UNIVERSAL BIVARIANT ALGEBRAIC K-THEORIES

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Abstract. To any admissible category of algebras and a family of fibrations on it a universal bivariant excisive homotopy invariant algebraic K-theory is associated. Also, Morita invariant and stable universal bivariant K-theories are studied. We introduce an additive category of correspondencies for non-unital algebras and study the problem of when stable bivariant K-groups can be computed by means of correspondences.

In relation with the Atiyah-Singer theorem, Kasparov introduced bivariant $\mathbb{Z}/2$-graded abelian groups $KK_*(A,B)$ on the level of C^*-algebras with remarkable formal properties, one of which comes from the associative product $KK_*(A,B)\times KK_*(B,C)\to KK_*(A,C)$. After Cuntz [6] and Higson [17] one can define an additive category KK by taking separable C^*-algebras as objects and $KK_0(A,B)$ as set of morphisms between the objects A and B. In fact, KK can be regarded as a functor from the category of separable C^*-algebras with ordinary morphisms ($*$-homomorphisms) into the category KK whose objects are separable C^*-algebras and whose morphisms are $KK_0(A,B)$. Moreover, any morphism $f:A\to B$ naturally defines an element of $KK_0(A,B)$ and the Kasparov product extends the composition of morphisms. Now KK is the universal functor into an additive category $\mathcal{A}$ that is homotopy invariant, C^*-stable (i.e., $\mathcal{A}(A,\mathcal{K}\otimes B)\cong\mathcal{A}(A,B)$), and split exact (i.e., $\mathcal{A}(A,E)\cong\mathcal{A}(A,J)\oplus\mathcal{A}(A,B)$, if $0\to J\to E\to B\to 0$ is a split exact sequence of C^*-algebras).

Using abstract ideas from category theory, Higson constructed a new theory, later called E-theory. One takes the additive category KK and forms a category of fractions E with morphism sets $E(A,B)$ by inverting in KK all morphisms induced by an inclusion $I\to A$ of a closed ideal I into a C^*-algebra A, for which the quotient A/I is contractible. The category E is additive with a natural functor from the category of separable C^*-algebras into E (which factors over KK). In E, every extension of C^*-algebras (not necessarily admitting a completely positive splitting) induces long exact sequences in $E(-,D)$ and $E(D,-)$. Moreover, E is the universal functor into an additive category which is homotopy invariant, stable and half-exact. The categories KK and E as well as some other variants of bivariant theories can be viewed as triangulated categories (see [20, 22]).

Since the construction of KK-theory and E-theory used techniques which are quite specific to C^*-algebras it seemed for many years that similar theories for other topological algebras would be impossible. However, in [7, 8, 9] Cuntz constructed a bivariant two-periodic theory $kk_*^{lca}(A,B)$ on the category of locally convex algebras with all the desired properties, in particular the usual properties of (differentiable) homotopy invariance, long exact sequences associated with extensions and stability under tensoring by the

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algebra of rapidly decreasing matrices. Similar to the triangulated categories KK and E this bivariant theory can also be thought of as a triangulated category, denoted by kk^{lca}, whose objects are the locally convex algebras and Hom-sets are given by $kk_0^{lca}(A, B)$. There is as well a canonical functor from the category of locally convex algebras to the category kk^{lca}. This theory also allows to carry over ideas and techniques from locally convex algebras to its possible algebraic counterpart for the category of all algebras.

Motivated by ideas and work of J. Cuntz on bivariant K-theory of locally convex algebras [7, 8, 9], Cortiñas and Thom [5] define bivariant graded abelian groups $kk_*(A, B)$ on the category Alg_k of algebras over a unital ground ring k. The groups depend on a class $\mathfrak{F}$ of extensions of algebras, which is either the class $\mathfrak{F}_{\mathrm{surj}}$ of all surjective homomorphisms of k-algebras or the class of k-split surjective homomorphisms $\mathfrak{F}_{\mathrm{spl}}$. This bivariant theory can also be thought of as a triangulated category kk, whose objects are those of Alg_k and morphisms between two algebras $A, B \in \mathrm{Alg}_k$ are given by $kk_0(A, B)$. There is a canonical functor $j : \mathrm{Alg}_k \to kk$ sending an algebra to itself. Cortiñas–Thom [5] prove that j is the universal functor from Alg_k to a triangulated category $\mathcal{T}$ that is (polynomially) homotopy invariant, M_∞-invariant (i.e., $\mathcal{T}(A, M_\infty k \otimes_k B) \cong \mathcal{T}(A, B)$ with $M_\infty k$ the algebra of all finite matrices over k), and every extension $0 \to J \to E \to B \to 0$ in $\mathfrak{F}$ is mapped in a functorial way to a triangle $\Omega B \xrightarrow{\partial} J \to E \to B$ in $\mathcal{T}$. The triangulated category kk can be viewed as the desired algebraic counterpart of kk^{lca} in the sense of Cuntz.

Independent of [5] the author introduced in [10] various algebraic bivariant K-theories, all of which are realized in triangulated categories. In particular, he recovers the category kk of Cortiñas–Thom. However [10] does not study universal bivariant theories. The paper is to construct universal homotopy invariant, excisive algebraic bivariant K-theories. For this we use ideas and techniques developed in [10]. We start with a datum of an admissible category of algebras $\Re$ and a class $\mathfrak{F}$ of fibrations on it and then construct a triangulated category $D(\Re, \mathfrak{F})$ out of the datum $(\Re, \mathfrak{F})$ by inverting certain arrows which we call weak equivalences. There is a canonical functor $\Re \to D(\Re, \mathfrak{F})$. It is the universal functor from $\Re$ to a triangulated category $\mathcal{T}$ that is (polynomially) homotopy invariant and every extension $0 \to J \to E \to B \to 0$ in $\mathfrak{F}$ is mapped in a functorial way to a triangle $\Omega B \xrightarrow{\partial} J \to E \to B$ in $\mathcal{T}$ (Theorem 2.6).

It should be emphasized that we do not consider any matrix-invariance in general. This is one of the most important features of the paper. This is caused by the fact that many interesting admissible categories of algebras deserving to be considered separately like that of all commutative ones are not closed under matrices. The most important classes of fibrations in practice are $\mathfrak{F}_{\mathrm{spl}}$ or $\mathfrak{F}_{\mathrm{surj}}$. We call $D(\Re, \mathfrak{F}_{\mathrm{spl}})$ and $D(\Re, \mathfrak{F}_{\mathrm{surj}})$ the *unstable algebraic KK- and E-theories* respectively. One could also call them *"dematricized"* K-theories or K-theories *"without matrices"* as they do not involve any matrices at all.

We next introduce matrices into the game and study universal bivariant, excisive, homotopy invariant *and* "Morita invariant" algebraic K-theories. For this a triangulated category $D_{mor}(\Re, \mathfrak{F})$ is constructed in an explicit way. More precisely, the objects of $D_{mor}(\Re, \mathfrak{F})$ are those of $\Re$ and the set of morphisms between two algebras $A, B \in \Re$ is defined as the colimit of the sequence of abelian groups

$$D(\Re, \mathfrak{F})(A, B) \to D(\Re, \mathfrak{F})(A, M_2 B) \to D(\Re, \mathfrak{F})(A, M_3 B) \to \cdots$$

There is a canonical functor $\Re \to D_{mor}(\Re, \mathfrak{F})$. It is the universal functor from $\Re$ to a triangulated category $\mathcal{T}$ that is (polynomially) homotopy invariant, Morita invariant (i.e., $\mathcal{T}(A, M_nB) \cong \mathcal{T}(A, B)$ for all n), and every extension $0 \to J \to E \to B \to 0$ in $\mathfrak{F}$ is mapped in a functorial way to a triangle $\Omega B \xrightarrow{\partial} J \to E \to B$ in $\mathcal{T}$ (Theorem 6.5). We call $D_{mor}(\Re, \mathfrak{F}_{\mathrm{spl}})$ and $D_{mor}(\Re, \mathfrak{F}_{\mathrm{surj}})$ the *Morita stable algebraic KK- and E-theories* respectively.

