\Delta]$ to $R[x][\Delta]$ is a simplicial map

$$h : R[x][\Delta] \times \Delta^1 \to R[x][\Delta].$$

Recall that a $n$-simplex $v$ of $\Delta^1$ is nothing more than to give an integer $i$ with $-1 \leqslant i \leqslant n$, and send the integers $\{0, 1, \dots, i\}$ to 0, while the integers $\{i+1, i+2, \dots, n\}$ map to 1. So any homotopy is given by maps

$$h_v^{(n)} : R[x]\big[\Delta^n\big] \to R[x]\big[\Delta^n\big], \quad v \in \Delta^1,$$

which must be compatible with the face and degeneracy operators.

Given $v = v(i) \in \Delta^1$ let $h_v^{(n)}(f) = f$ if $f \in R[\Delta^n]$ and

$$x \mapsto \begin{cases} x(t_0 + \dots + t_i), & i \geqslant 0, \\ 0, & i = -1. \end{cases}$$

It is directly verified that the maps $h_v^{(n)}$ are compatible with the face and degeneracy operators. These maps define a simplicial homotopy between the identity map of $R[x][\Delta]$ and the composite

$$R[x][\Delta] \xrightarrow{x=0} R[\Delta] \subset R[x][\Delta].$$

This implies the claim. □

**Definition.** (Homotopization) Let $F$ be a functor from rings to simplicial sets. Its *homotopization* $Sing_*(F)$ is defined at each ring $R$ as the diagonal of the bisimplicial set $F(R[\Delta])$. Thus $Sing_*(F)$ is also a functor from rings to simplicial sets. If we consider $R$ as a constant simplicial ring, the natural map $R \to R[\Delta]$ yields a natural transformation $F \to Sing_*(F)$.

(Strict Homotopization) Let $F$ be a functor from rings to sets. Its *strict homotopization* $[F]$ is defined as the coequalizer of the evaluations at $t = 0, 1 : F(R[t]) \rightrightarrows F(R)$. The coequaliser can be constructed as follows. Given $x, y \in F(R)$, write $x \sim y$ if there is a $z \in F(R[t])$ such that $(t = 0)(z) = x$ and $(t = 1)(z) = y$. Then this relation is reflexive and symmetric (use the automorphism $R[t] \xrightarrow{t \mapsto 1-t} R[t]$). Its transitive closure determines an equivalence relation and then $[F](R)$ is the quotient of $F(R)$ with respect to this equivalence relation.

In fact, $[F]$ is a homotopy invariant functor and there is a universal transformation $F(R) \to [F](R)$. Moreover, if $F$ takes values in groups then so does $[F]$ (see Weibel [26]).

Given a functor $F$ from rings to simplicial sets, by $F[t]$ denote the functor which is defined as $F(R[t])$ at each ring $R$. The natural inclusion $R \to R[t]$ yields a natural transformation $F \to F[t]$.

**Proposition 3.2.** *Let $F$ be a functor from rings to simplicial sets. Then*:

(1) $Sing_*(F)$ *is a homotopy invariant functor*;

(2) *if $F$ is homotopy invariant then $F(R) \to Sing_*(F)(R)$ is a weak equivalence for all $R$ and $Sing_*(F) \to Sing_*(F)[t]$ is an objectwise homotopy equivalence, functorial in $R$*;
(3) *$\pi_0(Sing_*(F))$ is a strict homotopization $[F_0]$ of the functor $F_0(R) = \pi_0(F(R))$.*

**Proof.** Let us show that the inclusion of simplicial rings $R[\Delta] \subset R[x][\Delta]$ induces a weak equivalence $Sing_*(F)(R) \to Sing_*(F)(R[x])$. Actually we shall prove even more: the latter map turns out to be a homotopy equivalence of simplicial sets (also showing that $Sing_*(F) \to Sing_*(F)[t]$ is a homotopy equivalence).

Let

$$h_v^{(n)} : R[x]\big[\Delta^n\big] \to R[x]\big[\Delta^n\big], \quad v \in \Delta^1,$$

be the maps constructed in the proof of Lemma 3.1. We claim that the maps

$$H_v^{(n)} = F_n\big(h_v^{(n)}\big) : F_n\big(R[x]\big[\Delta^n\big]\big) \to F_n\big(R[x]\big[\Delta^n\big]\big), \quad v \in \Delta^1,$$

define a simplicial homotopy between the identity map of $Sing_*(F)(R[x])$ and the composite

$$Sing_*(F)\big(R[x]\big) \xrightarrow{x=0} Sing_*(F)(R) \to Sing_*(F)\big(R[x]\big).$$

For this, we must verify that the maps $H_v^{(n)}$ are compatible with the structure maps $w : [m] \to [n]$ in $\Delta$.

We already know that $w^* \circ h_v^{(n)} = h_{vw}^{(m)} \circ w^*$. One has a commutative diagram

$$\begin{array}{ccccc}
F_n(R[x][\Delta^n]) & \xrightarrow{F_n(w^*)} & F_n(R[x][\Delta^m]) & \xrightarrow{F_n(h_{vw}^{(m)})} & F_n(R[x][\Delta^m]) \\
\downarrow{\scriptstyle w_F^*} & \searrow{\scriptstyle w^*_{Sing_*(F)}} & \downarrow{\scriptstyle w_F^*} & & \downarrow{\scriptstyle w_F^*} \\
F_m(R[x][\Delta^n]) & \xrightarrow[F_m(w^*)]{} & F_m(R[x][\Delta^m]) & \xrightarrow[F_m(h_{vw}^{(m)}) = H_{vw}^{(m)}]{} & F_m(R[x][\Delta^m]).
\end{array}$$

Then,

$$\begin{aligned}
w^*_{Sing_*(F)} \circ H_v^{(n)} &= w_F^* \circ F_n(w^*) \circ F_n\big(h_v^{(n)}\big) = w_F^* \circ F_n\big(h_{vw}^{(m)}\big) \circ F_n(w^*) \\
&= H_{vw}^{(m)} \circ w_F^* \circ F_n(w^*) = H_{vw}^{(m)} \circ w^*_{Sing_*(F)}.
\end{aligned}$$

We have checked that the maps $H_v^{(n)}$ are compatible with the structure maps in $\Delta$, as claimed. These give the necessary simplicial homotopy.

Part (3) follows from the fact that, for any simplicial space $X.$, the group $\pi_0(|X.|)$ is the coequaliser of $\partial_0, \partial_1 : \pi_0(X_1) \rightrightarrows \pi_0(X_0)$. In this case $\pi_0(X_0) = \pi_0(F(R))$ and $\pi_0(X_1) = \pi_0(F(R[t]))$. □

Let $\Re$ be an admissible category of rings. In order to construct homology theories on $\Re$, we shall use the model category $U\Re$ of covariant functors from $\Re$ to simplicial sets (and not contravariant functors as usual). Note that this usage deviates from the usual notation and practice, e.g. as in Dugger [8]. We consider the Heller model structure on $U\mathfrak{R}$ instead of the most commonly used Bousfield–Kan model structure. It is a proper, simplicial, cellular model category with weak equivalences and cofibrations being defined objectwise, and fibrations being those maps having the right lifting property with respect to trivial cofibrations (see Dugger [8]). We consider the fully faithful contravariant functor

$$r:\mathfrak{R}\to U\mathfrak{R},\quad A\mapsto \operatorname{Hom}_{\mathfrak{R}}(A,-),$$

where $rA(B)=\operatorname{Hom}_{\mathfrak{R}}(A,B)$ is to be thought of as the constant simplicial set for any $B\in\mathfrak{R}$.

The model structure on $U\mathfrak{R}$ enjoys the following properties (see Dugger [8, p. 21]):

- ◇ every object is cofibrant;
- ◇ being fibrant implies being objectwise fibrant, but is stronger (there are additional diagrammatic conditions involving maps being fibrations, etc.);
- ◇ any object which is constant in the simplicial direction is fibrant.

If $F\in U\mathfrak{R}$ then $U\mathfrak{R}(rA\times\Delta^n,F)=F_n(A)$ (isomorphism of sets). Hence, if we look at simplicial mapping spaces we find

$$\operatorname{Map}(rA,F)=F(A)$$

(isomorphism of simplicial sets). This is a kind of "simplicial Yoneda Lemma."

**Definition.** Let $f,g:\mathscr{X}\to\mathscr{Y}$ be two maps of simplicial presheaves in $U\mathfrak{R}$. An *elementary $I$-homotopy* from $f$ to $g$ is a map $H:\mathscr{X}\to\mathscr{Y}[t]$ such that $\partial^0\circ H=f$ and $\partial^1\circ H=g$, where $\partial^{0;1}:\mathscr{Y}[t]\to\mathscr{Y}$ are the maps induced by $\partial_t^{0;1}:A[t]\to A$, $A\in\mathfrak{R}$. Two morphisms are said to be *$I$-homotopic* if they can be connected by a sequence of elementary $I$-homotopies. A map $f:\mathscr{X}\to\mathscr{Y}$ is called an *$I$-homotopy equivalence* if there is a map $g:\mathscr{Y}\to\mathscr{X}$ such that $fg$ and $gf$ are $I$-homotopic to $\mathrm{id}_{\mathscr{Y}}$ and $\mathrm{id}_{\mathscr{X}}$ respectively.

Let $A,B$ be two rings in $\mathfrak{R}$ and let $H:rB\to(rA)[t]$ be an elementary homotopy of representable functors. It follows that the map $\widehat{H}:=H_B(\mathrm{id}_B):A\to B[t]$ yields an elementary homotopy between $A$ and $B$. Moreover, for any ring $R\in\mathfrak{R}$ and any ring homomorphism $\alpha:B\to R$

$$H_R\circ\alpha_*(\mathrm{id}_B)=H_R(\alpha)=\alpha[t]\circ\widehat{H},\tag{3}$$

where $\alpha_*=\mathfrak{R}(B,\alpha):\mathfrak{R}(B,B)\to\mathfrak{R}(B,R)$ and $\alpha[t]:B[t]\to R[t]$, $\sum b_it^i\mapsto\sum\alpha(b_i)t^i$.

Conversely, suppose $\widehat{H}:A\to B[t]$ is an elementary homotopy in $\mathfrak{R}$, then the collection of maps

$$\bigl\{H_R(\alpha):=\alpha[t]\circ\widehat{H}\bigm| R\in\mathfrak{R},\ \alpha\in\mathfrak{R}(B,R)\bigr\}$$

gives rise to an elementary homotopy $H:rB\to(rA)[t]$.

**Corollary 3.3.** *Two maps $f,g:A\to B$ are elementary homotopic in $\mathfrak{R}$ if and only if the induced maps $f^*,g^*:rB\to(rA)[t]$ are elementary $I$-homotopic in $U\mathfrak{R}$. Furthermore, there is a bijection between elementary homotopies in $\mathfrak{R}$ and elementary $I$-homotopies in $U\mathfrak{R}$. This bijection is given by* (3).

*3.2. The model category $U\mathfrak{R}_I$*

Let $\mathfrak{R}$ be an admissible category of rings and let $I = \{i = i_A : r(A[t]) \to r(A) \mid A \in \mathfrak{R}\}$, where each $i_A$ is induced by the natural homomorphism $i : A \to A[t]$. We shall refer to the $I$-local equivalences as $I$-weak equivalences. The resulting model category $U\mathfrak{R}/I$ will be denoted by $U\mathfrak{R}_I$ and its homotopy category is denoted by $\mathrm{Ho}_I(\mathfrak{R})$. Notice that any homotopy invariant functor $F : \mathfrak{R} \to \mathit{Sets}$ is an $I$-local object in $U\mathfrak{R}$ (hence fibrant in $U\mathfrak{R}_I$).

The following lemma is straightforward.

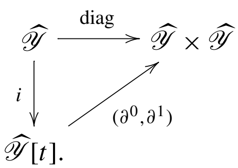
**Lemma 3.4.** *A fibrant object $\mathcal{X} \in U\mathfrak{R}$ is $I$-local if and only if the map $\mathcal{X} \to \mathcal{X}[t]$ is a weak equivalence in $U\mathfrak{R}$.*

**Lemma 3.5.** *If two maps $f, g : \mathcal{X} \to \mathcal{Y}$ in $U\mathfrak{R}$ are elementary $I$-homotopic, then they coincide in the $I$-homotopy category* $\mathrm{Ho}_I(\mathfrak{R})$.

**Proof.** By assumption there is a map $H : \mathcal{X} \to \mathcal{Y}[t]$ such that $\partial^0 H = f$ and $\partial^1 H = g$.

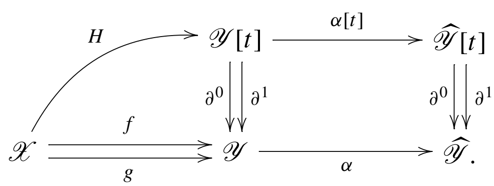
Let $\alpha : \mathcal{Y} \to \widehat{\mathcal{Y}}$ be a fibrant replacement of $\mathcal{Y}$ in $U\mathfrak{R}_I$. It follows that $\widehat{\mathcal{Y}}$ is an $I$-local object in $U\mathfrak{R}$. By Lemma 3.4 the map $i : \widehat{\mathcal{Y}} \to \widehat{\mathcal{Y}}[t]$ is a weak equivalence.

One has a commutative diagram




We see that $\widehat{\mathcal{Y}}[t]$ is a path object of $\widehat{\mathcal{Y}}$ in $U\mathfrak{R}_I$.

Consider the following diagram:




Here $\partial^\varepsilon \circ \alpha[t] = \alpha \circ \partial^\varepsilon$. Hence $\alpha f \overset{r}{\sim} \alpha g$. It follows from [13, 9.5.24; 9.5.15] that $\alpha f$ and $\alpha g$ represent the same map in the homotopy category. Since $\alpha$ is an isomorphism in $\mathrm{Ho}_I(\mathfrak{R})$, we deduce that $f = g$ in $\mathrm{Ho}_I(\mathfrak{R})$. □

**Lemma 3.6.** *Any $I$-homotopy equivalence is an $I$-weak equivalence.*

**Proof.** Let $f : \mathcal{X} \to \mathcal{Y}$ be an $I$-homotopy equivalence and $g$ be an $I$-homotopy inverse to $f$. We have to show that the compositions $fg$ and $gf$ are equal to the corresponding identity morphisms in the $I$-homotopy category $\mathrm{Ho}_I(\mathfrak{R})$. By definition, these maps are $I$-homotopic to the identity and it remains to show that two elementary $I$-homotopic morphisms coincide in the $I$-homotopy category. But this follows immediately from the preceding lemma. □

**Lemma 3.7.** *For any $\mathscr{X}$ the canonical morphism $\mathscr{X} \to \mathscr{X}[t]$ is an $I$-homotopy equivalence, and thus an $I$-weak equivalence.*

**Proof.** For any ring $R$ the natural homomorphism $i: R \to R[t]$ is an elementary homotopy equivalence, split by evaluation at $t = 0$. Indeed, the homomorphism $R[t] \to R[t, y]$ sending $t$ to $ty$ defines an elementary homotopy between the identity homomorphism and the composite

$$R[t] \xrightarrow{t=0} R \subset R[t].$$

Applying $\mathscr{X}$ to the elementary homotopy equivalence $i: R \to R[t]$, one gets an $I$-homotopy from $\mathscr{X}(i \circ (t = 0))$ and $\mathrm{id}_{\mathscr{X}[t]}$. Since $\mathscr{X}((t = 0) \circ i) = \mathrm{id}_{\mathscr{X}}$, the lemma is proven. □

**Corollary 3.8.** *For any $\mathscr{X}$ the canonical morphism $\mathscr{X} \to Sing_*(\mathscr{X})$ is an $I$-trivial cofibration.*

**Proof.** Since $R$ is a retract of $R[\Delta^n]$ for any ring $R$ the map of the assertion is plainly a cofibration. It remains to check that it is an $I$-weak equivalence.