And finally, we introduce and study universal bivariant, excisive, homotopy invariant *and* M_∞-invariant (or "stable") algebraic K-theories. For this a triangulated category $D_{st}(\Re, \mathfrak{F})$ is constructed in an explicit way. More precisely, the objects of $D_{st}(\Re, \mathfrak{F})$ are those of $\Re$ and the set of morphisms between two algebras $A, B \in \Re$ is defined as the colimit of the sequence of abelian groups

$$D(\Re, \mathfrak{F})(A, B) \to D(\Re, \mathfrak{F})(A, M_\infty k \otimes_k B) \to D(\Re, \mathfrak{F})(A, M_\infty k \otimes_k M_\infty k \otimes_k B) \to \cdots$$

There is a canonical functor $\Re \to D_{st}(\Re, \mathfrak{F})$. It is the universal functor from $\Re$ to a triangulated category $\mathcal{T}$ that is (polynomially) homotopy invariant, M_∞-invariant, and every extension $0 \to J \to E \to B \to 0$ in $\mathfrak{F}$ is mapped in a functorial way to a triangle $\Omega B \xrightarrow{\partial} J \to E \to B$ in $\mathcal{T}$ (Theorem 9.3). As above, we call $D_{st}(\Re, \mathfrak{F}_{\mathrm{spl}})$ and $D_{st}(\Re, \mathfrak{F}_{\mathrm{surj}})$ the *stable algebraic KK- and E-theories* respectively.

We show that morphisms in $D_{mor}(\Re, \mathfrak{F})$ and $D_{st}(\Re, \mathfrak{F})$ are closely related to additive categories of correspondences of Kassel [19] and Grayson [15]. We also introduce an additive category of correspondences kh for non-unital algebras and show that

$$\mathrm{colim}_n\, kh(k, \Sigma^n \Omega^n A) \cong KH_0(A)$$

for any $A \in \mathrm{Alg}_k$, where Ω, Σ are the loop and suspension functors, $KH_0(A)$ is the zeroth homotopy K-theory group in the sense of Weibel [24] (Theorem 10.2).

We also prove a result stating when $D_{st}(\Re, \mathfrak{F})(k, A)$ are computed by means of correspondences (Theorem 10.6). This is an extension of an important computational result by Cortiñas–Thom saying that

$$kk_*(k, A) \cong KH_*(A),$$

where the right hand side is homotopy K-theory in the sense of Weibel [24].

One should mention that our approach is entirely different from that of Cuntz–Cortiñas–Thom. In particular, it allows to consider very general classes of algebras $\Re$ and fibrations $\mathfrak{F}$ on them.

Throughout the paper k is a fixed commutative ring with unit and Alg_k is the category of non-unital k-algebras and non-unital k-homomorphisms. By Alg_k^u we denote the full subcategory in Alg_k of unital algebras. If there is no likelihood of confusion, we replace $\otimes_k$ by $\otimes$. If $\mathcal{C}$ is a category and A, B are objects of $\mathcal{C}$, we shall often write $\mathcal{C}(A, B)$ to denote the Hom-set $\mathrm{Hom}_{\mathcal{C}}(A, B)$.

In general, we shall not be very explicit about set-theoretical foundations, and we shall tacitly assume we are working in some fixed universe $\mathbb{U}$ of sets. Members of $\mathbb{U}$ are then called *small sets*, whereas a collection of members of $\mathbb{U}$ which does not itself belong to $\mathbb{U}$ will be referred to as a *large set* or a *proper class*.

1. Preliminaries

This section gives the preliminaries to the definition of triangulated categories for algebras which we shall introduce in the next section.

1.1. *Algebraic homotopy*

Following Gersten [12] a category of non-unital k-algebras $\Re$ is *admissible* if it is a full subcategory of Alg_k and

(1) R in $\Re$, I a (two-sided) ideal of R then I and R/I are in $\Re$;
(2) if R is in $\Re$, then so is $R[x]$, the polynomial algebra in one variable;
(3) given a cartesian square

$$\begin{tikzcd} D \arrow[r, "\rho"] \arrow[d, "\sigma"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd}$$

in Alg_k with A, B, C in $\Re$, then D is in $\Re$.

Observe that every algebra which is isomorphic to an algebra from $\Re$ belongs to $\Re$. One may abbreviate 1, 2, and 3 by saying that $\Re$ is closed under operations of taking ideals, homomorphic images, polynomial extensions in a finite number of variables, and pullbacks. For instance, the category of commutative k-algebras CAlg_k is admissible.

Recall that an algebra A is *square zero* if $A^2 = 0$. If we regard every k-module M as a non-unital k-algebra with trivial multiplication $m_1 \cdot m_2 = 0$ for all $m_1, m_2 \in M$, then $\mathrm{Mod}\, k$ is an admissible category of k-algebras coinciding with the category of square zero algebras.

If R is an algebra then the polynomial algebra $R[x]$ admits two homomorphisms onto R

$$\begin{tikzcd} R[x] \arrow[r, shift left, "\partial_x^0"] \arrow[r, shift right, "\partial_x^1"'] & R \end{tikzcd}$$

where

$$\partial_x^i|_R = 1_R, \quad \partial_x^i(x) = i, \quad i = 0, 1.$$

Of course, $\partial_x^1(x) = 1$ has to be understood in the sense that $\Sigma r_n x^n \mapsto \Sigma r_n$.

Definition. Two homomorphisms $f_0, f_1 : S \to R$ are *elementary homotopic*, written $f_0 \sim f_1$, if there exists a homomorphism

$$f : S \to R[x]$$

such that $\partial_x^0 f = f_0$ and $\partial_x^1 f = f_1$. A map $f : S \to R$ is called an *elementary homotopy equivalence* if there is a map $g : R \to S$ such that fg and gf are elementary homotopic to id_R and id_S respectively.

For example, let A be a $\mathbb{Z}_{n\geqslant 0}$-graded algebra, then the inclusion $A_0 \to A$ is an elementary homotopy equivalence. The homotopy inverse is given by the projection $A \to A_0$. Indeed, the map $A \to A[x]$ sending a homogeneous element $a_n \in A_n$ to $a_n t^n$ is a homotopy between the composite $A \to A_0 \to A$ and the identity id_A.

The relation "elementary homotopic" is reflexive and symmetric [12, p. 62]. One may take the transitive closure of this relation to get an equivalence relation (denoted by the symbol "$\simeq$"). Following notation of Gersten [13], the set of equivalence classes of morphisms $R \to S$ is written $[R, S]$.

Lemma 1.1 (Gersten [13])**.** *Given morphisms in* Alg_k

$$\begin{tikzcd} R \arrow[r, "f"] & S \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] & T \arrow[r, "h"] & U \end{tikzcd}$$

such that $g \simeq g'$*, then* $gf \simeq g'f$ *and* $hg \simeq hg'$*.*

Thus homotopy behaves well with respect to composition and we have category $\mathcal{H}(\mathrm{Alg}_k)$, the *homotopy category of k-algebras*, whose objects are k-algebras and $\mathrm{Hom}_{\mathcal{H}(\mathrm{Alg}_k)}(R, S) = [R, S]$. The homotopy category of an admissible category of algebras $\Re$ will be denoted by $\mathcal{H}(\Re)$. Call a homomorphism $s : A \to B$ an *I-weak equivalence* if its image in $\mathcal{H}(\Re)$ is an isomorphism.

The diagram in Alg_k

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is a short exact sequence if f is injective ($\equiv \operatorname{Ker} f = 0$), g is surjective, and the image of f is equal to the kernel of g.

Definition. An algebra R is *contractible* if $0 \sim 1$; that is, if there is a homomorphism $f : R \to R[x]$ such that $\partial_x^0 f = 0$ and $\partial_x^1 f = 1_R$.