Given a functor $F: \mathfrak{R} \to Sets$, the canonical morphism $F \to F[t_1, \ldots, t_n]$ is an $I$-weak equivalence by Lemma 3.7. Since for any ring $R$ and any $n \geqslant 0$ the ring $R[\Delta^n]$ is isomorphic to $R[t_1, \ldots, t_n]$, functorially in $R$, we see that the canonical morphism $F \to F[\Delta^n]$ is an $I$-weak equivalence, where $F[\Delta^n](R) := F(R[\Delta^n])$.

The canonical morphism $\mathscr{X} \to Sing_*(\mathscr{X})$ coincides objectwise with the canonical morphisms $\mathscr{X}_n \to \mathscr{X}_n[\Delta^n]$. It follows from [13, 18.5.3] that the map

$$\mathrm{hocolim}_{\Delta^{\mathrm{op}}} \mathscr{X}_n \to \mathrm{hocolim}_{\Delta^{\mathrm{op}}} \mathscr{X}_n\big[\Delta^n\big]$$

is an $I$-weak equivalence. By [13, 18.7.5] the canonical map $\mathrm{hocolim}_{\Delta^{\mathrm{op}}} \mathscr{X}_n \to \mathscr{X}$ (respectively $\mathrm{hocolim}_{\Delta^{\mathrm{op}}} \mathscr{X}_n[\Delta^n] \to Sing_*(\mathscr{X})$) is a weak equivalence in $U\mathfrak{R}$, whence the assertion follows. □

Let $\vartheta_{\mathscr{X}}: \mathscr{X} \to R(\mathscr{X})$ denote a fibrant replacement functor in $U\mathfrak{R}$. That is $R(\mathscr{X})$ is fibrant and the map $\vartheta_{\mathscr{X}}$ is a trivial cofibration in $U\mathfrak{R}$. Given a model category $\mathscr{C}$, we write $\mathscr{C}_\bullet$ to denote the model category under the terminal object [14, p. 4]. If $\mathscr{C} = U\mathfrak{R}$ we shall refer to the objects of $U\mathfrak{R}_\bullet$ as pointed simplicial functors.

**Theorem 3.9.** *The map $\mathscr{X} \mapsto R(Sing_*(\mathscr{X}))$ yields a fibrant replacement functor in $U\mathfrak{R}_I$. That is the object $R(Sing_*(\mathscr{X}))$ is $I$-local and the composition*

$$\mathscr{X} \to Sing_*(\mathscr{X}) \to R\big(Sing_*(\mathscr{X})\big)$$

*is an $I$-trivial cofibration. Furthermore, the natural map*

$$\pi_0\big(Sing_*(\mathscr{X})(A)\big) = [\mathscr{X}_0](A) \to \mathrm{Hom}_{\mathrm{Ho}_I(\mathfrak{R})}(rA, \mathscr{X})$$

*is a bijection for any $A \in \mathfrak{R}$. Moreover, if $\mathscr{X}$ is pointed, then for any integer $n \geqslant 0$ and any $A \in \mathfrak{R}$ the obvious map*

$$\pi_n\big(Sing_*(\mathscr{X})(A)\big) \to \mathrm{Hom}_{\mathrm{Ho}_{I,\bullet}(\mathfrak{R})}\big((rA_+) \wedge S^n, \mathscr{X}\big)$$

*is a bijection, where $rA_+ = rA \sqcup pt$.*

**Proof.** The fact that $R(Sing_*(\mathscr{X}))$ is an $I$-local object is a consequence of Proposition 3.2. The map $\mathscr{X} \mapsto R(Sing_*(\mathscr{X}))$ yields a fibrant replacement functor by Corollary 3.8.

The rest of the proof follows from the fact that for any $\mathscr{X} \in U\Re$ the function space of maps $\mathrm{Map}(rA, R(\mathscr{X}))$ may be identified with $R(\mathscr{X})(A)$, which is weakly equivalent to $\mathscr{X}(R)$ because $\mathscr{X} \to R(\mathscr{X})$ is an objectwise weak equivalence. □

**Corollary 3.10.** *Let $\Re$ be an admissible category of rings and let $\mathscr{H}\Re$ be its homotopy category. Then the functor*

$$r : \mathscr{H}\Re \to \mathrm{Ho}_I(\Re), \quad [A, B] \mapsto \mathrm{Ho}_I(\Re)(rB, rA)$$

*is a fully faithful contravariant embedding.*

**Proof.** This is a consequence of Proposition 3.2, Corollary 3.3, and Theorem 3.9. □

Call a ring homomorphism $s : A \to B$ an *$I$-weak equivalence* if its image in $U\Re$ is an $I$-weak equivalence.

**Corollary 3.11.** *Let $B$ be a ring in $\Re$ and consider a ring $B^I$ together with homomorphisms*

$$B \xrightarrow{s} B^I \xrightarrow{(d_0,d_1)} B \times B,$$

*where $s$ is an $I$-weak equivalence and the composite is the diagonal. Then for any homomorphism $H : A \to B^I$ the homomorphisms $d_0 \circ H$ and $d_1 \circ H$ coincide in $\mathscr{H}\Re$.*

**Proof.** Since $s$ is an $I$-weak equivalence, it follows that $rB^I$ is a cylinder object for $rB$. The proof now follows from Theorem 3.9 and Corollary 3.10. □

**Examples.** (1) Let $A \in \mathscr{R}ing$. The group $GL_n(A)$ is defined as $\mathrm{Ker}(GL_n(\varepsilon) : GL_n(A^+) \to GL_n(\mathbb{Z}))$. Here $A^+ = \mathbb{Z} \oplus A$ as a group and

$$(n, a)(m, b) = (nm, nb + ma + ab).$$

We put $\varepsilon : A^+ \to \mathbb{Z}$ to be the augmentation $\varepsilon(n, a) = n$ and $GL(A) := \mathrm{colim}_n\, GL_n(A)$. The associated functor $A \mapsto GL(A)$ in $U\Re_\bullet$, pointed at the unit element, denote by $\mathscr{G}l$.

By definition, the Karoubi–Villamayor $K$-theory is defined as

$$KV_n(A) = \pi_{n-1}\big(GL\big(A[\Delta]\big)\big), \quad n \geqslant 1.$$

It follows from Theorem 3.9 that

$$KV_n(A) = \mathrm{Hom}_{\mathrm{Ho}_{I,\bullet}(\Re)}\big((rA_+) \wedge S^{n-1}, \mathscr{G}l\big), \quad n \geqslant 1.$$

(2) Let $\mathbb{K}^B(R)$ be a non-connective $K$-theory (simplicial) $\Omega$-spectrum, functorial in $R$, where $R$ is a ring with unit. We can extend $\mathbb{K}^B$ to all rings by the rule

$$R \in \mathscr{R}ing \mapsto \mathit{fibre}\big(\mathbb{K}^B\big(R^+\big) \to \mathbb{K}^B(\mathbb{Z})\big).$$

If $R$ has a unit this definition is consistent because then $R^+ \cong \mathbb{Z} \times R$.

The *homotopy K-theory* of $R \in \mathscr{R}ing$ in the sense of Weibel [25] is given by the (fibrant) geometric realization $KH(R)$ of the simplicial spectrum $\mathbb{K}^B(R[\Delta])$. Note that $KH(R)$ is an $\Omega$-spectrum. For $n \in \mathbb{Z}$, we shall write $KH_n(R)$ for $\pi_n KH(R)$.

Let $K(A)$ denote the zeroth term of the spectrum $\mathbb{K}^B(A)$. The corresponding functor $[A \in \mathfrak{R} \mapsto K(A)] \in U\mathfrak{R}$ denote by $\mathscr{K}$. It is pointed at zero. It follows from Theorem 3.9 that

$$KH_n(A) = \operatorname{Hom}_{\mathrm{Ho}_{I,\bullet}(\mathfrak{R})}\big((rA_+) \wedge S^n, \mathscr{K}\big), \quad n \geqslant 0.$$

## 4. Homology theories on rings

In this section we shall construct homology theories on rings. Precisely, one will naturally associate to any pointed simplicial functor $\mathscr{X} \in U\mathfrak{R}_\bullet$ a homology theory $\{H_n = H_n^{\mathscr{X},\mathfrak{F}}\}_{n\geqslant 0}$: $\mathfrak{R} \to Sets$ depending on the family of fibrations $\mathfrak{F}$ of rings defined below. Such a homology theory is defined by means of an explicitly constructed pointed simplicial functor $Ex_{I,J}(\mathscr{X}) \in U\mathfrak{R}_\bullet$ and, by definition,

$$H_n(A) = \pi_n\big(Ex_{I,J}(\mathscr{X})(A)\big)$$

for any $A \in \mathfrak{R}$ and $n \geqslant 0$. Moreover, there is a natural transformation $\theta_{\mathscr{X}} : \mathscr{X} \to Ex_{I,J}(\mathscr{X})$, functorial in $\mathscr{X}$.

There is another formula for $H_n(A)$. A model category $U\mathfrak{R}_{I,J,\bullet}$ is constructed and then

$$H_n(A) = \mathrm{Ho}_{I,J,\bullet}\big(S^n \wedge rA, \mathscr{X}\big),$$

where $\mathrm{Ho}_{I,J,\bullet}$ stands for the homotopy category of $U\mathfrak{R}_{I,J,\bullet}$.

Roughly speaking, we turn any pointed simplicial functor into a homology theory. In this way the important simplicial functors $\mathscr{G}l$ and $\mathscr{K}$ give rise to the homology theories $\{KV_n \mid \mathfrak{F} = GL\text{-fibrations}\}$ and $\{KH_n \mid \mathfrak{F} = \text{surjective maps}\}$ respectively.

### 4.1. Fibrations of rings

**Definition.** Let $\mathfrak{R}$ be an admissible category of rings. A family $\mathfrak{F}$ of surjective homomorphisms of $\mathfrak{R}$ is called *fibrations* if it meets the following axioms:

(Ax 1) for each $R$ in $\mathfrak{R}$, $R \to 0$ is in $\mathfrak{F}$;
(Ax 2) $\mathfrak{F}$ is closed under composition and any isomorphism is a fibration;
(Ax 3) if the diagram

$$\begin{array}{ccc} D & \xrightarrow{\rho} & A \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{g} & C \end{array}$$

is cartesian in $\mathfrak{R}$ and $g \in \mathfrak{F}$, then $\rho \in \mathfrak{F}$. Call such squares *distinguished*. We also require that the "degenerate square" with only one entry, 0, in the upper left-hand corner be a distinguished square;

(Ax 4) any map $u$ in $\mathfrak{R}$ can be factored $u = pi$, where $p$ is a fibration and $i$ is an $I$-weak equivalence.

Notice that the axioms imply that $\mathfrak{R}$ is closed under finite direct products. We call a short exact sequence in $\mathfrak{R}$

$$A \xrightarrow{g} B \xrightarrow{f} C$$

with $f \in \mathfrak{F}$ a *$\mathfrak{F}$-fibre sequence*.

$\mathfrak{F}$ is said to be *saturated* if the homomorphism $\partial_x^1 : EA \to A$ is a fibration for any $A \in \mathfrak{R}$.

The trivial case is $\mathfrak{R} = \mathfrak{F} = 0$. A non-trivial example, $\mathfrak{R} \neq 0$, of fibrations is given by the surjective homomorphisms. Indeed, the axioms (Ax 1)–(Ax 3) are trivial and (Ax 4) follows from Lemma 4.1 below.

Another important example of fibrations is defined by any left exact functor. Recall that a functor $F : \mathscr{R}ing \to Sets$ is *left exact* if $F$ preserves finite limits. In particular, if $A \to B \to C$ is a short exact sequence in $\mathscr{R}ing$, then

$$0 \to FA \to FB \to FC$$

is an exact sequence of pointed sets (since the zero ring is a zero object in $\mathscr{R}ing$, it determines a unique element of $FA$). Furthermore $F$ preserves cartesian squares.

For instance, any representable functor is left exact as well as the functor (see Gersten [9])

$$R \in \mathscr{R}ing \mapsto GL(R).$$

**Definition.** A surjective map $g : B \to C$ is said to be a *$F$-fibration* (where $F : \mathscr{R}ing \to Sets$ is a functor) if $F(E^n(g)) : FE^nB \to FE^nC$ is surjective for all $n > 0$. Observe that nothing is said about $F(g) : FB \to FC$. It follows that if the composite $fg$ of two maps is a $F$-fibration, then so is $f$. If $F = GL$ we refer to $F$-fibrations as *GL-fibrations*. We also note that the family of all surjective homomorphisms is the family of $F$-fibrations with $F$ sending a ring $A$ to itself.

**Lemma 4.1.** *The collection of $F$-fibrations, where $F : \mathfrak{R} \to Sets$ is left exact, enjoys the axioms* (Ax 1)–(Ax 4) *for fibrations on $\mathfrak{R}$ and is saturated.*

**Proof.** The axioms (Ax 1)–(Ax 3) and the fact that $\mathfrak{F}$ is saturated follow from Gersten [10]. Let us check (Ax 4).

Let $u : A \to B$ be a homomorphism in $\mathfrak{R}$. Consider the following commutative diagram

$$\begin{array}{ccccc} EB & \overset{\nu}{\rightarrowtail} & A' & \overset{\iota_2}{\longrightarrow} & A \\ \| & & \downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle u} \\ EB & \overset{\mu}{\rightarrowtail} & B[x] & \overset{\partial_x^0}{\longrightarrow} & B \end{array}$$

with $A' = A \times_B B[x]$. The map $i : A \to A'$, $a \mapsto (a, u(a))$, is split, $\iota_2 i = 1_A$, and obviously an elementary homotopy equivalence. Hence it is an $I$-weak equivalence.

Put $p := \partial_x^1 \circ \iota_1$. Then $p$ is surjective, because any element $b \in B$ is the image of $(0, bx)$. By [10, 2.3] $F(E^n(p\nu)) = F(E^n(\partial_x^1 \mu))$, $n > 0$, is a surjective map. It follows that $F(E^n(p))$ is surjective. We see that $p$ is a $F$-fibration. □

*4.2. The model category $U\mathfrak{R}_J$*

We now introduce the class of excisive functors on $\mathfrak{R}$. They look like flasque presheaves on a site defined by a cd-structure in the sense of Voevodsky [22, p. 14].

**Definition.** Let $\mathfrak{R}$ be an admissible category of rings and let $\mathfrak{F}$ be a family of fibrations. A simplicial functor $\mathscr{X} \in U\mathfrak{R}$ is called *excisive* with respect to $\mathfrak{F}$ if for any distinguished square in $\mathfrak{R}$

$$\begin{array}{ccc} D & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & C \end{array}$$

the square of simplicial sets

$$\begin{array}{ccc} \mathscr{X}(D) & \longrightarrow & \mathscr{X}(A) \\ \downarrow & & \downarrow \\ \mathscr{X}(B) & \longrightarrow & \mathscr{X}(C) \end{array}$$

is a homotopy pullback square. In the case of the degenerate square the latter condition has to be understood in the sense that $\mathscr{X}(0)$ is weakly equivalent to the homotopy pullback of the empty diagram and is contractible. It immediately follows from the definition that every pointed excisive object takes $\mathfrak{F}$-fibre sequences in $\mathfrak{R}$ to homotopy fibre sequences of simplicial sets.

**Examples.** Let $\mathfrak{F}$ be the family of *GL*-fibrations. It follows from [24] that the simplicial functor

$$A \in \mathfrak{R} \mapsto Sing_*(\mathscr{G}l)(A) = GL\big(A[\Delta]\big)$$

is excisive.

The same is valid (see Weibel [25, Excision Theorem 2.2]) for the homotopy $K$-theory simplicial functor

$$A \in \mathfrak{R} \mapsto Sing_*(\mathscr{K})(A)$$

if $\mathfrak{F}$ consists of all surjective homomorphisms.

Let $\alpha$ denote a distinguished square in $\mathfrak{R}$

$$\begin{array}{ccc} D & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & C \end{array}$$

and denote the pushout of the diagram

$$\begin{array}{ccc} rC & \longrightarrow & rA \\ \downarrow & & \\ rB & & \end{array}$$

by $P(\alpha)$. Notice that the obtained diagram is homotopy pushout. There is a natural map $P(\alpha) \to rD$, and both objects are cofibrant. In the case of the degenerate square this map has to be understood as the map from the initial object $\emptyset$ to $r0$.