For example, every square zero algebra $A \in \mathrm{Alg}_k$ is contractible by means of the homotopy $A \to A[x]$, $a \in A \mapsto ax \in A[x]$. Therefore every k-module, regarded as a k-algebra with trivial multiplication, is contractible.

Following Karoubi and Villamayor [18] we define ER, the *path algebra* on R, as the kernel of $\partial_x^0 : R[x] \to R$, so $ER \to R[x] \xrightarrow{\partial_x^0} R$ is a short exact sequence in Alg_k. Also $\partial_x^1 : R[x] \to R$ induces a surjection

$$\partial_x^1 : ER \to R$$

and we define the *loop algebra* ΩR of R to be its kernel, so we have a short exact sequence in Alg_k

$$\Omega R \to ER \xrightarrow{\partial_x^1} R.$$

We call it the *loop extension* of R. Clearly, ΩR is the intersection of the kernels of ∂_x^0 and ∂_x^1. By [12, 3.3] ER is contractible for any algebra R.

1.2. *Fibrations of algebras*

Definition. Let $\Re$ be an admissible category of algebras. A family $\mathfrak{F}$ of surjective homomorphisms of $\Re$ is called *fibrations* if it meets the following axioms:

Ax 1) for each R in $\Re$, $R \to 0$ is in $\mathfrak{F}$;
Ax 2) $\mathfrak{F}$ is closed under composition and any isomorphism is a fibration;
Ax 3) if the diagram

$$\begin{tikzcd} D \arrow[r, "\rho"] \arrow[d, "\sigma"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd}$$

is cartesian in $\mathfrak{R}$ and $g \in \mathfrak{F}$, then $\rho \in \mathfrak{F}$;

Ax 4) any map u in $\mathfrak{R}$ can be factored $u = pi$, where p is a fibration and i is an I-weak equivalence.

We call a short exact sequence in $\mathfrak{R}$

$$A \xrightarrow{g} B \xrightarrow{f} C$$

with $f \in \mathfrak{F}$ a $\mathfrak{F}$-*fibre sequence.*

$\mathfrak{F}$ is said to be *saturated* if the homomorphism $\partial_x^1 : EA \to A$ is a fibration for any $A \in \mathfrak{R}$. It is *tensor closed* if for any fibration p and any $D \in \mathfrak{R}$ the sequence

$$\operatorname{Ker} p \otimes D \xrightarrow{\ker p \otimes 1} B \otimes D \xrightarrow{p \otimes 1} C \otimes D$$

is a $\mathfrak{F}$-fibre sequence.

The trivial case is $\mathfrak{R} = \mathfrak{F} = 0$. A non-trivial example, $\mathfrak{R} \neq 0$, of fibrations is given by the surjective homomorphisms. Another important example of fibrations is defined by any left exact functor. Recall that a functor $F : \mathrm{Alg}_k \to \mathit{Sets}$ is *left exact* if F preserves finite limits. In particular, if $A \to B \to C$ is a short exact sequence in Alg_k, then

$$0 \to FA \to FB \to FC$$

is an exact sequence of pointed sets (since the zero algebra is a zero object in Alg_k, it determines a unique element of FA). Furthermore F preserves cartesian squares.

For instance, any representable functor is left exact as well as the functor (see Gersten [12])

$$R \in \mathrm{Alg}_k \longmapsto GL(R).$$

Definition. A surjective map $g : B \to C$ is said to be a F-*fibration* (where $F : \mathrm{Alg}_k \to \mathit{Sets}$ is a functor) if $F(E^n(g)) : FE^nB \to FE^nC$ is surjective for all $n > 0$. Observe that nothing is said about $F(g) : FB \to FC$. It follows that if the composite fg of two maps is a F-fibration, then so is f. If $F = GL$ we refer to F-fibrations as GL-*fibrations.* We also note that the family $\mathfrak{F}_{\mathrm{surj}}$ of all surjective homomorphisms is the family of F-fibrations with F sending an algebra A to its underlying set.

By [10, 4.1] the collection of F-fibrations, where $F : \mathfrak{R} \to \mathit{Sets}$ is left exact, enjoys the axioms Ax 1)-4) for fibrations on $\mathfrak{R}$ and is saturated. Similarly, it can be checked that the collection of surjective k-split homomorphisms $\mathfrak{F}_{\mathrm{spl}}$ forms a saturated family of fibrations. $\mathfrak{F}_{\mathrm{spl}}$ is plainly tensor closed. Observe that if k is a field then $\mathfrak{F}_{\mathrm{spl}} = \mathfrak{F}_{\mathrm{surj}}$.

There are plenty of fibrations between $\mathfrak{F}_{\mathrm{spl}}$ and $\mathfrak{F}_{\mathrm{surj}}$. For example, every proper class ω in the category of k-modules in the sense of [11] gives rise to a class of fibrations $\mathfrak{F}_{\mathrm{spl}} \subseteq \mathfrak{F}_\omega \subseteq \mathfrak{F}_{\mathrm{surj}}$. A basic example is the class $\mathfrak{F}_{\mathrm{pure}}$ of those k-algebra homomorphisms which are pure epimorphisms in $\operatorname{Mod} k$.

1.3. *Categories of fibrant objects*

Definition. Let $\mathcal{A}$ be a category with finite products and a final object e. Assume that $\mathcal{A}$ has two distinguished classes of maps, called *weak equivalences* and *fibrations.* A map is called a *trivial fibration* if it is both a weak equivalence and a fibration. We define a *path space* for an object B to be an object B^I together with maps

$$B \xrightarrow{s} B^I \xrightarrow{(d_0,d_1)} B \times B,$$

where s is a weak equivalence, (d_0, d_1) is a fibration, and the composite is the diagonal map.

Following Brown [2], we call $\mathcal{A}$ a *category of fibrant objects* or a *Brown category* if the following axioms are satisfied.

(A) Let f and g be maps such that gf is defined. If two of f, g, gf are weak equivalences then so is the third. Any isomorphism is a weak equivalence.

(B) The composite of two fibrations is a fibration. Any isomorphism is a fibration.

(C) Given a diagram

$$A \xrightarrow{u} C \xleftarrow{v} B,$$

with v a fibration (respectively a trivial fibration), the pullback $A \times_C B$ exists and the map $A \times_C B \to A$ is a fibration (respectively a trivial fibration).

(D) For any object B in $\mathcal{A}$ there exists at least one path space B^I (not necessarily functorial in B).

(E) For any object B the map $B \to e$ is a fibration.

2. Triangulated categories of algebras

In this section we want to introduce a triangulated category $D(\Re, \mathfrak{F}, \mathcal{W})$ associated with a triple $(\Re, \mathfrak{F}, \mathcal{W})$, where $\Re$ is an admissible category of algebras, $\mathfrak{F}$ is a saturated family of fibrations on it and $\mathcal{W}$ is an arbitrary class of weak equivalences containing $A \to A[t]$, $A \in \Re$, such that $(\Re, \mathfrak{F}, \mathcal{W})$ is a Brown category. The category $D(\Re, \mathfrak{F}, \mathcal{W})$ was first constructed in [10] for the case when $\mathcal{W}$ is the class of "$\mathfrak{F}$-quasi-isomorphisms". In practice we have to work with various families of weak equivalences. For this reason we need to introduce $D(\Re, \mathfrak{F}, \mathcal{W})$ for quite general classes of weak equivalences. Though the construction of $D(\Re, \mathfrak{F}, \mathcal{W})$ is very close to that in [10] for "$\mathfrak{F}$-quasi-isomorphisms", we give it here from scratch in order to be sure that nothing goes wrong. The main class of weak equivalences $\mathcal{W}_\triangle$ we work with is defined by means of excisive, homotopy invariant homology theories. We start with preparations.

2.1. *Left derived categories*

Recall that a pair (C, W) of a category C and a class of morphisms W is said to admit a calculus of right fractions if the following properties hold.