We can localize $U\mathfrak{R}$ (respectively $U\mathfrak{R}_\bullet$) at the family of maps

$$J = \{P(\alpha) \to rD \mid \alpha \text{ is a distinguished square}\}$$

(respectively $J = \{P(\alpha)_+ \to (rD)_+\}_\alpha$). The corresponding $J$-localization will be denoted by $U\mathfrak{R}_J$ (respectively $U\mathfrak{R}_{J,\bullet}$). The weak equivalences (trivial cofibrations) of $U\mathfrak{R}_J$ will be referred to as $J$-weak equivalences ($J$-trivial cofibrations).

It follows that the square "$r(\alpha)$"

$$\begin{array}{ccc} rC & \longrightarrow & rA \\ \downarrow & & \downarrow \\ rB & \longrightarrow & rD \end{array}$$

with $\alpha$ a distinguished square is a homotopy pushout square in $U\mathfrak{R}_J$.

The proof of the next lemma is straightforward.

**Lemma 4.2.** *For any two rings $A, B \in \mathfrak{R}$, the natural map*

$$rA \sqcup rB \to r(A \times B)$$

*is a $J$-weak equivalence. Therefore the simplicial set $\mathcal{X}(A) \times \mathcal{X}(B)$ is weakly equivalent to the simplicial set $\mathcal{X}(A \times B)$ for any $J$-local object $\mathcal{X}$. In particular, the natural map*

$$rA \sqcup pt = rA \sqcup r0 \to rA$$

*is a $J$-weak equivalence.*

**Lemma 4.3.** *A simplicial functor $\mathcal{X}$ in $U\mathfrak{R}$ (respectively $U\mathfrak{R}_\bullet$) is $J$-local if and only if it is fibrant and excisive.*

**Proof.** Straightforward. □

Now we define the mapping cylinder $\mathrm{cyl}(f)$ of a map $f : A \to B$ between cofibrant objects in a simplicial model category $\mathscr{M}$. Let $A \otimes \Delta^1$ denote the standard cylinder object for $A$. One has a commutative diagram

$$\begin{array}{ccc} A \sqcup A & \xrightarrow{\nabla} & A \\ {\scriptstyle i=i_0 \sqcup i_1}\downarrow & \nearrow_{\sigma} & \\ A \otimes \Delta^1 & & \end{array}$$

in which $i$ is a cofibration and $\sigma$ is a weak equivalence [13, 9.5.14]. Each $i_\varepsilon$ must be a trivial cofibration.

Form the pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle i_{0*}} \\ A \otimes \Delta^1 & \xrightarrow{f_*} & \mathrm{Cyl}(f). \end{array}$$

Then $(f\sigma) \circ i_0 = f$, and so there is a unique map $q : \mathrm{Cyl}(f) \to B$ such that $qf_* = f\sigma$ and $qi_{0*} = 1_B$. Put $\mathrm{cyl}(f) = f_* i_1$; then $f = q \circ \mathrm{cyl}(f)$.

Since the objects $A, B, A \otimes \Delta^1$ are cofibrant in $\mathscr{M}$, it follows from [11, II.8.1] that $\mathrm{Cyl}(f)$ is a cofibrant object. Observe also that $q$ is a weak equivalence.

The map $\mathrm{cyl}(f)$ is a cofibration, since the diagram

$$\begin{array}{ccc} A \sqcup A & \xrightarrow{f \sqcup 1_A} & B \sqcup A \\ {\scriptstyle i_0 \sqcup i_1}\downarrow & & \downarrow{\scriptstyle i_{0*} \sqcup \mathrm{cyl}(f)} \\ A \otimes \Delta^1 & \xrightarrow{f_*} & \mathrm{Cyl}(f). \end{array}$$

is a pushout.

Given a distinguished square $\alpha$ let $P(\alpha) \to D_\alpha$ denote the cofibration $\mathrm{cyl}(P(\alpha) \to rD)$. We shall consider the following set of maps

$$\Lambda(J) = \left\{ P(\alpha) \times \Delta^n \coprod_{P(\alpha) \times \partial\Delta^n} D_\alpha \times \partial\Delta^n \to D_\alpha \times \Delta^n \right\}_{n \geqslant 0, \alpha}.$$

In the pointed case one considers the set

$$\Lambda(J) = \left\{ P(\alpha)_+ \wedge \Delta^n_+ \coprod_{P(\alpha)_+ \wedge \partial\Delta^n_+} D_\alpha \wedge \partial\Delta^n_+ \to D_\alpha \wedge \Delta^n_+ \right\}_{n \geqslant 0, \alpha}$$

with $P(\alpha)_+ \to D_\alpha$ the cofibration $\mathrm{cyl}(P(\alpha)_+ \to (rD)_+)$. It follows from [13, 9.3.7(3)] that each map of $\Lambda(J)$ is a $J$-trivial cofibration. Let $\mathscr{C}$ be a generating set of trivial cofibrations in $U\mathfrak{R}$ and put $\Lambda := \Lambda(J) \cup \mathscr{C}$.

**Proposition 4.4.** *A simplicial functor $\mathscr{X}$ in $U\mathfrak{R}$ (respectively $U\mathfrak{R}_\bullet$) is $J$-local if and only if the map $\mathscr{X} \to *$ has the right lifting property with respect to every element of $\Lambda$.*

**Proof.** The proof is similar to [13, 4.2.4]. Use as well [13, 9.4.7]. □

Observe that if an object $\mathscr{X} \in U\mathfrak{R}$ has the right lifting property with respect to every element of $\Lambda(J)$ then it is excisive (again use [13, 9.4.7]).

### 4.3. The model category $U\mathfrak{R}_{I,J}$

In this paragraph we shall construct the model category $U\mathfrak{R}_{I,J}$. It is the localization of $U\mathfrak{R}$ with respect to the maps from $I \cup J$. We start with definitions.

**Definition.** Let $\mathfrak{R}$ be an admissible category of rings and let $\mathfrak{F}$ be a family of fibrations. A simplicial functor $\mathscr{X} \in U\mathfrak{R}$ is called *quasi-fibrant* with respect to $\mathfrak{F}$ if it is homotopy invariant and excisive.

Let $J$ be as above. The model category $U\mathfrak{R}_{I,J}$ is, by definition, the Bousfield localization of $U\mathfrak{R}$ with respect to $I \cup J$. The homotopy category of $U\mathfrak{R}_{I,J}$ will be denoted by $\mathrm{Ho}_{I,J}(\mathfrak{R})$. The weak equivalences (trivial cofibrations) of $U\mathfrak{R}_{I,J}$ will be referred to as $(I, J)$-weak equivalences ($(I, J)$-trivial cofibrations).

An $(I, J)$*-resolution functor* is a pair $(Ex_{I,J}, \theta)$ consisting of a functor $Ex_{I,J} : U\mathfrak{R} \to U\mathfrak{R}$ and a natural transformation $\theta : 1 \to Ex_{I,J}$ such that for any $\mathscr{X}$ the object $Ex_{I,J}(\mathscr{X})$ is quasi-fibrant and the morphism $\mathscr{X} \to Ex_{I,J}(\mathscr{X})$ is an $(I, J)$-trivial cofibration.

**Lemma 4.5.** *A simplicial functor $\mathscr{X} \in U\mathfrak{R}$ is $(I, J)$-local if and only if it is fibrant, homotopy invariant and excisive.*

**Proof.** Straightforward. □

*An "explicit" $(I, J)$-resolution functor*. The purpose of this paragraph is to construct an explicit $(I, J)$-resolution functor. It is constructed inductively as follows (cf. Morel–Voevodsky [16, p. 92]).

Given $\mathscr{X} \in U\mathfrak{R}$, let $\Lambda(J)$ be the set of $J$-trivial cofibrations defined above and let $S$ be the set of all commutative diagrams of the following form

$$\begin{array}{ccc} P(\alpha) \times \Delta^n \coprod_{P(\alpha)\times\partial\Delta^n} D_\alpha \times \partial\Delta^n & \longrightarrow & Sing_*(\mathscr{X}) \\ \downarrow & & \downarrow \\ D_\alpha \times \Delta^n & \longrightarrow & *, \end{array}$$

where $\alpha$ runs over distinguished squares. Construct a pushout square

$$\begin{array}{ccc} \coprod_S [P(\alpha) \times \Delta^n \coprod_{P(\alpha)\times\partial\Delta^n} D_\alpha \times \partial\Delta^n] & \longrightarrow & Sing_*(\mathscr{X}) \\ \downarrow & & \downarrow{\scriptstyle \xi_0} \\ \coprod_S D_\alpha \times \Delta^n & \longrightarrow & \mathscr{X}_1. \end{array}$$

Since the left arrow is a $J$-trivial cofibration, then so is $\xi_0$. We get a sequence of cofibrations

$$\mathscr{X} \xrightarrow{\chi_0} Sing_*(\mathscr{X}) \xrightarrow{\xi_0} \mathscr{X}_1 \xrightarrow{\chi_1} Sing_*(\mathscr{X}_1)$$

with $\chi_0, \chi_1$ $I$-trivial cofibrations. Repeating this procedure, one obtains an infinite sequence of alternating $I$-trivial cofibrations and $J$-trivial cofibrations respectively,

$$\cdots Sing_*(\mathscr{X}_n) \xrightarrow{\xi_n} \mathscr{X}_{n+1} \xrightarrow{\chi_{n+1}} Sing_*(\mathscr{X}_{n+1}) \cdots \tag{4}$$

**Proposition 4.6.** *Let $Ex_{I,J}(\mathscr{X})$ denote a colimit of* (4) *and let $\theta_{\mathscr{X}} : \mathscr{X} \to Ex_{I,J}(\mathscr{X})$ be the natural inclusion which is functorial in $\mathscr{X}$. Then the pair $(Ex_{I,J}, \theta)$ yields an $(I, J)$-resolution functor.*

**Proof.** The map $\theta_{\mathscr{X}}$ is a $(I, J)$-trivial cofibration by [13, 17.9.1]. $Ex_{I,J}(\mathscr{X})$ is plainly homotopy invariant. To show that it is excisive, it is enough to check that the map $Ex_{I,J}(\mathscr{X}) \to *$ has the right lifting property with respect to all maps from $\Lambda(J)$. For this it suffices to observe that both domains and codomains of maps in $\Lambda(J)$ commute with a colimit of (4). □

An $(I, J)$-resolution functor $Ex_{I,J}(\mathscr{X})$ with $\mathscr{X} \in U\Re_\bullet$ is constructed in a similar way. The following computes a fibrant replacement functor in $U\Re_{I,J}$ (respectively in $U\Re_{I,J,\bullet}$).

**Proposition 4.7.** *Let $\vartheta_{\mathscr{X}} : \mathscr{X} \to R(\mathscr{X})$ denote a fibrant replacement functor in $U\Re$ (respectively in $U\Re_\bullet$). Then the map $\vartheta_{Ex_{I,J}(\mathscr{X})} \circ \theta_{\mathscr{X}} : \mathscr{X} \mapsto R(Ex_{I,J}(\mathscr{X}))$ yields a fibrant replacement functor in $U\Re_{I,J}$ (respectively in $U\Re_{I,J,\bullet}$). That is the object $R(Ex_{I,J}(\mathscr{X}))$ is $(I, J)$-local and the composition*

$$\mathscr{X} \to Ex_{I,J}(\mathscr{X}) \to R\big(Ex_{I,J}(\mathscr{X})\big)$$

*is an $(I, J)$-trivial cofibration.*

**Proof.** $R(Ex_{I,J}(\mathscr{X}))$ is plainly homotopy invariant. Given a distinguished square $\alpha$, the square of simplicial sets

$$\begin{array}{ccc} Ex_{I,J}(\mathscr{X})(D) & \longrightarrow & Ex_{I,J}(\mathscr{X})(A) \\ \downarrow & & \downarrow \\ Ex_{I,J}(\mathscr{X})(B) & \longrightarrow & Ex_{I,J}(\mathscr{X})(C) \end{array}$$

is a homotopy pullback square by Proposition 4.6. This square is weakly equivalent to the square

$$\begin{array}{ccc} R(Ex_{I,J}(\mathscr{X}))(D) & \longrightarrow & R(Ex_{I,J}(\mathscr{X}))(A) \\ \downarrow & & \downarrow \\ R(Ex_{I,J}(\mathscr{X}))(B) & \longrightarrow & R(Ex_{I,J}(\mathscr{X}))(C), \end{array}$$

and hence the latter square is a homotopy pullback square by [13, 13.3.13]. Lemma 4.5 completes the proof. □

If we consider $rA$ as a pointed (at zero) simplicial functor then the natural map $rA_+ \to rA$ is a $J$-weak equivalence in $U\Re_\bullet$ (see Lemma 4.2). The proof of the following statement is like that of Theorem 3.9.

**Proposition 4.8.** *The natural map*

$$\pi_0\big(Ex_{I,J}(\mathscr{X})(A)\big) \to \mathrm{Hom}_{\mathrm{Ho}_{I,J}(\Re)}(rA, \mathscr{X})$$

*is a bijection for any $A \in \Re$ and $\mathscr{X} \in U\Re$. Moreover, if $\mathscr{X}$ is pointed, then for any integer $n \geqslant 0$ and any $A \in \Re$ the obvious map*

$$\pi_n\big(Ex_{I,J}(\mathscr{X})(A)\big) \to \mathrm{Hom}_{\mathrm{Ho}_{I,J,\bullet}(\Re)}\big(rA \wedge S^n, \mathscr{X}\big)$$

*is a bijection, where $rA$ is supposed to be pointed at zero.*

**Corollary 4.9.** (1) *Suppose $\mathfrak{F}$ consists of the GL-fibrations. Then for any ring $A \in \Re$*

$$KV_n(A) = \mathrm{Hom}_{\mathrm{Ho}_{I,J,\bullet}(\Re)}\big(rA \wedge S^{n-1}, \mathscr{G}l\big), \quad n \geqslant 1.$$

(2) *Suppose $\mathfrak{F}$ consists of all surjective homomorphisms. Then for any ring $A \in \Re$*

$$KH_n(A) = \mathrm{Hom}_{\mathrm{Ho}_{I,J,\bullet}(\Re)}\big(rA \wedge S^n, \mathscr{K}\big), \quad n \geqslant 0.$$

*4.4. The Puppe sequence*

Throughout this paragraph the family of fibrations $\mathfrak{F}$ is supposed to be saturated. Let $g : B \to C$ be a ring homomorphism in $\Re$. Consider the pullback of $g$ along the map $\partial_x^1 : EC = xC[x] \to C$,

$$\begin{array}{ccc} P(g) & \xrightarrow{g'} & EC \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle \partial_x^1} \\ B & \xrightarrow{g} & C. \end{array}$$

Given a pointed quasi-fibrant simplicial functor $\mathscr{X}$, the following lemma computes the homotopy type for *fibre*$(\mathscr{X}(B) \to \mathscr{X}(C))$.

**Lemma 4.10.** *If $\mathfrak{F}$ is saturated and $\mathscr{X}$ is a pointed quasi-fibrant simplicial functor, then the square of pointed simplicial sets*

$$\begin{array}{ccc} \mathscr{X}(P(g)) & \longrightarrow & \mathscr{X}(EC) \simeq * \\ \downarrow & & \downarrow \\ \mathscr{X}(B) & \longrightarrow & \mathscr{X}(C) \end{array}$$

*is homotopy pullback. In particular, it determines an exact sequence of pointed sets at the middle point of the diagram*

$$[\mathscr{X}_0]\big(P(g)\big) \xrightarrow{g_{1*}} [\mathscr{X}_0](B) \xrightarrow{g_*} [\mathscr{X}_0](C).$$

**Proof.** Easy. □

**Corollary 4.11.** *If $\mathfrak{F}$ is saturated, $\mathscr{X}$ is a pointed quasi-fibrant simplicial functor and $\Omega A = (x^2-x)A[x] = \mathrm{Ker}(EA \xrightarrow{\partial_x^1} A)$, $A \in \Re$, then $|\mathscr{X}(\Omega A)|$ has the homotopy type of $\Omega|\mathscr{X}(A)|$. In particular, $\pi_n(\mathscr{X}(A)) = \pi_0(\mathscr{X}(\Omega^n A))$ for any $n \geqslant 0$.*

Clearly one can iterate the construction of $P(g)$ to get the diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & P(g_3) & \overset{g_4}{\Longrightarrow} & P(g_2) & \overset{g_2'}{\longrightarrow} & EP(g) & & \\
 & {\scriptstyle g_3'}\downarrow & & {\scriptstyle g_3}\Downarrow & & \downarrow{\scriptstyle g_3''} & & \\
 & EP(g_1) & \overset{g_4''}{\longrightarrow} & P(g_1) & \overset{g_2}{\Longrightarrow} & P(g) & \overset{g'}{\longrightarrow} & EC \\
 & & & {\scriptstyle g_1'}\downarrow & & {\scriptstyle g_1}\Downarrow & & \downarrow{\scriptstyle g_1''} \\
 & & & EB & \overset{g_2''}{\longrightarrow} & B & \overset{g}{\Longrightarrow} & C.
\end{array}$$

The latter diagram determines the *Puppe sequence of g*

$$\cdots \to P(g_n) \xrightarrow{g_{n+1}} P(g_{n-1}) \xrightarrow{g_n} \cdots \to P(g) \xrightarrow{g_1} B \xrightarrow{g} C. \tag{5}$$

If we factor $g$ as $fi$ with $i$ a quasi-isomorphism and $f$ a fibration, then using [11, II.9.10] it is easy to show that the Puppe sequence of $g$ is quasi-isomorphic to the Puppe sequence of $f$.

**Proposition 4.12.** *(Cf. Gersten [10].) If $\mathfrak{F}$ is saturated and $\mathscr{X}$ is a pointed quasi-fibrant simplicial functor, then* (5) *gives rise to an exact sequence of pointed sets*

$$\begin{aligned}\cdots \to [\mathscr{X}_0]\big(P(g_n)\big) \to [\mathscr{X}_0]\big(P(g_{n-1})\big) \to \cdots \\ \to [\mathscr{X}_0]\big(P(g)\big) \to [\mathscr{X}_0](B) \to [\mathscr{X}_0](C).\end{aligned}$$

Gersten [10, 2.9] constructs the same long exact sequence for group valued left exact functors.

*4.5. Homology theories*

**Definition.** (Cf. Gersten [9].) Let $\Re$ be an admissible category of rings and let $\mathfrak{F}$ be a family of fibrations. A *homology theory* $H_*$ on $\Re$ relative to $\mathfrak{F}$ consists of

(1) a family $\{H_n, n \geqslant 0\}$ of functors $H_n : \Re \to \mathit{Sets}_\bullet$ with $H_{n\geqslant 1}(A)$ a group,

(2) for every $\mathfrak{F}$-fibre sequence

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

with $g \in \mathfrak{F}$, morphisms

$$H_{n+1}(C) \xrightarrow{\partial_{n+1}(g)} H_n(A), \quad n \geqslant 0,$$

(we shall often write simply $\partial_{n+1}$ if $g$ is understood) satisfying axioms

(Ax 1) $H_n(u) = H_n(v)$ for any homotopic homomorphisms $u, v$ and any $n \geqslant 0$,

(Ax 2) the morphism $\partial_{n+1}(g)$ of (2) is natural in the sense that given a commutative diagram in $\Re$ with rows $\mathfrak{F}$-fibre sequences

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array}$$

and with $g, g' \in \mathfrak{F}$, then the diagram

$$\begin{array}{ccc} H_{n+1}(C) & \xrightarrow{\partial_{n+1}(g)} & H_n(A) \\ {\scriptstyle H_{n+1}(c)}\downarrow & & \downarrow{\scriptstyle H_n(a)} \\ H_{n+1}(C') & \xrightarrow{\partial_{n+1}(g')} & H_n(A') \end{array}$$

is commutative for $n \geqslant 0$;

(Ax 3) if $A \xrightarrow{f} B \xrightarrow{g} C$ is an $\mathfrak{F}$-fibre sequence with $g \in \mathfrak{F}$, then we have a long exact sequence of pointed sets

$$\cdots \to H_{n+1}(A) \xrightarrow{H_{n+1}(f)} H_{n+1}(B) \xrightarrow{H_{n+1}(g)} H_{n+1}(C)$$
$$\xrightarrow{\partial_{n+1}(g)} H_n(A) \to \cdots \to H_0(B) \to H_0(C)$$

in the sense that the kernel (defined as the preimage of the basepoint) is equal to the image at each spot.

We are now in a position to prove the following

**Theorem 4.13.** *To any pointed simplicial functor $\mathcal{X}$ on $\Re$ and any family of fibrations $\mathfrak{F}$ one naturally associates a homology theory. It is defined as*

$$H_n(A) := \pi_n\big(Ex_{I,J}(\mathcal{X})(A)\big) = \mathrm{Ho}_{I,J,\bullet}\big(S^n \wedge rA, \mathcal{X}\big)$$

*for any $A \in \Re$ and $n \geqslant 0$. Moreover, if $\mathfrak{F}$ is saturated then $H_n(A) = H_0(\Omega^n A)$. We also say that this homology theory is* represented by $\mathcal{X}$.

**Proof.** It follows from Proposition 4.6 that $Ex_{I,J}(\mathscr{X})$ is a quasi-fibrant object. Now our assertion easily follows from Proposition 4.8 and Corollary 4.11. □

It follows from Corollary 4.9 that the homology theories associated to the functors $\mathscr{G}l$ and $\mathscr{K}$ are the $KV$- and $KH$-theories respectively.

## 5. Derived categories of rings

In this section we introduce and study the left derived category $D^-(\Re, \mathfrak{F})$ associated to any family of fibrations $\mathfrak{F}$ on $\Re$. It is obtained from the homotopy category $\mathscr{H}\Re$ by inverting the quasi-isomorphisms introduced below. For this we should first define a structure which is a bit weaker than the model category structure on $\Re$ with respect to fibrations and quasi-isomorphisms. Following Brown [3] this structure is called the *category of fibrant objects*. It shares many properties with model categories. If $\mathfrak{F}$ is saturated (which is always the case in practice), it follows from Theorem 5.6 that $D^-(\Re, \mathfrak{F})$ is naturally left triangulated. The category of left triangles meets the axioms which are versions for the axioms of a triangulated category. The left triangulated structure as such is a tool for producing homology theories on rings. The special case when $\mathfrak{F}$ consists of the surjective homomorphisms will be discussed in Section 7.

### 5.1. Categories of fibrant objects

**Definition.** I. Let $\mathscr{A}$ be a category with finite products and a final object $e$. Assume that $\mathscr{A}$ has two distinguished classes of maps, called *weak equivalences* and *fibrations*. A map is called a *trivial fibration* if it is both a weak equivalence and a fibration. We define a *path space* for an object $B$ to be an object $B^I$ together with maps

$$B \xrightarrow{s} B^I \xrightarrow{(d_0,d_1)} B \times B,$$

where $s$ is a weak equivalence, $(d_0, d_1)$ is a fibration, and the composite is the diagonal map.

Following Brown [3], we call $\mathscr{A}$ a *category of fibrant objects* if the following axioms are satisfied.

(A) Let $f$ and $g$ be maps such that $gf$ is defined. If two of $f$, $g$, $gf$ are weak equivalences then so is the third. Any isomorphism is a weak equivalence.
(B) The composite of two fibrations is a fibration. Any isomorphism is a fibration.
(C) Given a diagram

$$A \xrightarrow{u} C \xleftarrow{v} B,$$

with $v$ a fibration (respectively a trivial fibration), the pullback $A \times_C B$ exists and the map $A \times_C B \to A$ is a fibration (respectively a trivial fibration).
(D) For any object $B$ in $\mathscr{A}$ there exists at least one path space $B^I$ (not necessarily functorial in $B$).
(E) For any object $B$ the map $B \to e$ is a fibration.

Note that if the final object $e$ is also initial, then the opposite category $\mathscr{A}^{\text{op}}$ is a saturated Waldhausen category (for precise definitions see [20,23]). The "gluing axiom" follows from [11, II.9.10]. If $\mathscr{A}$ is small the associated Waldhausen $K$-theory space of $\mathscr{A}^{\mathrm{op}}$ (see Waldhausen [23]) will be denoted by $K\mathscr{A}$.

**Definition.** Let $\Re$ be an admissible category of rings and let $\mathfrak{F}$ be a family of fibrations. A homomorphism $A \to B$ in $\Re$ is said to be a *$\mathfrak{F}$-quasi-isomorphism* or just a *quasi-isomorphism* if the map $rB \to rA$ is an $(I, J)$-weak equivalence. This is equivalent to saying that the induced map $H_*(A) \to H_*(B)$ is an isomorphism for every representable homology theory $H_*$.

**Proposition 5.1.** *Let $\Re$ be an admissible category of rings and let $\mathfrak{F}$ be a family of fibrations. Then it enjoys the axioms* (A)–(E) *for a category of fibrant objects, where fibrations are the elements of $\mathfrak{F}$ and weak equivalences are quasi-isomorphisms.*

**Proof.** Clearly, the axioms (A), (B), (E) are satisfied. The axiom (D) is a consequence of (Ax 4). Indeed, let $B$ be a ring in $\Re$ and consider homomorphisms

$$B \xrightarrow{i} B[x] \xrightarrow{(\partial_x^0, \partial_x^1)} B \times B,$$

where $i$ is an $I$-weak equivalence and the composite is the diagonal. By (Ax 4), $(\partial_x^0, \partial_x^1)$ can be factored $(\partial_x^0, \partial_x^1) = (d_0, d_1) \circ s'$, where $s'$ is an $I$-weak equivalence and $(d_0, d_1)$ is a fibration. Put $s := s'i$; then the diagonal can be factored $diag = (d_0, d_1) \circ s$, hence the axiom (D).

A pullback of a fibration is, by definition, a fibration. It remains to check that a pullback of a trivial fibration is a trivial fibration.

Suppose the square $\alpha$

$$\begin{array}{ccc} D & \xrightarrow{\rho} & A \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{g} & C \end{array}$$

is distinguished in $\Re$ and $f$ is a trivial fibration. We must show that $\sigma$ is a trivial fibration.

Since the morphism $r(f)$ is an $(I, J)$-trivial cofibration, then so is the morphism $rB \to P(\alpha) = rA \bigsqcup_{rC} rB$ by [13, 7.2.12]. By definition, the morphism $P(\alpha) \to rD$ is a $J$-weak equivalence, whence our assertion follows. □

**Definition.** Let $\Re$ be an admissible category of rings and let $\mathfrak{F}$ be a family of fibrations. The *left derived category $D^-(\Re, \mathfrak{F})$* of $\Re$ with respect to $\mathfrak{F}$ is the category obtained from $\Re$ by inverting quasi-isomorphisms.

**Proposition 5.2.** *The family of quasi-isomorphisms in the category $\mathscr{H}\Re$ admits a calculus of right fractions. The derived category $D^-(\Re, \mathfrak{F})$ is obtained from $\mathscr{H}\Re$ by inverting the quasi-isomorphisms.*

**Proof.** Let $C$ be a ring in $\Re$. By the proof of Proposition 5.1 one can choose a path space $C^I$

$$C \xrightarrow{s} C^I \xrightarrow{(d_0, d_1)} C \times C$$

with $s$ an $I$-weak equivalence. Consider a diagram

$$B \xrightarrow{f} C \xleftarrow{t} A \tag{6}$$

with $t$ a quasi-isomorphism. Let $D$ be the limit of the diagram of the solid arrows

$$\begin{array}{ccccccc} & & & D & & & \\ & \swarrow^{t'} & & \downarrow{\scriptstyle h} & & \searrow^{f'} & \\ B & & & C^I & & & A \\ & \searrow^{f} & \swarrow^{d_0} & & \searrow^{d_1} & \swarrow^{t} & \\ & C & & & & C. & \end{array}$$

It follows from [3, Lemma 3] that $t'$ is a trivial fibration. By Corollary 3.11, $tf' = ft'$ in $\mathscr{H}\mathfrak{R}$. Thus (6) fits into a commutative diagram in $\mathscr{H}\mathfrak{R}$,

$$\begin{array}{ccc} D & \xrightarrow{f'} & A \\ {\scriptstyle t'}\downarrow & & \downarrow{\scriptstyle t} \\ B & \xrightarrow{f} & C \end{array}$$

with $t'$ a quasi-isomorphism.

Given $f, g : A \rightrightarrows B$, suppose there is a quasi-isomorphism $t : B \to C$ such that $tf = tg$ in $\mathscr{H}\mathfrak{R}$. By [3, Propositions 1–2] there is a quasi-isomorphism $t' : A' \to A$ such that $ft'$ is homotopic to $gt'$ by a homotopy $h : A' \to C^I$. It follows from Corollary 3.11 that $ft' = gt'$ in $\mathscr{H}\mathfrak{R}$, and hence $\mathscr{H}\mathfrak{R}$ admits a calculus of right fractions. □

**Remark.** There is a generalization, due to Cisinski [4], for the notion of a category of fibrant objects: the "catégorie dérivable à gauche". For such a category Cisinski describes (similar to Brown [3]) its derived category. We can also conform his construction to admissible categories of rings, but we shall leave this to the interested reader.

**Question.** Let $X$ be an object of $D^-(\mathfrak{R}, \mathfrak{F})$. Is it true that the functor

$$[X, -] = \operatorname{Hom}_{D^-(\mathfrak{R}, \mathfrak{F})}(X, -)$$

is represented by $rX$? It is equivalent to the problem of whether the functor $D^-(\mathfrak{R}, \mathfrak{F}) \to \mathrm{Ho}_{I,J,\bullet}(\mathfrak{R})$, induced by $A \mapsto rA$, is fully faithful.

By Brown [3, Theorem 1] there is a functor $\Omega' : D^-(\mathfrak{R}, \mathfrak{F}) \to D^-(\mathfrak{R}, \mathfrak{F})$ such that for any ring $B$ and any path space $B^I$, $\Omega' B$ can be canonically identified with the fibre of $(d_0, d_1) : B^I \to B \times B$. Furthermore, $\Omega' B$ has a natural group structure. Let $p : E \to B$ be a fibration with fibre $F$. By [3, Proposition 3] there is a natural map $a : F \times \Omega' B \to F$ in $D^-(\mathfrak{R}, \mathfrak{F})$ which defines a right action of the group $\Omega' B$ on $F$.

Following Quillen [17], we now define a *fibration sequence* to be a diagram $F \to E \to B$ in $D^-(\Re, \mathfrak{F})$ together with an action $m : F \times \Omega' B \to F$ in $D^-(\Re, \mathfrak{F})$ which are isomorphic to the diagram and action obtained from a fibration in $\Re$. Let $A \in \Re$; the map $m_* : [A, F] \times [A, \Omega' B] \to [A, F]$ will be denoted by $(\alpha, \lambda) \mapsto \alpha \cdot \lambda$.

**Theorem 5.3.** *(Quillen [17], Brown [3]) Given a fibration sequence*

$$F \xrightarrow{i} E \xrightarrow{p} B, \qquad F \times \Omega' B \to F,$$

*there is an exact sequence in* $D^-(\Re, \mathfrak{F})$

$$\cdots \to \Omega' E \to \Omega' B \to F \to E \to B,$$

*where exactness is interpreted as in* [17, p. I.3.8]. *The induced sequence*

$$\cdots \to [A, \Omega' E] \xrightarrow{(\Omega' p)_*} [A, \Omega' B] \xrightarrow{\partial_*} [A, F] \xrightarrow{i_*} [A, E] \xrightarrow{p_*} [A, B]$$

*meets the following properties*:

(1) $(p_*)^{-1}(0) = \operatorname{Im} i_*$;
(2) $i_* \partial_* = 0$ *and* $i_* \alpha_1 = i_* \alpha_2 \Leftrightarrow \alpha_2 = \alpha_1 \cdot \lambda$ *for some* $\lambda \in [A, \Omega' B]$;
(3) $\partial_* (\Omega' p)_* = 0$ *and* $\partial_* \lambda_1 = \partial_* \lambda_2 \Leftrightarrow \lambda_2 = (\Omega' p)_* \mu \lambda_1$ *for some* $\mu \in [A, \Omega' E]$ *under the product in the group* $[A, \Omega' B]$;
(4) *the sequence of group homomorphisms from* $[A, \Omega' E]$ *to the left is exact in the usual sense.*

**Corollary 5.4.** *Let* $\mathfrak{F}$ *be a saturated family of fibrations. Then* $\Omega A$ *is canonically isomorphic to* $\Omega' A$ *for every* $A \in \Re$. *In particular,* $\Omega A$ *is a group object in* $D^-(\Re, \mathfrak{F})$.