- W contains all identities and is closed under composition.
- Given an arrow $v : x \to z$ in W and any arrow $f : y \to z$, there is an arrow $v' : w \to y$ in W and an arrow $f' : w \to x$ in C such that $fv' = vf'$.
- Given an arrow $v : y \to z$ in W and a pair of parallel morphisms $f, g : x \to y$ such that $vf = vg$, there is an arrow $u : w \to x$ in W such that $fu = gu$.

If $(C^{\mathrm{op}}, W^{\mathrm{op}})$ admits a calculus of right fractions, we say that (C, W) admits a calculus of left fractions.

Let $\mathfrak{F}$ be a saturated family of fibrations in an admissible not necessarily small category of algebras $\Re$ and let $\mathcal{W}$ be a class of weak equivalences containing homomorphisms $A \to A[t]$, $A \in \Re$, such that the triple $(\Re, \mathfrak{F}, \mathcal{W})$ is a Brown category.

Definition. The *left derived category* $D^-(\Re, \mathfrak{F}, \mathcal{W})$ of $\Re$ with respect to $(\mathfrak{F}, \mathcal{W})$ is the category obtained from $\Re$ by inverting the weak equivalences.

Proposition 2.1. *The family of weak equivalences in the category $\mathcal{H}\Re$ admits a calculus of right fractions. The left derived category $D^-(\Re, \mathfrak{F}, \mathcal{W})$ is obtained from $\mathcal{H}\Re$ by inverting the weak equivalences.*

Proof. The proof is similar to that of [10, Prop. 5.2]. □

We should remark that if the category $\Re$ is not small then the left derived category $D^-(\Re, \mathfrak{F}, \mathcal{W})$ is possibly "large", i.e. for any given pair of algebras $A, B \in \Re$ the equivalence classes of fractions $D^-(\Re, \mathfrak{F}, \mathcal{W})(A, B)$ do not form a small set.

Below we shall need the following

Lemma 2.2. *Let $A \overset{i}{\rightarrowtail} B \overset{p}{\twoheadrightarrow} C$ be a $\mathfrak{F}$-fibre sequence with C weakly equivalent to zero. Then i is a weak equivalence.*

Proof. Consider a commutative diagram

$$\begin{tikzcd} B \arrow[r, two heads, "p"] \arrow[d, equal] & C \arrow[d] & 0 \arrow[l] \arrow[d, equal] \\ B \arrow[r] & 0 & 0 \arrow[l] \end{tikzcd}$$

with left vertical arrows fibrations. Our assertion now follows from [14, II.9.10]. □

The endofunctor $\Omega : \Re \to \Re$ respects weak equivalences. Indeed, let $f : A \to B$ be a weak equivalence. Consider the following commutative diagram:

$$\begin{tikzcd} \Omega A \arrow[r, tail] \arrow[d, "\Omega f"'] & EA \arrow[r, two heads, "\partial_x^1"] \arrow[d, "E(f)"'] & A \arrow[d, "f"] \\ \Omega B \arrow[r, tail] & EB \arrow[r, two heads, "\partial_x^1"] & B \end{tikzcd}$$

Since EA, EB are isomorphic to zero in $D^-(\Re, \mathfrak{F}, \mathcal{W})$, it follows that $E(f)$ is a weak equivalence. Then Ωf is a weak equivalence by [2, §4, Lemma 3]. Thus Ω can be regarded as an endofunctor of $D^-(\Re, \mathfrak{F}, \mathcal{W})$. It can be shown similar to [10, 5.4] that if $\mathfrak{F}$ is a saturated family of fibrations, then ΩA is a group object in $D^-(\Re, \mathfrak{F}, \mathcal{W})$. We also refer the reader to [2, §4, Cor. 2].

A group structure on ΩA is explicitly constructed as follows (see [10]). Let $B[x] \times_B B[x] := \{(f(x), g(x)) \mid f(1) = g(0)\}$ and let $\widetilde{\Omega} B$ be the kernel of $(d_0, d_1) : B[x] \times_B B[x] \to B \times B$, $(f(x), g(x)) \mapsto (f(0), g(1))$. Consider the homomorphism $\alpha : \Omega B \to \widetilde{\Omega} B$, $f(x) \longmapsto (f(x), 0)$. Using Lemma 2.2 and the proof of [10, 6.1], it is a weak equivalence. Let $\omega : \Omega A \times \Omega A \to \widetilde{\Omega} A$ be the evident map induced by the pullback property. Then

$$\mu_A := \alpha^{-1}\omega : \Omega A \times \Omega A \to \Omega A$$

determines a group structure on ΩA (see [10, section 6]). For any $n \geqslant 2$ the algebra $\Omega^n A$ is an abelian group object in $D^-(\Re, \mathfrak{F}, \mathcal{W})$ (see [10, 6.5]).

Given a fibration $g : A \to B$ with kernel F, consider the commutative diagram as follows:

$$\begin{tikzcd}
 & \Omega B \arrow[r, equal] \arrow[d, tail, "j"'] & \Omega B \arrow[d, tail] \\
F \arrow[r, tail, "i"] \arrow[d, equal] & P(g) \arrow[r, two heads] \arrow[d, "g_1"] & EB \arrow[d, "\partial_x^1"] \\
F \arrow[r, tail, "\iota"] & A \arrow[r, two heads, "g"] & B
\end{tikzcd}$$

Since EB is contractible, it follows from Lemma 2.2 that i is a weak equivalence. We deduce the sequence in $D^-(\Re, \mathfrak{F}, \mathcal{W})$

$$\Omega B \xrightarrow{i^{-1} \circ j} F \xrightarrow{\iota} A \xrightarrow{g} B. \tag{1}$$

We shall refer to such sequences as *standard left triangles*. Any diagram in $D^-(\Re, \mathfrak{F}, \mathcal{W})$ which is isomorphic to the latter sequence will be called a *left triangle*. One must be careful to note that $\Omega B' \to F' \longrightarrow A' \longrightarrow B'$ is isomorphic to a standard triangle (1) if and only if there is a commutative diagram

$$\begin{tikzcd}
\Omega B \arrow[r] \arrow[d, "\Omega b"'] & F \arrow[r] \arrow[d, "f"'] & A \arrow[r] \arrow[d, "a"] & B \arrow[d, "b"] \\
\Omega B' \arrow[r] & F' \arrow[r] & A' \arrow[r] & B'
\end{tikzcd}$$

with f, a, b isomorphisms in $D^-(\Re, \mathfrak{F}, \mathcal{W})$.

It follows that the diagram

$$\Omega B \xrightarrow{j} P(g) \xrightarrow{g_1} A \xrightarrow{g} B \tag{2}$$

is a left triangle. If g is not a fibration then g is factored as $g = g'\ell$ with g' a fibration and ℓ a weak equivalence. We get a commutative diagram

$$\begin{tikzcd}
\Omega B \arrow[r] \arrow[d, equal] & P(g) \arrow[r] \arrow[d, "t"] & A \arrow[r, "g"] \arrow[d, "\ell"] & B \arrow[d, equal] \\
\Omega B \arrow[r] & P(g') \arrow[r] & A' \arrow[r, "g'"] & B.
\end{tikzcd}$$

If $\mathfrak{F}$ is a saturated family of fibrations, then the arrow t is a weak equivalence by [14, II.9.10]. Hence the upper sequence of the diagram is a left triangle. This also verifies that any map in $D^-(\Re, \mathfrak{F})$ fits into a left triangle.

For any algebra A the automorphism $\sigma = \sigma_A : \Omega A \to \Omega A$ takes a polynomial $a(x)$ to $a(1-x)$. Notice that σ is functorial in A and $\sigma^2 = 1$. Given a morphism α in $D^-(\Re, \mathfrak{F}, \mathcal{W})$, by $-\Omega\alpha$ denote the morphism $\Omega\alpha \circ \sigma = \sigma \circ \Omega\alpha$. For any $n \geqslant 1$ the morphism $(-1)^n \Omega\alpha$ means $\sigma^n \Omega\alpha$.