**Proof.** The proof is straightforward, using the preceding theorem and the exact sequence $\Omega A \to EA \to A$ (recall that $EA$ is contractible). □

Let $K(\Re, \mathfrak{F})$ denote the Waldhausen $K$-theory space associated to a family of fibrations $\mathfrak{F}$. Recall from [20, p. 261] that the group $K_0(\Re, \mathfrak{F})$ is abelian and it is the free group on generators $[A]$ as $A$ runs over the objects of $\Re$, modulo the two relations

- ◇ $[A] = [B]$ if there is a quasi-isomorphism $A \xrightarrow{\sim} B$.
- ◇ $[E] = [F] + [B]$ for all $\mathfrak{F}$-fibre sequences $F \rightarrowtail E \twoheadrightarrow B$.

The *Grothendieck group* $K_0(D^-(\Re, \mathfrak{F}))$ of $D^-(\Re, \mathfrak{F})$ is the free group on generators $(A)$ as $(A)$ runs over the iso-classes of objects in $D^-(\Re, \mathfrak{F})$, modulo the relation: $(E) = (F) + (B)$ for all fibration sequences $F \to E \to B$ in $D^-(\Re, \mathfrak{F})$.

By [3, §4, Proposition 4] there is a fibration sequence $\Omega' A \to 0 \to A$ for any $A \in \Re$, hence $(\Omega' A) = -(A)$ in $K_0(D^-(\Re, \mathfrak{F}))$. It follows that $(B) - (A) = (B \times \Omega' A)$ and thus every element of $K_0(D^-(\Re, \mathfrak{F}))$ is the class $(A)$ of some $A$ in $\Re$. We leave to the reader to check that the natural map

$$K_0(\Re, \mathfrak{F}) \to K_0\big(D^-(\Re, \mathfrak{F})\big)$$

is an isomorphism of abelian groups.

### *5.2. The left triangulated structure*

Fix a saturated family of fibrations $\mathfrak{F}$ on $\Re$. In this paragraph we define and study abstract properties of left triangles in the derived category $D^-(\Re, \mathfrak{F})$.

The endofunctor

$$\Omega : \Re \to \Re, \quad A \mapsto \Omega A = \left(x^2 - x\right)A[x]$$

respects quasi-isomorphisms. Indeed, let $f : A \to B$ be a quasi-isomorphism. Consider the following commutative diagram:

$$\begin{array}{ccccc} \Omega A & \rightarrowtail & EA & \overset{\partial_x^1}{\twoheadrightarrow} & A \\ {\scriptstyle \Omega f}\downarrow & & {\scriptstyle E(f)}\downarrow & & \downarrow{\scriptstyle f} \\ \Omega B & \rightarrowtail & EB & \overset{\partial_x^1}{\twoheadrightarrow} & B. \end{array}$$

Since $EA$, $EB$ are isomorphic to zero in $D^-(\Re, \mathfrak{F})$, it follows that $E(f)$ is a quasi-isomorphism. Then $\Omega f$ is a quasi-isomorphism by [3, §4, Lemma 3]. Thus $\Omega$ can be regarded as an endofunctor of $D^-(\Re, \mathfrak{F})$.

Given a fibration $g : A \to B$ with fibre $F$, consider the commutative diagram as follows:

$$\begin{array}{ccccc} & & \Omega B & = & \Omega B \\ & & {\scriptstyle j}\downarrow & & \downarrow \\ F & \overset{i}{\rightarrowtail} & P(g) & \twoheadrightarrow & EB \\ \| & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle \partial_x^1} \\ F & \overset{\iota}{\rightarrowtail} & A & \overset{g}{\twoheadrightarrow} & B. \end{array}$$

Since $EB$ is isomorphic to zero in $D^-(\Re, \mathfrak{F})$, it follows from Theorem 5.3 that $i$ is a quasi-isomorphism. We deduce the sequence in $D^-(\Re, \mathfrak{F})$

$$\Omega B \xrightarrow{i^{-1} \circ j} F \xrightarrow{\iota} A \xrightarrow{g} B. \tag{7}$$

We shall refer to such sequences as *standard left triangles*. Any diagram in $D^-(\Re, \mathfrak{F})$ which is isomorphic to the latter sequence will be called a *left triangle*. One must be careful to note that $\Omega B' \to F' \to A' \to B'$ is isomorphic to a standard triangle (7) if and only if there is a commutative diagram

$$\begin{array}{ccccccc} \Omega B & \longrightarrow & F & \longrightarrow & A & \longrightarrow & B \\ {\scriptstyle \Omega b}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ \Omega B' & \longrightarrow & F' & \longrightarrow & A' & \longrightarrow & B' \end{array}$$

with $f, a, b$ isomorphisms in $D^-(\Re, \mathfrak{F})$.

It follows that the diagram

$$\Omega B \xrightarrow{j} P(g) \xrightarrow{g_1} A \xrightarrow{g} B$$

is a left triangle. If $g$ is not a fibration then $g$ is factored as $g = g'\ell$ with $g'$ a fibration and $\ell$ a quasi-isomorphism. We get a commutative diagram

$$\begin{array}{ccccccc} \Omega B & \longrightarrow & P(g) & \longrightarrow & A & \xrightarrow{g} & B \\ \| & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \ell} & & \| \\ \Omega B & \longrightarrow & P(g') & \longrightarrow & A' & \xrightarrow{g'} & B. \end{array}$$

The arrow $t$ is a quasi-isomorphism by [11, II.9.10]. Hence the upper sequence of the diagram is a left triangle. This also verifies that any map in $D^-(\mathfrak{R}, \mathfrak{F})$ fits into a left triangle.

For any ring $A$ the automorphism $\sigma = \sigma_A : \Omega A \to \Omega A$ takes a polynomial $a(x)$ to $a(1-x)$. Notice that $\sigma$ is functorial in $A$ and $\sigma^2 = 1$. Given a morphism $\alpha$ in $D^-(\mathfrak{R}, \mathfrak{F})$, by $-\Omega\alpha$ denote the morphism $\Omega\alpha \circ \sigma = \sigma \circ \Omega\alpha$. For any $n \geqslant 1$ the morphism $(-1)^n \Omega\alpha$ means $\sigma^n \Omega\alpha$. Now we want to check that for a standard left triangle

$$\Omega B \xrightarrow{i^{-1}\circ j} F \xrightarrow{\iota} A \xrightarrow{g} B$$

the sequence

$$\Omega A \xrightarrow{-\Omega g} \Omega B \xrightarrow{i^{-1}\circ j} F \xrightarrow{\iota} A$$

is a left triangle, too.

Consider the following diagram in $D^-(\mathfrak{R}, \mathfrak{F})$:

$$\begin{array}{ccccccc} \Omega A & \xrightarrow{-\Omega g} & \Omega B & \xrightarrow{i^{-1}\circ j} & F & \xrightarrow{\iota} & A \\ \| & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle i} & & \| \\ \Omega A & \xrightarrow{\kappa} & P(g_1) & \xrightarrow{g_2} & P(g) & \xrightarrow{g_1} & A, \end{array}$$

where $P(g_1) = P(g) \times_A EA$ and $\nu : \Omega B \to P(g_1)$ is the natural inclusion taking $b(x) \in \Omega B$ to $((0, b(x)), 0)$. Moreover, $\nu$ is a quasi-isomorphism. The homomorphism $\kappa$ takes $a(x) \in \Omega A$ to $((0,0), a(x)) \in P(g_1)$.

The right and the central squares are commutative. We want to check that so is the left square. For this it is enough to show that $\nu \circ \Omega g \circ \sigma$ is homotopic to $\kappa$. The desired (elementary) homotopy is given by the homomorphism

$$a(x) \in \Omega A \mapsto \big(\big(a(1-y), g\big(a(1-xy)\big)\big), a\big(x(1-y)\big)\big) \in P(g_1)[y].$$

It follows that the upper sequence is isomorphic to the lower which is a left triangle by above.

Since every left triangle

$$\Omega B' \xrightarrow{\gamma} F' \xrightarrow{\beta} A' \xrightarrow{\alpha} B'$$

is, by definition, isomorphic to a standard left triangle of the form

$$\Omega B \xrightarrow{i^{-1}\circ j} F \xrightarrow{\iota} A \xrightarrow{g} B,$$

we infer from above that the sequence

$$\Omega A' \xrightarrow{-\Omega\alpha} \Omega B' \xrightarrow{\gamma} F' \xrightarrow{\beta} A'$$

is a left triangle.

Let $g : A \to B$ be a homomorphism in $\Re$ and let $g$ be factored as $fi$ with $i : A \to A'$ a quasi-isomorphism and $f : A' \to B$ a fibration. Then the Puppe sequence of $g$

$$\cdots \to P(g_n) \xrightarrow{g_{n+1}} P(g_{n-1}) \xrightarrow{g_n} \cdots \to P(g) \xrightarrow{g_1} A \xrightarrow{g} B$$

is quasi-isomorphic to the Puppe sequence of $f$. We also infer from above that the latter is naturally isomorphic in $D^-(\Re, \mathfrak{F})$ to the sequence

$$\cdots \to \Omega F \xrightarrow{-\Omega\iota} \Omega A' \xrightarrow{-\Omega f} \Omega B \xrightarrow{j\circ i^{-1}} F \xrightarrow{\iota} A' \xrightarrow{f} B.$$

This isomorphism can be depicted as the following commutative diagram in $D^-(\Re, \mathfrak{F})$ with the vertical arrows quasi-isomorphisms (for simplicity we assume that $g$ is a fibration).

$$\begin{array}{ccccccccccccccc}
\cdots & \longrightarrow & \Omega^2 B & \xrightarrow{-\Omega(i^{-1}j)} & \Omega F & \xrightarrow{-\Omega\iota} & \Omega A & \xrightarrow{-\Omega g} & \Omega B & \xrightarrow{i^{-1}j} & F & \xrightarrow{\iota} & A & \xrightarrow{g} & B \\
 & & \downarrow{\scriptstyle \Omega\nu} & & \downarrow{\scriptstyle \Omega i} & & \| & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle i} & & \| & & \| \\
\cdots & \longrightarrow & \Omega P(g_1) & \xrightarrow{-\Omega g_2} & \Omega P(g) & \xrightarrow{-\Omega g_1} & \Omega A & \xrightarrow{\kappa} & P(g_1) & \xrightarrow{g_2} & P(g) & \xrightarrow{g_1} & A & \xrightarrow{g} & B \\
 & & \| & & \| & & \downarrow{\scriptstyle i_1} & & \| & & \| & & \| & & \| \\
\cdots & \longrightarrow & \Omega P(g_1) & \xrightarrow{-\Omega g_2} & \Omega P(g) & \xrightarrow{j_1} & P(g_2) & \xrightarrow{g_3} & P(g_1) & \xrightarrow{g_2} & P(g) & \xrightarrow{g_1} & A & \xrightarrow{g} & B \\
 & & \| & & \downarrow{\scriptstyle i_2} & & \| & & \| & & \| & & \| & & \| \\
\cdots & \longrightarrow & \Omega P(g_1) & \xrightarrow{j_2} & P(g_3) & \xrightarrow{g_4} & P(g_2) & \xrightarrow{g_3} & P(g_1) & \xrightarrow{g_2} & P(g) & \xrightarrow{g_1} & A & \xrightarrow{g} & B \\
 & & \downarrow{\scriptstyle i_3} & & \| & & \| & & \| & & \| & & \| & & \| \\
\cdots & \longrightarrow & P(g_4) & \xrightarrow{g_5} & P(g_3) & \xrightarrow{g_4} & P(g_2) & \xrightarrow{g_3} & P(g_1) & \xrightarrow{g_2} & P(g) & \xrightarrow{g_1} & A & \xrightarrow{g} & B \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

Now we are going to check the following property. Suppose we are given two left triangles $\Omega B \xrightarrow{\gamma} F \xrightarrow{\beta} A \xrightarrow{\alpha} B$ and $\Omega B' \xrightarrow{\gamma'} F' \xrightarrow{\beta'} A' \xrightarrow{\alpha'} B'$ and two morphisms $\varphi : A \to A'$ and $\psi : B \to B'$ in $D^-(\mathfrak{R}, \mathfrak{F})$ with $\psi\alpha = \alpha'\varphi$. We claim that there exists a morphism $\chi : F \to F'$ such that the triple $(\chi, \varphi, \psi)$ is a morphism from the first triangle to the second in the usual sense. It will follow from the construction that $\chi$ is an isomorphism whenever $\varphi$, $\psi$ are.

Without loss of generality we can assume that the first left triangle is the sequence

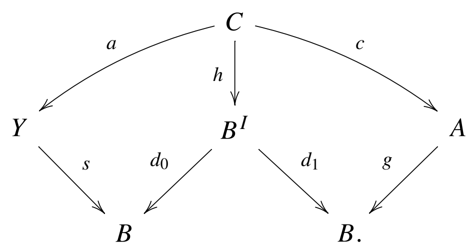
$$\Omega B \xrightarrow{j} P(g) \xrightarrow{g_1} A \xrightarrow{g} B$$

and the second one is

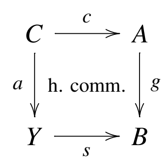
$$\Omega B' \xrightarrow{j'} P(g') \xrightarrow{g_1'} A' \xrightarrow{g'} B'.$$

Let $\psi = us^{-1}$ and $\varphi = vt^{-1}$.

Given two morphisms $g : A \to B$ and $s : Y \to B$ and a path space $B^I$ of $B$, let $C := Y \times_B B^I \times_B A$. One has a commutative diagram

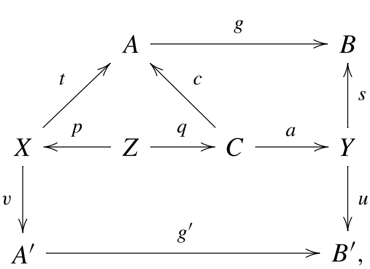



The square



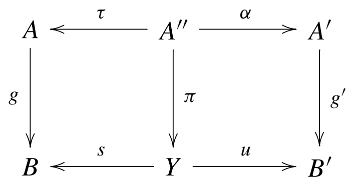

is homotopy commutative by the homotopy $h$. Moreover, $c$ is a quasi-isomorphism whenever $s$ is [3, §2, Lemma 3]. Maps from a ring $D$ to $C$ correspond bijectively to data $(\ell, p, k)$ where $\ell : D \to Y$ and $p : D \to A$ are maps and $k : D \to B^I$ is a homotopy $gp \sim s\ell : D \to B$.

We can now construct the following commutative diagram in $D^-(\mathfrak{R}, \mathfrak{F})$:



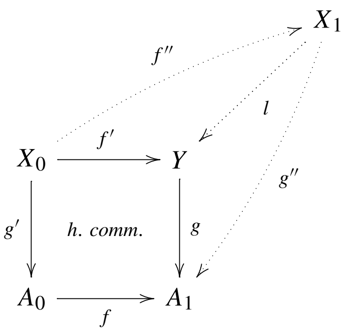
where $C := Y \times_B B^I \times_B A$ and $Z := X \times_A A^I \times_A C$. Let $B'^I$ be a path space of $B'$. It follows from [3, §2] that there exists a quasi-isomorphism $\ell : A'' \to Z$ such that $gcq\ell \sim saq\ell$ by a homotopy $k : A'' \to B^I$ and $g'vp\ell \sim uaq\ell$ by a homotopy $k' : A'' \to B'^I$.