The proof of the following result literally repeats that of [10, 5.6].

Theorem 2.3. *Let $\mathfrak{F}$ be a saturated family of fibrations in an admissible not necessarily small category of algebras $\Re$ and let $\mathcal{W}$ be a class of weak equivalences containing homomorphisms $A \to A[t]$, $A \in \Re$, such that the triple $(\Re, \mathfrak{F}, \mathcal{W})$ is a Brown category. Denote by $\mathcal{L}tr(\Re, \mathfrak{F}, \mathcal{W})$ the category of left triangles having the usual set of morphisms from $\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$ to $\Omega C' \xrightarrow{f'} A' \xrightarrow{g'} B' \xrightarrow{h'} C'$. Then $\mathcal{L}tr(\Re, \mathfrak{F}, \mathcal{W})$ is* *a left triangulation in the sense of Beligiannis-Marmaridis* [1] *of the (possibly "large") category $D^-(\mathfrak{R}, \mathfrak{F}, \mathcal{W})$, i.e. it is closed under isomorphisms and enjoys the following four axioms:*

(LT1) *for any $A \in \mathfrak{R}$ the left triangle $0 \xrightarrow{0} A \xrightarrow{1_A} A \xrightarrow{0} 0$ belongs to $\mathcal{L}tr(\mathfrak{R}, \mathfrak{F}, \mathcal{W})$ and for any morphism $h : B \to C$ there is a left triangle in $\mathcal{L}tr(\mathfrak{R}, \mathfrak{F}, \mathcal{W})$ of the form $\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$;*

(LT2) *for any left triangle $\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$ in $\mathcal{L}tr(\mathfrak{R}, \mathfrak{F}, \mathcal{W})$, the diagram $\Omega B \xrightarrow{-\Omega h} \Omega C \xrightarrow{f} A \xrightarrow{g} B$ is also in $\mathcal{L}tr(\mathfrak{R}, \mathfrak{F}, \mathcal{W})$;*

(LT3) *for any two left triangles $\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$, $\Omega C' \xrightarrow{f'} A' \xrightarrow{g'} B' \xrightarrow{h'} C'$ in $\mathcal{L}tr(\mathfrak{R}, \mathfrak{F}, \mathcal{W})$ and any two morphisms $\beta : B \to B'$ and $\gamma : C \to C'$ of $D^-(\mathfrak{R}, \mathfrak{F}, \mathcal{W})$ with $\gamma h = h'\beta$, there is a morphism $\alpha : A \to A'$ of $D^-(\mathfrak{R}, \mathfrak{F}, \mathcal{W})$ such that the triple (α, β, γ) gives a morphism from the first triangle to the second;*

(LT4) *any two morphisms $B \xrightarrow{h} C \xrightarrow{k} D$ of $D^-(\mathfrak{R}, \mathfrak{F}, \mathcal{W})$ can be fitted into a commutative diagram*

$$\begin{tikzcd}
 & \Omega E \arrow[d, "f\circ\Omega\ell"] & & \\
\Omega C \arrow[r, "f"] \arrow[d, "\Omega k"] & A \arrow[r, "g"] \arrow[d, "\alpha"] & B \arrow[r, "h"] \arrow[d, "1_B"'] & C \arrow[d, "k"] \\
\Omega D \arrow[r, "j"] \arrow[d, "1_{\Omega D}"] & F \arrow[r, "m"] \arrow[d, "\beta"] & B \arrow[r, "kh"] \arrow[d, "h"'] & D \arrow[d, "1_D"] \\
\Omega D \arrow[r, "i"] & E \arrow[r, "\ell"] & C \arrow[r, "k"] & D
\end{tikzcd}$$

in which the rows and the second column from the left are left triangles in $\mathcal{L}tr(\mathfrak{R}, \mathfrak{F}, \mathcal{W})$.

The axiom (LT4) is a version of Verdier's octahedral axiom for left triangles in $D^-(\mathfrak{R}, \mathfrak{F}, \mathcal{W})$.

2.2. *Stabilization*

Let $\mathfrak{R}$ be an admissible category of algebras and $\mathfrak{F}$ a saturated family of fibrations. There is a general method of stabilizing the loop functor Ω (see Heller [16]) and producing a triangulated (possibly "large") category $D(\mathfrak{R}, \mathfrak{F}, \mathcal{W})$ from the left triangulated structure on $D^-(\mathfrak{R}, \mathfrak{F}, \mathcal{W})$.

An object of $D(\mathfrak{R}, \mathfrak{F}, \mathcal{W})$ is a pair (A, m) with $A \in D^-(\mathfrak{R}, \mathfrak{F}, \mathcal{W})$ and $m \in \mathbb{Z}$. If $m, n \in \mathbb{Z}$ then we consider the directed set $I_{m,n} = \{k \in \mathbb{Z} \mid m, n \leqslant k\}$. The set of morphisms between (A, m) and $(B, n) \in D(\mathfrak{R}, \mathfrak{F}, \mathcal{W})$ is defined by

$$D(\mathfrak{R}, \mathfrak{F}, \mathcal{W})[(A, m), (B, n)] := \varinjlim_{k \in I_{m,n}} D^-(\mathfrak{R}, \mathfrak{F}, \mathcal{W})(\Omega^{k-m}(A), \Omega^{k-n}(B)).$$

Morphisms of $D(\mathfrak{R}, \mathfrak{F}, \mathcal{W})$ are composed in the obvious fashion. We define the *loop* automorphism on $D(\mathfrak{R}, \mathfrak{F}, \mathcal{W})$ by $\Omega(A, m) := (A, m - 1)$. There is a natural functor $S : D^-(\mathfrak{R}, \mathfrak{F}, \mathcal{W}) \to D(\mathfrak{R}, \mathfrak{F}, \mathcal{W})$ defined by $A \longmapsto (A, 0)$.

Since $\Omega^n B$ is an abelian group object for $n \geqslant 2$ it follows that $D(\Re, \mathfrak{F}, \mathcal{W})[(A, m), (B, n)]$ is an abelian group and the category $D(\Re, \mathfrak{F}, \mathcal{W})$ is preadditive. Since it has finite direct products then it is additive. We define a triangulation $\mathcal{T}r(\Re, \mathfrak{F}, \mathcal{W})$ of the pair $(D(\Re, \mathfrak{F}, \mathcal{W}), \Omega)$ as follows. A sequence

$$\Omega(A, l) \to (C, n) \to (B, m) \to (A, l)$$

belongs to $\mathcal{T}r(\Re, \mathfrak{F}, \mathcal{W})$ if there is an even integer k and a left triangle of representatives $\Omega(\Omega^{k-l}(A)) \to \Omega^{k-n}(C) \to \Omega^{k-m}(B) \to \Omega^{k-l}(A)$ in $D^-(\Re, \mathfrak{F}, \mathcal{W})$. Clearly, the functor S takes left triangles in $D^-(\Re, \mathfrak{F}, \mathcal{W})$ to triangles in $D(\Re, \mathfrak{F}, \mathcal{W})$.

Theorem 2.4. *Let $\mathfrak{F}$ be a saturated family of fibrations in $\Re$. Then $\mathcal{T}r(\Re, \mathfrak{F}, \mathcal{W})$ is a triangulation of $D(\Re, \mathfrak{F}, \mathcal{W})$ in the classical sense of Verdier* [23].

Proof. See [10, 6.7]. $\square$

2.3. *Universal properties*

Let $\mathfrak{F}$ be a saturated family of fibrations in an admissible not necessarily small category of algebras $\Re$ and let $\mathcal{E}$ be the class of all $\mathfrak{F}$-fibre sequences of k-algebras

$$(E) : A \to B \to C. \tag{3}$$

Definition. Following Cortiñas–Thom [5] a *($\mathfrak{F}$-)excisive homology theory* on $\Re$ with values in a triangulated category $(\mathcal{T}, \Omega)$ consists of a functor $X : \Re \to \mathcal{T}$, together with a collection $\{\partial_E : E \in \mathcal{E}\}$ of maps $\partial_E^X = \partial_E \in \mathcal{T}(\Omega X(C), X(A))$. The maps ∂_E are to satisfy the following requirements.

(1) For all $E \in \mathcal{E}$ as above,

$$\Omega X(C) \xrightarrow{\partial_E} X(A) \xrightarrow{X(f)} X(B) \xrightarrow{X(g)} X(C)$$

is a distinguished triangle in $\mathcal{T}$.

(2) If

$$\begin{tikzcd}
(E): & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[d, "\gamma"'] \\
(E'): & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C'
\end{tikzcd}$$

is a map of $\mathfrak{F}$-fibre sequences, then the following diagram commutes

$$\begin{tikzcd}
\Omega X(C) \arrow[r, "\partial_E"] \arrow[d, "\Omega X(\gamma)"'] & X(A) \arrow[d, "X(\alpha)"] \\
\Omega X(C') \arrow[r, "\partial_{E'}"'] & X(A).
\end{tikzcd}$$

We say that the functor $X : \Re \to \mathcal{T}$ is *homotopy invariant* if it maps homotopic homomomorphisms to equal maps, or equivalently, if for every $A \in \mathrm{Alg}_k$, X maps the inclusion $A \subset A[t]$ to an isomorphism.

We shall denote the class of homomorphisms f such that $X(f)$ is an isomorphism for any excisive, homotopy invariant homology theory $X : \Re \to \mathcal{T}$ by $\mathcal{W}_\triangle$ ("$\triangle$" has the meaning that this class is determined by triangulated categories).

Lemma 2.5. *The triple* $(\mathfrak{R}, \mathfrak{F}, \mathcal{W}_\triangle)$ *is a Brown category.*

Proof. We have to verify axioms (A)-(E) from subsection 1.3. Axiom (A) is obvious. Axioms (B), (D), (E) follow from axioms Ax 2, Ax 4, and Ax 1 respectively (see subsection 1.2). Ax 3 implies that a pullback of a fibration is a fibration. The fact that a pullback of a trivial fibration is a trivial fibration follows from standard facts for triangulated categories. □

In what follows we shall write $D^-(\mathfrak{R}, \mathfrak{F})$ and $D(\mathfrak{R}, \mathfrak{F})$ to denote the categories $D^-(\mathfrak{R}, \mathfrak{F}, \mathcal{W}_\triangle)$ and $D(\mathfrak{R}, \mathfrak{F}, \mathcal{W}_\triangle)$, dropping $\mathcal{W}_\triangle$ from notation. In this paragraph we discuss universal properties of $D^-(\mathfrak{R}, \mathfrak{F})$ and $D(\mathfrak{R}, \mathfrak{F})$.

Theorem 2.6. *Let* $X : \mathfrak{R} \to \mathcal{T}$ *be an excisive, homotopy invariant homology theory. Then the following statements are true:*

(1) *there is a unique functor* $\bar{X} : D^-(\mathfrak{R}, \mathfrak{F}) \to \mathcal{T}$ *such that* $X = \bar{X} \circ j$ *with* $j : \mathfrak{R} \to D^-(\mathfrak{R}, \mathfrak{F})$ *the canonical functor. Moreover,* $\bar{X}$ *takes left triangles in* $D^-(\mathfrak{R}, \mathfrak{F})$ *to triangles in* $\mathcal{T}$*;*

(2) *there is a unique functor* $\widetilde{X} : D(\mathfrak{R}, \mathfrak{F}) \to \mathcal{T}$ *such that* $\bar{X} = \widetilde{X} \circ S$ *with* $S : D^-(\mathfrak{R}, \mathfrak{F}) \to D(\mathfrak{R}, \mathfrak{F})$ *the canonical stabilization functor. Moreover,* $\widetilde{X}$ *is triangulated, i.e. it is additive and takes triangles in* $D(\mathfrak{R}, \mathfrak{F})$ *to triangles in* $\mathcal{T}$*.*

Proof. (1). Let $X : \mathfrak{R} \to \mathcal{T}$ be an excisive, homotopy invariant homology theory. By definition of $\mathcal{W}_\triangle$ the theory X takes each element of $\mathcal{W}_\triangle$ to an isomorphism, and hence there is a unique functor $\bar{X} : D^-(\mathfrak{R}, \mathfrak{F}) \to \mathcal{T}$ such that $X = \bar{X} \circ j$. Since the functor S takes left triangles to triangles, the fact that $\bar{X}$ takes left triangles to triangles follows from the second assertion we are going to prove.

(2). The fact that there is a unique functor $\widetilde{X} : D(\mathfrak{R}, \mathfrak{F}) \to \mathcal{T}$ such that $\bar{X} = \widetilde{X} \circ S$ follows from [16, 1.1]. It follows from the definition of an excisive, homotopy invariant homology theory that $\widetilde{X}$ is additive.

To show that it is triangulated, we follow some part of the proof of [5, 6.6.2]. We recall it here for convenience of the reader. Recall that the rotation axiom for triangulated categories says that a triangle in $\mathcal{T}$

$$\Omega W \xrightarrow{f} U \xrightarrow{g} V \xrightarrow{h} W$$

is distinguished if and only if so is the triangle

$$\Omega V \xrightarrow{-\Omega(h)} \Omega W \xrightarrow{-f} U \xrightarrow{-g} V.$$

In view of the fact that every left triangle is isomorphic to a left triangle of the form (2) and of the rotation axiom, it is enough to prove that if $g \in \mathrm{Hom}_{\mathrm{Alg}_k}(A, B)$, then X maps

$$\Omega A \xrightarrow{\Omega g} \Omega B \xrightarrow{j} P(g) \xrightarrow{g_1} A \qquad (4)$$

to a distinguished triangle in $\mathcal{T}$. Consider the $\mathfrak{F}$-fibre sequence E formed by j and g_1; we have a diagram

$$\begin{tikzcd}
& \Omega X(B) \arrow[d, "\partial_l"] & & \\
\Omega X(A) \arrow[ur, "\Omega X(g)"] \arrow[r, "\partial_E"'] \arrow[d, "\partial_l"] & X(\Omega B) \arrow[r, "X(j)"'] & X(P(g)) \arrow[r, "X(g_1)"] & X(A) \\
X(\Omega A) \arrow[ur, "X(\Omega g)"'] & & &
\end{tikzcd}$$

where ∂_l is the connecting map associated with the $\mathfrak{F}$-fibre sequence

$$\Omega B \rightarrowtail EB \overset{\partial_x^1}{\twoheadrightarrow} B.$$

In the diagram above the row is a distinguished triangle and the square on the left commutes, as do its upper and lower halves. It follows from this and the axioms of a triangulated category, that X applied to (4) is a distinguished triangle. $\square$

Corollary 2.7. *Let $\mathfrak{R}$ be an admissible category of algebras and let $\mathfrak{F}$ be a saturated family of fibrations. Then the canonical functor $j : \mathfrak{R} \to D^-(\mathfrak{R}, \mathfrak{F})$ reflects weak equivalences from $\mathcal{W}_\triangle$, that is if a homomorphism is an isomorphism in $D^-(\mathfrak{R}, \mathfrak{F})$ then it is in $\mathcal{W}_\triangle$.*

Proof. Let f be an arrow in $\mathfrak{R}$ such that $j(f)$ is an isomorphism. Theorem 2.6(1) implies that for any excisive, homotopy invariant homology theory $X : \mathfrak{R} \to \mathcal{T}$ the arrow $X(f) = \bar{X} \circ j(f)$ is an isomorphism, hence $f \in \mathcal{W}_\triangle$. $\square$

Corollary 2.8. *If $\mathfrak{F}$ is a tensor closed collection of fibrations then the tensor product $\otimes_k$ of algebras induces a tensor product on $D^-(\mathfrak{R}, \mathfrak{F})$ and $D(\mathfrak{R}, \mathfrak{F})$ making these symmetric monoidal categories. Moreover, $D \otimes -, - \otimes D$, $D \in \mathfrak{R}$, respect weak equivalences from $\mathcal{W}_\triangle$ and take (left) triangles to (left) triangles.*