We get the following commutative diagram in $\mathscr{H}\mathfrak{R}$:



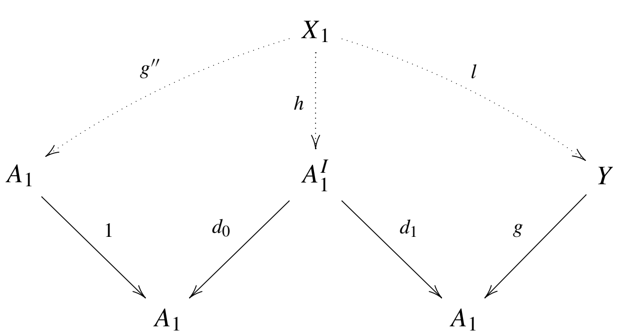

with $\tau = cq\ell$ a quasi-isomorphism, $\pi = aq\ell$, and $\alpha = vp\ell$.

**Lemma 5.5.** *Suppose we are given a homotopy commutative square with entries* $(X_0, Y, A_0, A_1)$




*and* $gf' \sim fg'$ *by a homotopy* $k : X_0 \to A_1^I$. *Then there is a* $X_1$ *and the dotted arrows* $l, f'', g''$ *such that the square with entries* $(X_0, X_1, A_0, A_1)$ *is genuinely commutative,* $lf'' = f'$, *and* $g'' \sim gl$ *by a homotopy* $h : X_1 \to A_1^I$. *Moreover,* $l$ *is a quasi-isomorphism.*

**Proof.** Let $X_1$ be the limit of the diagram of the solid arrows




The arrow $f''$ corresponds to the triple $(fg', f', k)$. Our assertion now follows immediately. □

By Lemma 5.5 one can construct a diagram in $\mathfrak{R}$

$$
\begin{array}{ccccc}
 & & & & W \\
 & & \nearrow^{\gamma} & \swarrow_{w} & \\
A'' & \xrightarrow{\pi} & Y & & \downarrow{\scriptstyle \delta} \\
{\scriptstyle \tau}\downarrow & \text{h. comm.} & \downarrow{\scriptstyle s} & \swarrow & \\
A & \xrightarrow[g]{} & B & &
\end{array}
$$

resulting the diagram

$$
\begin{array}{ccccc}
A & \xleftarrow{\tau} & A'' & \xrightarrow{\alpha} & A' \\
{\scriptstyle g}\downarrow & \text{comm.} & {\scriptstyle \gamma}\downarrow & \text{h. comm.} & \downarrow{\scriptstyle g'} \\
B & \xleftarrow{\delta} & W & \xrightarrow{uw} & B'.
\end{array}
$$

In a similar way, one can construct a diagram

$$
\begin{array}{ccccc}
 & & & & B'' \\
 & & \nearrow^{g''} & \swarrow_{\iota} & \\
A'' & \xrightarrow{\gamma} & W & & \downarrow{\scriptstyle z} \\
{\scriptstyle \alpha}\downarrow & \text{h. comm.} & \downarrow{\scriptstyle uw} & \swarrow & \\
A' & \xrightarrow[g']{} & B' & &
\end{array}
$$

resulting the commutative diagram in $\mathfrak{R}$

$$
\begin{array}{ccccc}
A & \xleftarrow{\tau} & A'' & \xrightarrow{\alpha} & A' \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g''} & & \downarrow{\scriptstyle g'} \\
B & \xleftarrow{\delta\iota} & B'' & \xrightarrow{z} & B'.
\end{array}
$$

Observe that $\varphi = \alpha\tau^{-1}$ and $\psi = z(\delta\iota)^{-1}$. We thus obtain the following commutative diagram in $\Re$:

$$\begin{array}{ccccccc}
\Omega B & \xrightarrow{j} & P(g) & \xrightarrow{g_1} & A & \xrightarrow{g} & B \\
\uparrow{\scriptstyle \Omega(\delta\iota)} & & \uparrow & & \uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \delta\iota} \\
\Omega B'' & \xrightarrow{j''} & P(g'') & \xrightarrow{g_1''} & A'' & \xrightarrow{g''} & B'' \\
\downarrow{\scriptstyle \Omega z} & & \downarrow & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle z} \\
\Omega B' & \xrightarrow{j'} & P(g') & \xrightarrow{g_1'} & A' & \xrightarrow{g'} & B'
\end{array}$$

verifying the desired property.

We are now in a position to formulate the main result of the paragraph.

**Theorem 5.6.** *Let $\mathfrak{F}$ be a saturated family of fibrations in $\Re$. Denote by $\mathscr{L}tr(\Re, \mathfrak{F})$ the category of left triangles having the usual set of morphisms from $\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$ to $\Omega C' \xrightarrow{f'} A' \xrightarrow{g'} B' \xrightarrow{h'} C'$. Then $\mathscr{L}tr(\Re, \mathfrak{F})$ is a left triangulation of $D^-(\Re, \mathfrak{F})$ in the sense of Beligiannis–Marmaridis* [2], *i.e. it is closed under isomorphisms and enjoys the following four axioms:*

- (LT1) *for any ring $A \in \Re$ the left triangle $0 \xrightarrow{0} A \xrightarrow{1_A} A \xrightarrow{0} 0$ belongs to $\mathscr{L}tr(\Re, \mathfrak{F})$ and for any morphism $h : B \to C$ there is a left triangle in $\mathscr{L}tr(\Re, \mathfrak{F})$ of the form $\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$;*
- (LT2) *for any left triangle $\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$ in $\mathscr{L}tr(\Re, \mathfrak{F})$, the diagram $\Omega B \xrightarrow{-\Omega h} \Omega C \xrightarrow{f} A \xrightarrow{g} B$ is also in $\mathscr{L}tr(\Re, \mathfrak{F})$;*
- (LT3) *for any two left triangles $\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$, $\Omega C' \xrightarrow{f'} A' \xrightarrow{g'} B' \xrightarrow{h'} C'$ in $\mathscr{L}tr(\Re, \mathfrak{F})$ and any two morphisms $\beta : B \to B'$ and $\gamma : C \to C'$ of $D^-(\Re, \mathfrak{F})$ with $\gamma h = h'\beta$, there is a morphism $\alpha : A \to A'$ of $D^-(\Re, \mathfrak{F})$ such that the triple $(\alpha, \beta, \gamma)$ gives a morphism from the first triangle to the second;*
- (LT4) *any two morphisms $B \xrightarrow{h} C \xrightarrow{k} D$ of $D^-(\Re, \mathfrak{F})$ can be fitted into a commutative diagram*

$$\begin{array}{ccccccc}
 & & \Omega E & & & & \\
 & & \downarrow{\scriptstyle f\circ\Omega\ell} & & & & \\
\Omega C & \xrightarrow{f} & A & \xrightarrow{g} & B & \xrightarrow{h} & C \\
\downarrow{\scriptstyle \Omega k} & & \downarrow{\scriptstyle \alpha} & & {\scriptstyle 1_B}\downarrow & & \downarrow{\scriptstyle k} \\
\Omega D & \xrightarrow{j} & F & \xrightarrow{m} & B & \xrightarrow{kh} & D \\
\downarrow{\scriptstyle 1_{\Omega D}} & & \downarrow{\scriptstyle \beta} & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle 1_D} \\
\Omega D & \xrightarrow{i} & E & \xrightarrow{\ell} & C & \xrightarrow{k} & D
\end{array}$$

*in which the rows and the second column from the left are left triangles in $\mathscr{L}tr(\Re, \mathfrak{F})$.*

The axiom (LT4) is a version of Verdier's octahedral axiom for left triangles in $D^-(\Re, \mathfrak{F})$.

**Proof.** The axioms (LT1)–(LT3) are already checked above. Notice that the morphism $\alpha$ in the axiom (LT3) is, by construction, an isomorphism whenever $\beta, \gamma$ are. It remains to show (LT4).

Since every morphism in $D^-(\mathfrak{R}, \mathfrak{F})$ is of the form $p \circ i \circ s^{-1}$ with $p$ a fibration and $i, s$ quasi-isomorphisms, it follows that the composable morphisms $h, k$ fit into a commutative diagram in $D^-(\mathfrak{R}, \mathfrak{F})$

$$\begin{array}{ccccc}
B & \xrightarrow{h} & C & \xrightarrow{k} & D \\
{\scriptstyle\cong}\downarrow & & {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle 1_D} \\
B' & \xrightarrow{p} & C' & \xrightarrow{q} & D
\end{array}$$

with the vertical maps isomorphisms and $p, q$ fibrations in $\mathfrak{R}$. It is routine to verify that (LT4) follows from the following fact we are going to prove: any two fibrations $B \xrightarrow{h} C \xrightarrow{k} D$ of $\mathfrak{R}$ can be fitted into a commutative diagram in $D^-(\mathfrak{R}, \mathfrak{F})$

$$\begin{array}{ccccccc}
 & & \Omega E & & & & \\
 & & \downarrow{\scriptstyle f\circ\Omega\ell} & & & & \\
\Omega C & \xrightarrow{f} & A & \xrightarrow{g} & B & \xrightarrow{h} & C \\
\downarrow{\scriptstyle \Omega k} & & \downarrow{\scriptstyle \alpha} & & {\scriptstyle 1_B}\downarrow & & \downarrow{\scriptstyle k} \\
\Omega D & \xrightarrow{v} & F & \xrightarrow{m} & B & \xrightarrow{kh} & D \\
\downarrow{\scriptstyle 1_{\Omega D}} & & \downarrow{\scriptstyle \beta} & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle 1_D} \\
\Omega D & \xrightarrow{u} & E & \xrightarrow{\ell} & C & \xrightarrow{k} & D
\end{array}$$

in which the rows are standard left triangles and the second column from the left is a left triangle in $\mathscr{L}tr(\mathfrak{R}, \mathfrak{F})$.

The horizontal standard triangles are constructed in a natural way and then $\alpha, \beta$ exist by the universal property of pullback diagrams. Note that $\beta$ is a fibration, because it is base extension of the fibration $h$ along $\ell$. Moreover, the sequence $A \xrightarrow{\alpha} F \xrightarrow{\beta} E$ is short exact. We have to show that the sequence $\Omega E \xrightarrow{f\circ\Omega\ell} A \xrightarrow{\alpha} F \xrightarrow{\beta} E$ is a left triangle in $\mathscr{L}tr(\mathfrak{R}, \mathfrak{F})$.

Recall that the map $f$ equals $i^{-1} \circ j$ with $i, j$ being constructed as

$$\begin{array}{ccccc}
 & & \Omega C & = & \Omega C \\
 & & \downarrow{\scriptstyle j} & & \downarrow \\
A & \xrightarrow{i} & P(h) & \twoheadrightarrow & EC \\
\| & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle \partial_x^1} \\
A & \xrightarrow{g} & B & \xrightarrow{h} & C.
\end{array}$$

Let us construct a commutative diagram as follows:

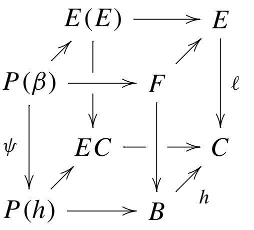
$$\begin{array}{ccccc}
 & & \Omega E & = & \Omega E \\
 & & {\scriptstyle\gamma}\downarrow & & \downarrow \\
A & \xrightarrow{\delta} & P(\beta) & \twoheadrightarrow & E(E) \\
\| & & \downarrow{\scriptstyle\beta_1} & & \downarrow{\scriptstyle\partial_x^1} \\
A & \xrightarrow{\alpha} & F & \xrightarrow{\beta} & E.
\end{array}$$

It follows that $\delta$ is a quasi-isomorphism. Our assertion would be proved if we show that the diagram

$$\begin{array}{ccccccc}
\Omega E & \xrightarrow{i^{-1}\circ j\circ\Omega\ell} & A & \xrightarrow{\alpha} & F & \xrightarrow{\beta} & E \\
\| & & \downarrow{\scriptstyle\delta} & & \| & & \| \\
\Omega E & \xrightarrow{\gamma} & P(\beta) & \xrightarrow{\beta_1} & F & \xrightarrow{\beta} & E
\end{array}$$

is commutative in $D^-(\mathfrak{R},\mathfrak{F})$, because the lower sequence is a left triangle. The left and central squares are commutative. It remains to verify that $\delta\circ i^{-1}\circ j\circ\Omega\ell=\gamma$.

By the universal property of pullback diagrams there exists a homomorphism $\psi : P(\beta)\to P(h)$ making the diagram




commutative. By construction, $\psi(f,e(x))=(m(f),\ell(e(x)))$ for $(f,e(x))\in P(\beta)$. It follows that $\psi\gamma=j\circ\Omega\ell$ and $\psi\delta=i$. Since $\delta, i$ are quasi-isomorphisms, then so is $\psi$. We have:

$$\delta\circ i^{-1}\circ j\circ\Omega\ell=\delta i^{-1}\psi\gamma=\delta\delta^{-1}\gamma=\gamma.$$

Our theorem is proved. □

**Corollary 5.7.** *Let $\mathfrak{F}$ be a saturated family of fibrations and $A\in\mathfrak{R}$. Then the representable functor*

$$[A,-]=\mathrm{Hom}_{D^-(\mathfrak{R},\mathfrak{F})}(A,-)$$

*gives rise to a homology theory $H_*$ on $\Re$ with $H_n(B) = [A, \Omega^n B]$, $n \geqslant 0$, and*

$$H_n(f) = \begin{cases} [A, (-1)^n \Omega(f)], & n \geqslant 1, \\ [A, f], & n = 0. \end{cases}$$

**Proof.** By Corollary 5.4, $H_{n \geqslant 1}(B)$ is a group. Our assertion would be proved if we show that for any left triangle $\Omega B \xrightarrow{f} F \xrightarrow{g} E \xrightarrow{h} B$ the induced sequence

$$[A, \Omega B] \xrightarrow{\partial_1 := f_*} [A, F] \xrightarrow{g_*} [A, E] \xrightarrow{h_*} [A, B] \tag{8}$$

is an exact sequence of pointed sets. Since any left triangle is, by definition, isomorphic to that induced by an $\mathfrak{F}$-fibre sequence, (8) is exact at the term $[A, E]$ by Theorem 5.3. By (LT2) $\Omega E \xrightarrow{-\Omega h} \Omega B \xrightarrow{f} F \xrightarrow{g} E$ is a left triangle. The same argument shows that (8) is exact at $[A, F]$. □

## 6. Stabilization

Throughout this section $\Re$ is an admissible category of rings and $\mathfrak{F}$ is a saturated family of fibrations. There is a general method of stabilizing the loop functor $\Omega$ (see Heller [12]) and producing a triangulated category $D(\Re, \mathfrak{F})$ from the left triangulated structure on $D^-(\Re, \mathfrak{F})$. We use stabilization to define a $\mathbb{Z}$-graded bivariant homology theory $k_*(A, B)$ on $\Re$, i.e. it is contravariant in the first variable and covariant in the second and produces long exact sequences in each variable out of $\mathfrak{F}$-fibre sequences.

We start with preparations. First let us verify that $\Omega^{n \geqslant 2} A$ are abelian group objects.

Let $B[x] \times_B B[x] := \{(f(x), g(x)) \mid f(1) = g(0)\}$ and let $\widetilde{\Omega} B$ be the kernel of $(d_0, d_1) : B[x] \times_B B[x] \to B[x]$, $(f(x), g(x)) \mapsto (f(0), g(1))$. Denote by $\widetilde{E}$ the fibred product of the diagram

$$E \xrightarrow{\partial_x^1} B \xleftarrow{\partial_x^0} B[x].$$

Since $\partial_x^1$ is a fibration then so is $pr_2 : \widetilde{E} \to B[x]$.