Proof. To any $\mathfrak{F}$-fibre sequence $F \xrightarrow{\iota} A \xrightarrow{g} B$ in $\mathfrak{R}$ one associates a standard triangle

$$\Omega B \xrightarrow{\partial} F \xrightarrow{\iota} A \xrightarrow{g} B.$$

$\mathfrak{F}$ is tensor closed by assumption, and hence $F \otimes D \xrightarrow{\iota \otimes 1} A \otimes D \xrightarrow{g \otimes 1} B \otimes D$ is a $\mathfrak{F}$-fibre sequence in $\mathfrak{R}$ and

$$\Omega(B \otimes D) = \Omega B \otimes D \xrightarrow{\partial \otimes 1} F \otimes D \xrightarrow{\iota \otimes 1} A \otimes D \xrightarrow{g \otimes 1} B \otimes D$$

is a standard triangle. One easily sees that if

$$\begin{tikzcd}
(E): & F \arrow[r, "\iota"] \arrow[d, "\alpha"'] & A \arrow[r, "g"] \arrow[d, "\beta"'] & B \arrow[d, "\gamma"'] \\
(E'): & F' \arrow[r, "\iota'"] & A' \arrow[r, "g'"] & B'
\end{tikzcd}$$

is a map of $\mathfrak{F}$-fibre sequences, then the following diagram commutes

$$\begin{tikzcd}
\Omega(B \otimes D) \arrow[r, "\partial\otimes 1"] \arrow[d, "\Omega(\gamma\otimes 1)"'] & F \otimes D \arrow[d, "\alpha\otimes 1"] \\
\Omega(B' \otimes D) \arrow[r, "\partial\otimes 1"'] & A \otimes D.
\end{tikzcd}$$

It follows that if $X : \Re \to \mathcal{T}$ is an excisive, homotopy invariant homology theory, then so is $X \circ (- \otimes D) : \Re \to \mathcal{T}$. By the preceding corollary the functor $- \otimes D$ respects weak equivalences from $\mathcal{W}_\triangle$. Therefore $j \circ (- \otimes D) : \Re \to D^-(\Re, \mathfrak{F})$ (respectively $S \circ j \circ (- \otimes D) : \Re \to D(\Re, \mathfrak{F})$) factors through $D^-(\Re, \mathfrak{F})$ (respectively $D(\Re, \mathfrak{F})$). Clearly, $- \otimes D$ takes (left) triangles to (left) triangles. $\square$

2.4. *Unstable algebraic KK- and E-theories*

We would like to discuss separately the most important in practice cases when $\mathfrak{F}$ is either $\mathfrak{F}_{\mathrm{spl}}$ or $\mathfrak{F}_{\mathrm{surj}}$. Throughout this section $\Re$ is an admissible category of algebras. We assume fixed an underlying category $\mathcal{U}$, which can be a full subcategory of either the category of sets *Sets* or $\mathrm{Mod}\, k$. The category $\mathcal{U}$ will depend on $\mathfrak{F}$. Namely, we shall assume that $\mathcal{U} \subseteq Sets$ if $\mathfrak{F} = \mathfrak{F}_{\mathrm{surj}}$ and $\mathcal{U} \subseteq \mathrm{Mod}\, k$ if $\mathfrak{F} = \mathfrak{F}_{\mathrm{spl}}$.

Definition. Let $\Re$ be an admissible category of algebras and let $\mathfrak{F}$ be either $\mathfrak{F}_{\mathrm{spl}}$ or $\mathfrak{F}_{\mathrm{surj}}$. The pair $(\Re, \mathfrak{F})$ is said to be *T-closed* if we have a faithful forgetful functor $F : \Re \to \mathcal{U}$ and a functor $\widetilde{T} : \mathcal{U} \to \Re$ such that $\widetilde{T}$ is left adjoint to F.

Throughout this section $\Re$ is supposed to be T-closed.

Examples. (1) Let $\Re = \mathrm{Alg}_k$ and $\mathfrak{F} = \mathfrak{F}_{\mathrm{spl}}$. Given an algebra A, consider the algebraic tensor algebra

$$TA = A \oplus A \otimes A \oplus A^{\otimes^3} \oplus \cdots$$

with the usual product given by concatenation of tensors. In Cuntz's treatment of bivariant K-theory [7, 8, 9], tensor algebras play a prominent role.

There is a canonical k-linear map $A \to TA$ mapping A into the first direct summand. Every k-linear map $s : A \to B$ into an algebra B induces a homomorphism $\gamma_s : TA \to B$ defined by

$$\gamma_s(x_1 \otimes \cdots \otimes x_n) = s(x_1)s(x_2)\cdots s(x_n).$$

The pair $(\Re, \mathfrak{F})$ is plainly T-closed.

(2) If $\Re = \mathrm{CAlg}_k$ and $\mathfrak{F} = \mathfrak{F}_{\mathrm{spl}}$ then

$$T(A) = Sym(A) = \oplus_{n\geqslant 1} S^n A, \quad S^n A = A^{\otimes n}/\langle a_1 \otimes \cdots \otimes a_n - a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}\rangle,$$

the symmetric algebra of A, and the pair $(\Re, \mathfrak{F})$ is T-closed.

(3) Let $\Re = \mathrm{Alg}_k$ and $\mathfrak{F} = \mathfrak{F}_{\mathrm{surj}}$. Given an algebra A, let TA be the algebra consisting of those polynomials in the non-commuting variables x_a, $a \in A$, which have no constant term. Then the pair $(\Re, \mathfrak{F})$ is T-closed. Observe that $E(k) = T(0)$.

(4) Let $\Re = \mathrm{CAlg}_k$ and $\mathfrak{F} = \mathfrak{F}_{\mathrm{surj}}$. Given an algebra A, let TA be the algebra consisting of those polynomials in the commuting variables x_a, $a \in A$, which have no constant term. Then the pair $(\Re, \mathfrak{F})$ is T-closed.

Recall that Kasparov's bivariant K-theory KK is a bifunctor on pairs of C^*-algebras, associating to (A, B) a $\mathbb{Z}/2\mathbb{Z}$-graded group $KK_*(A, B)$. The Kasparov product

$$KK_i(A, B) \otimes KK_j(B, C) \to KK_{i+j}(A, C)$$

allows one to view the KK-groups as morphisms in a category whose objects are all separable C^*-algebras. KK is a triangulated category [20] and is universal for C^*-stable, split-exact homology theories on the category of C^*-algebras.

In turn, E-theory of C^*-algebras developed by Connes and Higson in [3, 17] is the universal bivariant homology theory satisfying excision for all extensions and stability. The triangulated category corresponding to E-theory is studied in [22].

Stabilization for non-unital algebras is the operation $A \in \mathrm{Alg}_k \mapsto M_\infty(A) = \cup_n M_n(A) \in \mathrm{Alg}_k$. It is not available for some interesting admissible categories of algebras like CAlg_k. Therefore the triangulated category $D(\Re, \mathfrak{F})$ can be regarded as a sort of unstable universal bivariant homology theory.

All these remarks justify the following

Definition. (1) The *unstable algebraic KK-theory* for $\Re$ is the triangulated category $D(\Re, \mathfrak{F}_{\mathrm{spl}})$. The *unstable algebraic KK-groups* are, by definition,

$$\mathfrak{K}\mathfrak{K}_n^{\mathrm{unst}}(A, B) := D(\Re, \mathfrak{F}_{\mathrm{spl}})(A, \Omega^n B), \quad n \in \mathbb{Z},\ A, B \in \Re.$$

(2) The *unstable algebraic E-theory* for $\Re$ is the triangulated category $D(\Re, \mathfrak{F}_{\mathrm{surj}})$. The *unstable algebraic E-groups* are, by definition,

$$\mathfrak{E}_n^{\mathrm{unst}}(A, B) := D(\Re, \mathfrak{F}_{\mathrm{surj}})(A, \Omega^n B), \quad n \in \mathbb{Z},\ A, B \in \Re.$$

In the following sections we shall be introducing matrix stabilization into the play. We shall first invert the homomorphisms $A \to M_n(A)$, $n \geqslant 1$ ("Morita stabilization") and then invert the maps $A \to M_\infty(A) = \cup_n M_n(A)$ ("stabilization"). An effect of the stabilization is that the loop functor Ω becomes invertible.