**Lemma 6.1.** *The homomorphism $\alpha : \Omega B \to \widetilde{\Omega} B$, $f(x) \mapsto (f(x), 0)$ is a quasi-isomorphism.*

**Proof.** Consider a commutative diagram in $\Re$

$$\begin{array}{ccccc}
\Omega B & \longrightarrow & \widetilde{E} & \xrightarrow{pr_2} & B[x] \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \partial_x^1} \\
\widetilde{\Omega} B & \longrightarrow & \widetilde{E} & \xrightarrow{p} & B \\
\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle 1} \\
F & \xrightarrow{l} & B[x] & \xrightarrow{\partial_x^1} & B
\end{array}$$

in which the rows are short exact. Note that $\gamma$ is a fibration, because it is base extension of the fibration $pr_2$ along $l$. Thus the left column is a $\mathfrak{F}$-fibre sequence.

The ring $F$ is quasi-isomorphic to 0, because it is isomorphic to the contractible ring $E$. Therefore $\alpha$ is a quasi-isomorphism by Theorem 5.3. □

Let us factorize $(\partial_x^0, \partial_x^1): B[x] \to B \times B$ as $B[x] \xrightarrow{i} B^I \xrightarrow{q} B \times B$, where $i$ is a $I$-weak equivalence and $q$ is a fibration. Denote by $\Omega' B$ the fibre of $(d_0, d_1): B^I \to B \times B$. The map $i$ induces a map $u: \Omega B \to \Omega' B$. By Brown [3, p. 430–431] one can regard $\Omega' B$ as an object of $D^-(\Re, \mathfrak{F})$ well defined up to canonical isomorphism. Moreover, we have a functor $\Omega': \Re \to D^-(\Re, \mathfrak{F})$. This functor preservers quasi-isomorphisms and so can be regarded as a functor $\Omega': D^-(\Re, \mathfrak{F}) \to D^-(\Re, \mathfrak{F})$.

Let $B^{2I} := B^I \times_B B^I$ and let $\widetilde{\Omega}' B$ denote the fibre of $(d_0, d_1): B^{2I} \to B \times B$. The map $(i, i): B[x] \times_B B[x] \to B^{2I}$ yields a map $v: \widetilde{\Omega} B \to \widetilde{\Omega}' B$. The maps of path spaces $(1, sd_1), (sd_0, 1): B^I \to B^{2I}$ induce two quasi-isomorphisms $a, b: \Omega' B \to \widetilde{\Omega}' B$ taking $f \in \Omega' B$ to $(f, 0)$ and $(0, f)$ respectively. It follows from [3, §4, Lemma 4] that $a = b$ in $D^-(\Re, \mathfrak{F})$.

**Lemma 6.2.** *The homomorphisms*

$$u: \Omega B \to \Omega' B \quad \text{and} \quad v: \widetilde{\Omega} B \to \widetilde{\Omega}' B$$

*are quasi-isomorphisms.*

**Proof.** Consider a commutative diagram in $\Re$ with exact rows

$$\begin{array}{ccccc} E & \longrightarrow & B[x] & \xrightarrow{\partial_x^0} & B \\ \downarrow{\scriptstyle e} & & {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle 1} \\ E' & \longrightarrow & B^I & \xrightarrow{d_0} & B. \end{array}$$

Since $E, E'$ are quasi-isomorphic to zero, it follows that $e$ is a quasi-isomorphism. Consider now a commutative diagram in $\Re$ with exact rows

$$\begin{array}{ccccc} \Omega B & \longrightarrow & E & \xrightarrow{\partial_x^1} & B \\ \downarrow{\scriptstyle u} & & \downarrow & & \downarrow{\scriptstyle 1} \\ \Omega' B & \longrightarrow & E' & \xrightarrow{d_1} & B. \end{array}$$

By the proof of the factorization lemma in Brown [3] the map $d_1$ is a fibration. It follows from [3, §4, Lemma 3] that $u$ is a quasi-isomorphism. Since $v\alpha = au$ and $\alpha, a, u$ are quasi-isomorphisms (see Lemma 6.1), then so is $v$. □

Denote by $\beta: \Omega B \to \widetilde{\Omega} B$ the map taking $f \in \Omega B$ to $(0, f) \in \widetilde{\Omega} B$. Since $bu = v\beta$, $au = v\alpha$, $u$ and $v$ are quasi-isomorphisms by the preceding lemma, and $a = b$ in $D^-(\Re, \mathfrak{F})$ we deduce the following

**Corollary 6.3.** $\alpha = \beta$ *in* $D^-(\Re, \mathfrak{F})$. *In particular,* $\beta$ *is a quasi-isomorphism.*

We now construct the following commutative diagram:

$$\begin{array}{ccccc} \Omega B \times \Omega B & \xrightarrow{\omega} & \widetilde{\Omega} B & \xleftarrow{\alpha} & \Omega B \\ \Big\downarrow{\scriptstyle (u,u)} & & \Big\downarrow{\scriptstyle v} & & \Big\downarrow{\scriptstyle u} \\ \Omega' B \times \Omega' B & \xrightarrow{w} & \widetilde{\Omega}' B & \xleftarrow{a} & \Omega' B, \end{array}$$

where $\omega, w$ are obvious maps and the vertical maps are quasi-isomorphisms by Lemma 6.2. Recall from Brown [3, p. 431] that the map

$$m_B := a^{-1} w : \Omega' B \times \Omega' B \to \Omega' B$$

gives a group structure for $\Omega' B$ in $D^-(\Re, \mathfrak{F})$. The map

$$\mu_B := \alpha^{-1} \omega : \Omega B \times \Omega B \to \Omega B$$

gives a group structure for $\Omega B$ in $D^-(\Re, \mathfrak{F})$, because $\mu_B$ is isomorphic in $D^-(\Re, \mathfrak{F})$ to $m_B$ by above.

**Lemma 6.4.** *For any ring $B \in \Re$ the homomorphism $\tau : \Omega^2 B \to \Omega^2 B$, $\sum a_{ij} x^i y^j \mapsto \sum a_{ij} x^j y^i$, is elementary homotopic to the identity.*

**Proof.** Any polynomial $f(x, y) \in \Omega^2 B$ can be written as $f(x, y) = (x^2 - x)(y^2 - y) f'(x, y)$ for some (unique) polynomial $f'(x, y)$. The desired elementary homotopy $H : \Omega^2 B \to \Omega^2 B[t]$ is defined by

$$\big(x^2 - x\big)\big(y^2 - y\big) f'(x, y) \overset{H}{\longmapsto} \big(x^2 - x\big)\big(y^2 - y\big) f'\big(tx + (1 - t)y, (1 - t)x + ty\big).$$

It follows that $d_0 H = \tau$ and $d_1 H = \mathrm{id}$. □

We are now in a position to prove the following

**Proposition 6.5.** *Let $\Re$ be an admissible category of rings and let $\mathfrak{F}$ be a saturated family of fibrations. Then for any ring $B \in \Re$ and any $n \geqslant 2$ the ring $\Omega^n B$ is an abelian group object in $D^-(\Re, \mathfrak{F})$.*

**Proof.** It will be sufficient to prove the claim for $\Omega^2 B$. We use the second coordinate to get the multiplication $\Omega \mu_B : \Omega^2 B \times \Omega^2 B \to \Omega^2 B$. First let us show that $\Omega \mu_B = \mu_{\Omega B}$, i.e. the multiplications in both coordinates agree.

The ring $\Omega \Omega' B$ is by construction consists of the polynomials of the form $(x^2 - x) \cdot [\sum_i (f_i(y), g_i(y)) x^i]$ with each $(f_i(y), g_i(y)) \in \Omega' B$. The ring $\Omega' \Omega B$ is by construction consists of the pairs $((y^2 - y) \cdot [\sum_i f_i(x) y^i], (y^2 - y) \cdot [\sum_i g_i(x) y^i])$ with each $(f_i(x), g_i(x)) \in \Omega' B$.

Let $\tau'$ denote the homomorphism

$$\big(x^2 - x\big) \cdot \Big[\sum_i \big(f_i(y), g_i(y)\big) x^i\Big] \mapsto \Big(\big(y^2 - y\big) \cdot \Big[\sum_i f_i(x) y^i\Big], \big(y^2 - y\big) \cdot \Big[\sum_i g_i(x) y^i\Big]\Big).$$

Then the following diagram is commutative:

$$\begin{array}{ccc} \Omega^2 B \times \Omega^2 B & \xrightarrow{\tau\times\tau} & \Omega^2 B \times \Omega^2 B \\ {\scriptstyle \Omega\omega_B}\downarrow & & \downarrow{\scriptstyle \omega_{\Omega B}} \\ \Omega\Omega' B & \xrightarrow{\tau'} & \Omega'\Omega B \\ {\scriptstyle \Omega\alpha_B}\uparrow & & \uparrow{\scriptstyle \alpha_{\Omega B}} \\ \Omega^2 B & \xrightarrow{\tau} & \Omega^2 B. \end{array}$$

By Lemma 6.4 the upper and lower arrows equal identity in $D^-(\Re, \mathfrak{F})$, hence $\Omega\mu_B = \mu_{\Omega B}$.

To verify that $\Omega^2 B$ is an abelian group object in $D^-(\Re, \mathfrak{F})$, one has to show that the diagram

$$\begin{array}{ccc} \Omega^2 B \times \Omega^2 B & \xrightarrow{T} & \Omega^2 B \times \Omega^2 B \\ & {\scriptstyle \mu_{\Omega B}}\searrow \quad \swarrow{\scriptstyle \mu_{\Omega B}} & \\ & \Omega^2 B, & \end{array}$$

in which $T$ is the isomorphism $(f, g) \mapsto (g, f)$, is commutative.

Let $T' : \Omega\Omega' B \to \Omega'\Omega B$ denote the homomorphism

$$\left(x^2 - x\right)\left[\sum_i \bigl(f_i(y), g_i(y)\bigr)x^i\right] \mapsto \left(\left(y^2 - y\right)\left[\sum_i g_i(1-x)(1-y)^i\right],\right.$$
$$\left.\left(y^2 - y\right)\left[\sum_i f_i(1-x)(1-y)^i\right]\right).$$

Consider the diagram

$$\begin{array}{ccc} \Omega^2 B \times \Omega^2 B & \xrightarrow{T} & \Omega^2 B \times \Omega^2 B \\ {\scriptstyle \Omega\omega_B}\downarrow & & \downarrow{\scriptstyle \omega_{\Omega B}} \\ \Omega\Omega' B & \xrightarrow{T'} & \Omega'\Omega B \\ {\scriptstyle \Omega\alpha_B}\uparrow & & \uparrow{\scriptstyle \beta_{\Omega B}} \\ \Omega^2 B & \xrightarrow{\mathrm{id}} & \Omega^2 B. \end{array} \tag{9}$$

We claim that is commutative in $D^-(\Re, \mathfrak{F})$.

Let $\sigma_x : \Omega^2 B \to \Omega^2 B$ (respectively $\sigma_y$) be the homomorphism mapping $(x, y)$ to $(1 - x, y)$ (respectively $(x, 1 - y)$). We have

$$\mu_{\Omega B}(\sigma_x \times \sigma_x) = \Omega\mu_B(\sigma_y \times \sigma_y),$$

and hence $\mu_{\Omega B} = \Omega\mu_B(\sigma_x\sigma_y \times \sigma_x\sigma_y) = \mu_{\Omega B}(\sigma_x\sigma_y \times \sigma_x\sigma_y)$. Then

$$\beta_{\Omega B}^{-1}\omega_{\Omega B}T = \mu_{\Omega B}T = \mu_{\Omega B}T(\sigma_x\sigma_y \times \sigma_x\sigma_y) = \beta_{\Omega B}^{-1}T'\Omega\omega_B,$$

and so $\omega_{\Omega B}T = T'\Omega\omega_B$.

We also have

$$T' \circ \Omega\alpha_B = T'\Omega\omega_B(1,0)^t = \omega_{\Omega B}T(1,0)^t = \omega_{\Omega B}(0,1)^t = \beta_{\Omega B}.$$

Here $(0,1)^t$ and $(1,0)^t$ denote the corresponding injections $\Omega^2 B \to \Omega^2 B \times \Omega^2 B$. The fact that $\beta_{\Omega B} = \alpha_{\Omega B}$ (see Corollary 6.3) and that (9) is commutative imply the desired abelian group structure on $\Omega^2 B$. □

**Corollary 6.6.** *Given two rings $A, B \in \Re$ and $n \geqslant 2$, the group $D^-(\Re, \mathfrak{F})(A, \Omega^n(B))$ is abelian.*

We recall the construction of $D(\Re, \mathfrak{F})$ from Heller [12], which consists of formally inverting the endofunctor $\Omega$. An object of $D(\Re, \mathfrak{F})$ is a pair $(A, m)$ with $A \in D^-(\Re, \mathfrak{F})$ and $m \in \mathbb{Z}$. If $m, n \in \mathbb{Z}$ then we consider the directed set $I_{m,n} = \{k \in \mathbb{Z} \mid m, n \leqslant k\}$. The set of morphisms between $(A, m)$ and $(B, n) \in D(\Re, \mathfrak{F})$ is defined by

$$D(\Re, \mathfrak{F})\big[(A,m),(B,n)\big] := \varinjlim_{k \in I_{m,n}} D^-(\Re, \mathfrak{F})\big(\Omega^{k-m}(A), \Omega^{k-n}(B)\big).$$

Morphisms of $D(\Re, \mathfrak{F})$ are composed in the obvious fashion. We define the *loop* automorphism on $D(\Re, \mathfrak{F})$ by $\Omega(A,m) := (A, m-1)$. There is a natural functor $S : D^-(\Re, \mathfrak{F}) \to D(\Re, \mathfrak{F})$ defined by $A \mapsto (A, 0)$.

It follows from above that the category $D(\Re, \mathfrak{F})$ is preadditive. Since it has finite direct products then it is additive. We define a triangulation $\mathscr{T}r(\Re, \mathfrak{F})$ of the pair $(D(\Re, \mathfrak{F}), \Omega)$ as follows. A sequence

$$\Omega(A,l) \to (C,n) \to (B,m) \to (A,l)$$

belongs to $\mathscr{T}r(\Re, \mathfrak{F})$ if there is an even integer $k$ and a left triangle of representatives $\Omega(\Omega^{k-l}(A)) \to \Omega^{k-n}(C) \to \Omega^{k-m}(B) \to \Omega^{k-l}(A)$ in $D^-(\Re, \mathfrak{F})$. Clearly, the functor $S$ takes left triangles in $D^-(\Re, \mathfrak{F})$ to triangles in $D(\Re, \mathfrak{F})$.

We are now in a position to prove the main result of the section.

**Theorem 6.7.** *Let $\mathfrak{F}$ be a saturated family of fibrations in $\Re$. Then $\mathscr{T}r(\Re, \mathfrak{F})$ is a triangulation of $D(\Re, \mathfrak{F})$ in the classical sense of Verdier* [21].

**Proof.** It is easy to see that $D(\Re, \mathfrak{F})$ is left triangulated, i.e. $\mathscr{T}r(\Re, \mathfrak{F})$ meets the axioms (LT1)–(LT4) of Theorem 5.6. By [2, p. 5], $D(\Re, \mathfrak{F})$ is triangulated, because it is additive and the endofunctor $\Omega$ is invertible. □

We use the triangulated category $D(\Re, \mathfrak{F})$ to define a $\mathbb{Z}$-graded bivariant homology theory depending on $(\Re, \mathfrak{F})$ as follows:

$$k_n(A,B) = k_n^{\Re,\mathfrak{F}}(A,B) := D(\Re, \mathfrak{F})\big((A,0),(B,n)\big), \quad n \in \mathbb{Z}.$$

**Corollary 6.8.** *For any $\mathfrak{F}$-fibre sequence $A \to B \to C$ and any $D \in \mathfrak{R}$, we have long exact sequences of abelian groups*

$$\cdots \to k_{n+1}(D, C) \to k_n(D, A) \to k_n(D, B) \to k_n(D, C) \to \cdots$$

*and*

$$\cdots \to k_{n+1}(A, D) \to k_n(C, D) \to k_n(B, D) \to k_n(A, D) \to \cdots$$

## 7. The triangulated category *kk*

Motivated by ideas and work of J. Cuntz on bivariant $K$-theory of locally convex algebras (see [6,7]), Cortiñas and Thom [5] construct a bivariant homology theory $kk_*(A, B)$ for algebras over a unital ground ring $H$. It is Morita invariant, homotopy invariant, excisive $K$-theory of algebras, which is universal in the sense that it maps uniquely to any other such theory. This bivariant $K$-theory is defined similar to the bivariant homology theory $k_*(A, B)$ discussed in the previous section. Namely, a triangulated category $kk$ whose objects are the $H$-algebras without unit is constructed and then set $kk_n(A, B) = kk(A, \Omega^n B)$, $n \in \mathbb{Z}$.

We make use of our machinery developed in the preceding sections to study various triangulated structures on admissible categories of rings which are not necessarily small. As an application, we give another description of the triangulated category $kk$. Throughout this section the class $\mathfrak{F}$ of fibrations consists of the surjective homomorphisms.

Let $\mathfrak{R}$ be an arbitrary not necessarily small admissible category of rings and let $\mathfrak{W}$ be any subcategory of homomorphisms containing the $I$-weak equivalences such that the triple $(\mathfrak{R}, \mathfrak{W}, \mathfrak{F})$ is a Brown category. Let $D^-(\mathfrak{R}, \mathfrak{W})$ be the category obtained from $\mathfrak{R}$ by inverting the weak equivalences. Then $\Omega : \mathfrak{R} \to \mathfrak{R}$ yields a loop functor on $D^-(\mathfrak{R}, \mathfrak{W})$. Let us define the category of left triangles $\mathscr{L}tr(\mathfrak{R}, \mathfrak{W})$ similar to $\mathscr{L}tr(\mathfrak{R}, \mathfrak{F})$. Then the following is true.

**Theorem 7.1.** *$\mathscr{L}tr(\mathfrak{R}, \mathfrak{W})$ determines a left triangulation of $D^-(\mathfrak{R}, \mathfrak{W})$. The stabilization procedure of the loop functor $\Omega$ described in the previous section yields a triangulated category $D(\mathfrak{R}, \mathfrak{W})$ whose objects and morphisms are defined similar to those of $D(\mathfrak{R}, \mathfrak{F})$.*

**Proof.** The proof repeats those of Theorems 5.6 and 6.7 word for word if we replace in appropriate places path spaces $B^I$ with the functorial path space $B[x]$ for $B \in (\mathfrak{R}, \mathfrak{W}, \mathfrak{F})$. □

**Remark.** Theorem 7.1 says that construction of $D(\mathfrak{R}, \mathfrak{F})$ is formal and can be defined in a more general setting whenever $\mathfrak{F}$ consists of the surjective homomorphisms.

**Corollary 7.2.** $k_*(A, B) := k_*^{\mathfrak{R}, \mathfrak{W}}(A, B) = D(\mathfrak{R}, \mathfrak{W})(A, \Omega^* B)$ *determines a bivariant homology theory on $\mathfrak{R}$. Moreover, for any short exact sequence $A \to B \to C$ and any $D \in \mathfrak{R}$, we have long exact sequences of abelian groups*

$$\cdots \to k_{n+1}(D, C) \to k_n(D, A) \to k_n(D, B) \to k_n(D, C) \to \cdots$$

*and*

$$\cdots \to k_{n+1}(A, D) \to k_n(C, D) \to k_n(B, D) \to k_n(A, D) \to \cdots .$$

We consider associative, not necessarily unital or central algebras over a fixed unital, not necessarily commutative ring $H$; we write $\mathrm{Alg}_H$ for the category of such algebras. By forgetting structure, we can embed $\mathrm{Alg}_H$ faithfully into each of the categories of $H$-bimodules, abelian groups and sets. Fix one of these underlying categories, call it $\mathscr{U}$, and let $F : \mathrm{Alg}_H \to \mathscr{U}$ be the forgetful functor. Let $\mathscr{E}$ be the class of all exact sequences of $H$-algebras

$$(E): 0 \to A \to B \to C \to 0 \tag{10}$$

such that $F(B) \to F(C)$ is a split surjection.

**Definition.** (Cortiñas–Thom [5]) Given a triangulated category $(\mathscr{T}, \Omega)$, an *excisive homology theory* on $\mathrm{Alg}_H$ with values in $\mathscr{T}$ consists of a functor $X : \mathrm{Alg}_H \to \mathscr{T}$, together with a collection $\{\partial_E\colon E \in \mathscr{E}\}$ of maps

$$\partial_E^X = \partial_E \in \mathscr{T}\big(\Omega X(C), X(A)\big).$$

The maps $\partial_E$ are to satisfy the following requirements.

(i) For all $E \in \mathscr{E}$ as above,

$$\Omega X(C) \xrightarrow{\partial_E} X(A) \xrightarrow{X(f)} X(B) \xrightarrow{X(g)} X(C)$$

is a distinguished triangle in $\mathscr{T}$.

(ii) If

$$\begin{array}{cccccc} (E): & A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ (E'): & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array}$$

is a map of extensions, then the following diagram commutes

$$\begin{array}{ccc} \Omega X(C) & \xrightarrow{\partial_E} & X(A) \\ {\scriptstyle\Omega X(\gamma)}\downarrow & & \downarrow{\scriptstyle X(\alpha)} \\ \Omega X(C') & \xrightarrow[\partial_{E'}]{} & X(A). \end{array}$$

Let $\iota_\infty : A \to M_\infty A$ be the natural inclusion from $A$ to $M_\infty A = \bigcup_n M_n A$, the union of matrix rings. An excisive, homotopy invariant homology theory $X : \mathrm{Alg}_H \to \mathscr{T}$ is $M_\infty$*-stable* if for every $A \in \mathrm{Alg}_H$, it maps the inclusion $\iota_\infty : A \to M_\infty A$ to an isomorphism. Note that if $X$ is $M_\infty$-stable, and $n \geqslant 1$, then $X$ maps the inclusion $\iota_n : A \to M_n A$ to an isomorphism.

The homotopy invariant, $M_\infty$-stable, excisive homology theories form a category, where a homomorphism between the theories $X:\mathrm{Alg}_H \to \mathscr{T}$ and $Y:\mathrm{Alg}_H \to \mathscr{S}$ is a triangulated functor $G:\mathscr{T} \to \mathscr{S}$ such that

$$\begin{array}{ccc} \mathrm{Alg}_H & \xrightarrow{X} & \mathscr{T} \\ & \searrow^{Y} & \downarrow{\scriptstyle G} \\ & & \mathscr{S} \end{array}$$

commutes, and such that for every extension (10), the natural isomorphism $\varphi: G(\Omega X(C)) \to \Omega Y(C)$ makes the following into a commutative diagram:

$$\begin{array}{ccc} G(\Omega X(C)) & \xrightarrow{G(\partial_E^X)} & Y(A) \\ {\scriptstyle\varphi}\downarrow & \nearrow_{\partial_E^Y} & \\ \Omega Y(C). & & \end{array} \tag{11}$$

**Theorem 7.3.** *(Cortiñas–Thom [5]) The category of homotopy invariant, $M_\infty$-stable, excisive homology theories has an initial object $\ell:\mathrm{Alg}_H \to kk$. The triangulated category kk has the same objects and the same endofunctor $\Omega$ as $\mathrm{Alg}_H$. Furthermore,*

$$kk_*(A, B) = kk(A, \Omega^* B)$$

*gives rise to a $M_\infty$-stable, homotopy invariant, excisive, bivariant homology theory of algebras.*

The preceding theorem is used to define a natural map

$$kk_*(A, B) \to KH_*(A, B),$$

where $KH_*(A, B)$ is the bivariant theory generated by the homotopy $K$-theory $KH$. A result of Cortiñas–Thom [5] states that this map is an isomorphism when $A = H$ is commutative and $B$ is a central $H$-algebra, i.e. $kk_*(H, B) = KH_*(B)$. When $H$ is a field of characteristic zero and $A$, $B$ are central $H$-algebras, they also obtain in this way a product preserving Chern character to bivariant periodic cyclic cohomology

$$ch_*: kk_*(A, B) \to HP^*(A, B).$$

Let $\mathfrak{W}_{CT}$ be the class of homomorphisms $f$ in $\mathrm{Alg}_H$ such that $X(f)$ is an isomorphism for any homotopy invariant, $M_\infty$-stable, excisive homology theory $X:\mathrm{Alg}_H \to \mathscr{T}$. It is directly verified that the triple $(\mathrm{Alg}_H, \mathfrak{W}_{CT}, \mathfrak{F})$ meets the axioms for a Brown category.

We are now in a position to prove the main result of this section.

**Theorem 7.4** *(Comparison). There is a natural triangulated equivalence of the triangulated categories kk and $D(\mathrm{Alg}_H, \mathfrak{W}_{CT})$.*

**Proof.** Let $\alpha : \mathfrak{R} \to D^-(\mathfrak{R}, \mathfrak{W})$, $S : D^-(\mathfrak{R}, \mathfrak{W}) \to D(\mathfrak{R}, \mathfrak{W})$ be the canonical functors and let $E$ be the extension (10). Define $\partial_E \in D(\mathfrak{R}, \mathfrak{W})(\Omega C, A)$ as the class of the canonically defined morphism (7), $i^{-1} \circ j \in D^-(\mathfrak{R}, \mathfrak{W})(\Omega C, A)$. Then $\iota := S\alpha : \mathrm{Alg}_H \to D(\mathrm{Alg}_H, \mathfrak{W}_{CT})$, together with $\{\partial_E\}_{E \in \mathscr{E}}$ is a homotopy invariant, $M_\infty$-stable, excisive homology theory. By Theorem 7.3 there is a unique morphism $G : kk \to D(\mathfrak{R}, \mathfrak{W})$ of homology theories such that $G\ell = \iota$. We claim that $G$ is an equivalence of categories.

Since $\ell$ takes weak equivalences to isomorphisms, there is a unique functor $F : D^-(\mathrm{Alg}_H, \mathfrak{W}_{CT}) \to kk$ such that $F \circ \alpha = \ell$. We have $S\alpha = G\ell = GF\alpha$. It follows that $S = GF$, and hence $G$ is full.

By [12, 1.1], $F$ is uniquely extended to a functor $H : D(\mathrm{Alg}_H, \mathfrak{W}_{CT}) \to \mathscr{T}$ such that $H \circ S = F$. By [5, 6.5.1] a diagram

$$\Omega C \longrightarrow A \longrightarrow B \longrightarrow C$$

of morphisms in *kk* is a distinguished triangle if it is isomorphic in *kk* to the path sequence

$$\Omega B' \xrightarrow{\ell(j)} P(f) \xrightarrow{\ell(\pi_f)} A' \xrightarrow{\ell(f)} B'$$

associated with a homomorphism $f : A' \to B'$ of $H$-algebras. We see that $F$ takes left triangles in $D^-(\mathrm{Alg}_H, \mathfrak{W}_{CT})$ to triangles in *kk*. Therefore $H$ takes triangles in $D(\mathrm{Alg}_H, \mathfrak{W}_{CT})$ to triangles. The same argument as in the proof of [5, 6.6.2] shows that $H$ must be a morphism of homotopy invariant, $M_\infty$-stable, excisive homology theories. By uniqueness $HG = \mathrm{id}_{kk}$, and so $G$ is faithful, as was to be proved. □

**Corollary 7.5.** *$\mathfrak{W}_{CT}$ is the smallest class of weak equivalences containing the homomorphisms of $H$-algebras $A \to A[x]$, $\iota_\infty : A \to M_\infty A$, that is $\mathfrak{W}_{CT} \subseteq \mathfrak{W}$ with $\mathfrak{W}$ being any class of weak equivalences containing $A \to A[x]$, $\iota_\infty : A \to M_\infty A$ such that the triple $(\mathrm{Alg}_H, \mathfrak{W}, \mathfrak{F})$ is a Brown category.*

**Proof.** Let $\mathfrak{W}$ be the smallest class of weak equivalences containing $A \to A[x]$, $\iota_\infty : A \to M_\infty A$ such that the triple $(\mathrm{Alg}_H, \mathfrak{W}, \mathfrak{F})$ is a Brown category. Then $\mathfrak{W} \subseteq \mathfrak{W}_{CT}$. By Theorem 7.1 the canonical functor $\mathrm{Alg}_H \to D(\mathrm{Alg}_H, \mathfrak{W})$ yields a homotopy invariant, $M_\infty$-stable, excisive homology theory. Therefore $\mathfrak{W}_{CT} \subseteq \mathfrak{W}$. □

We infer from the preceding theorem that *kk* does not depend on the choice of the underlying category $\mathscr{U}$. For further properties of the category *kk* we refer the reader to [5].

## 8. Addendum

When $\mathscr{M}$ is a model category and $S$ is a set of maps between cofibrant objects, we shall produce a new model structure on $\mathscr{M}$ in which the maps $S$ are weak equivalences. The new model structure is called the *Bousfield localization* or just localization of the old one. A theorem of Hirschhorn says that when $\mathscr{M}$ is a "sufficiently nice" model category one can localize at any set of maps. "Sufficiently nice" entails being cofibrantly generated together with having certain other finiteness properties; the exact notion is that of a *cellular* model category. We do not recall the definition here, but refer the reader to [13]. Suffice it to say that all the model categories we encounter in this paper are cellular.

For simplicity we shall from now on assume that all model categories are simplicial. This is not strictly necessary, but it allows us to avoid a certain machinery required for dealing with the general case (see [13] for details).

Since all model categories we shall consider are simplicial, we do not make use of the homotopy function complex $\mathrm{map}(X, Y)$ defined in [13]. Indeed, let $\mathscr{M}$ be a simplicial model category with simplicial mapping object Map, and let $X$ and $Y$ be two objects of $\mathscr{M}$. If $\widetilde{X} \to X$ is a cofibrant replacement of $X$ and $Y \to \widehat{Y}$ is a fibrant replacement of $Y$, then $\mathrm{map}(X, Y)$ is homotopy equivalent to $\mathrm{Map}(\widetilde{X}, \widehat{Y})$. Consequently, one can recast the localization theory of $\mathscr{M}$ in terms of the simplicial mapping object instead of the homotopy function complex.

**Definition.** Let $\mathscr{M}$ be a simplicial model category and let $S$ be a set of maps between cofibrant objects.

(1) An *$S$-local object* of $\mathscr{M}$ is a fibrant object $X$ such that for every map $A \to B$ in $S$, the induced map of $\mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$ is a weak equivalence of simplicial sets.
(2) An *$S$-local equivalence* is a map $A \to B$ such that $\mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$ is a weak equivalence for every $S$-local object $X$.

In words, the $S$-local objects are the ones which see every map in $S$ as if it were a weak equivalence. The $S$-local equivalences are those maps which are seen as weak equivalences by every $S$-local object.

**Theorem 8.1.** *(Hirschhorn [13]) Let $\mathscr{M}$ be a cellular, simplicial model category and let $S$ be a set of maps between cofibrant objects. Then there exists a new model structure on $\mathscr{M}$ in which*

(1) *the weak equivalences are the $S$-local equivalences*;
(2) *the cofibrations in $\mathscr{M}/S$ are the same as those in $\mathscr{M}$*;
(3) *the fibrations are the maps having the right-lifting-property with respect to cofibrations which are also $S$-local equivalences.*

*Left Quillen functors from $\mathscr{M}/S$ to $\mathscr{D}$ are in one-to-one correspondence with left Quillen functors $\Phi : \mathscr{M} \to \mathscr{D}$ such that $\Phi(f)$ is a weak equivalence for all $f \in S$. In addition, the fibrant objects of $\mathscr{M}$ are precisely the $S$-local objects, and this new model structure is again cellular and simplicial.*

The model category whose existence is guaranteed by the above theorem is called *$S$-localization* of $\mathscr{M}$. The underlying category is the same as that of $\mathscr{M}$, but there are more trivial cofibrations (and hence fewer fibrations). We sometimes use $\mathscr{M}/S$ to denote the $S$-localization.

Note that the identity maps yield a Quillen pair $\mathscr{M} \rightleftarrows \mathscr{M}/S$, where the left Quillen functor is the map $\mathrm{id} : \mathscr{M} \to \mathscr{M}/S$.

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