3. The category $\mathcal{H}_{mor}(\mathrm{Alg}_k)$

Many (bivariant) homology theories are Morita invariant. In order to construct universal bivariant Morita invariant theories we have to introduce a category structure for algebras whose morphisms are obtained from algebra homomorphisms by equating polynomially homotopic maps and inverting the maps

$$s_{n,A} : A \to M_n A, \quad n > 0,$$

sending $a \in A$ to the matrix (x_{ij}) with $x_{11} = a$ and the other entries zero.

We first prove the following statement (see as well [5, 4.1.1; 5.1.2]).

Proposition 3.1. *Let B be a k-algebra, $A \subset B$ a subalgebra, and $V, W \in B$ elements such that*

$$WA, AV \subset A, \quad aVWa' = aa' \quad (a, a' \in A).$$

Then

$$\varphi^{V,W} : A \to A, \quad a \mapsto WaV$$

is a k-algebra homomorphism, and $s_{2,A} : A \to M_2 A$ is homotopic to $s_{2,A}\varphi^{V,W}$.

Proof. Let $s'_{2,A} : A \to M_2A$ be the homomorphism sending $a \in A$ to the matrix $\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$. Consider an invertible matrix $T \in GL_2(k[x])$ such that

$$\partial_x^0(T) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \partial_x^1(T) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The matrix

$$T = \begin{pmatrix} 1-x^2 & x^3-2x \\ x & 1-x^2 \end{pmatrix}$$

is a concrete example with

$$T^{-1} = \begin{pmatrix} 1-x^2 & 2x-x^3 \\ -x & 1-x^2 \end{pmatrix}.$$

Let $H : A \to M_2A[x]$ be the homomorphism $a \mapsto Ts_{2,A}(a)T^{-1}$. Then $s_{2,A} = \partial_x^0 H \sim \partial_x^1 H = s'_{2,A}$.

Consider a homomorphism

$$\psi : \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2A \longmapsto \begin{pmatrix} W & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & 1 \end{pmatrix} \in M_2A.$$

Then $s_{2,A}\varphi^{V,W} = \psi s_{2,A} \sim \psi s'_{2,A} = s'_{2,A} \sim s_{2,A}$. $\square$

Given two algebras $A, B \in \mathrm{Alg}_k$, we set

$$[A,B]_{mor} := \varinjlim_n [A, M_nB],$$

where the colimit is taken over the homomorphisms $\iota_n : M_nA \to M_{n+1}A$ sending M_nA into the left upper corner of $M_{n+1}A$. Define a composition law

$$[A,B]_{mor} \times [B,C]_{mor} \to [A,C]_{mor} \tag{5}$$

by the rule

$$(\alpha : A \to M_nB, \beta : B \to M_lC) \longmapsto \beta \star \alpha := M_n(\beta) \circ \alpha : A \to M_{nl}C.$$

Here $M_n(\beta)$ is the composition of the homomorphism $M_nB \to M_nM_lC$, induced by β, and the natural isomorphism $M_nM_lC \cong M_{nl}C$.

The composition law (5) is consistent with polynomial homotopy. Precisely, if $\alpha \sim \alpha'$ and $\beta \sim \beta'$ then $\beta \star \alpha \sim \beta' \star \alpha'$, and hence $\beta \star \alpha = \beta' \star \alpha'$ in $[A,C]_{mor}$. It is also consistent with the colimit maps ι_n in the first argument. Namely if we replace α with $\iota_n\alpha$ then $\beta \star \alpha = \beta \star (\iota_n\alpha)$ in $[A,C]_{mor}$. To show that it is consistent with the colimit maps ι_l in the second argument, it is enough to observe that $M_n(\iota_l\beta) \circ \alpha$ is conjugate by a permutation matrix to $M_{n+1}(\beta)(\iota_n\alpha)$, and hence equal in $[A,C]_{mor}$ by the preceding proposition.

It is easy to see that the composition law is associative giving rise to a category which we denote by $\mathcal{H}_{mor}(\mathrm{Alg}_k)$. It is useful to give another description of $\mathcal{H}_{mor}(\mathrm{Alg}_k)$. Given k-modules $M_1, \ldots, M_s$, we set

$$M_1 \otimes \cdots \otimes M_s = M_1 \otimes (M_2 \otimes (\cdots \otimes (M_{s-1} \otimes M_s) \cdots)).$$

For any algebra A and $n > 0$ the homomorphism $s_{n,A}$ is naturally isomorphic to the homomorphism $A \to M_n(k) \otimes A$ taking $a \in A$ to $s_{n,k}(1) \otimes a$.

Let

$$\Sigma := \{\mathrm{id}_A\} \cup \{A \to M_{n_1}(k) \otimes \cdots \otimes M_{n_s}(k) \otimes A\}, \quad A \in \mathrm{Alg}_k, n_1, \ldots, n_s \in \mathbb{N}.$$

Definition. A functor $X : \mathrm{Alg}_k \to \mathcal{C}$ from Alg_k to a category $\mathcal{C}$ is *Morita invariant* if $X(s_{n,A}) : X(A) \to X(M_n A)$ is an isomorphism for every $n \geqslant 1$.

Theorem 3.2. *Σ admits a calculus of left fractions in $\mathcal{H}(\mathrm{Alg}_k)$ and there is a natural isomorphism of categories*

$$\mathcal{H}_{mor}(\mathrm{Alg}_k) \cong \mathcal{H}(\mathrm{Alg}_k)[\Sigma^{-1}].$$

Proof. Clearly, the identity of each object and the composition of two elements of Σ belong to Σ. Each diagram

$$M_{n_1}(k) \otimes \cdots \otimes M_{n_s}(k) \otimes A \xleftarrow{\sigma} A \xrightarrow{f} B$$

with $\sigma \in \Sigma$ can obviously be completed to a commutative square

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\sigma"'] & B \arrow[d, "\sigma'"] \\
M_{n_1}(k) \otimes \cdots \otimes M_{n_s}(k) \otimes A \arrow[r, "f'"] & M_{n_1}(k) \otimes \cdots \otimes M_{n_s}(k) \otimes B
\end{tikzcd}$$

with $\sigma' \in \Sigma$.

To show that Σ admits a calculus of left fractions, it remains to verify that if f, g are morphisms in $\mathcal{H}(\mathrm{Alg}_k)$ and $\sigma \in \Sigma$ is such that $f\sigma = g\sigma$ then there is $\sigma' \in \Sigma$ such that $\sigma' f = \sigma' g$. Without loss of generality it is enough to show that if $f, g : M_n A \to B$ are two homomorphisms with $f s_{n,A} = g s_{n,A}$ then $s_{l,B} f = s_{l,B} g$ for some l.

Consider the diagram

$$\begin{tikzcd}
A \arrow[r, "s_{n,A}"] & M_n A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, shift right, "M_n(s_{n,A})"'] \arrow[d, shift left, "s_{n,M_n A}"] & B \arrow[d, "s_{n,B}"] \\
& M_{n^2} A \arrow[r, shift left, "M_n(f)"] \arrow[r, shift right, "M_n(g)"'] \arrow[d, "s_{2,M_{n^2} A}"'] & M_n B \arrow[d, "s_{2,M_n B}"] \\
& M_{2n^2} A \arrow[r, shift left, "M_{2n}(f)"] \arrow[r, shift right, "M_{2n}(g)"'] & M_{2n} B.
\end{tikzcd}$$

Observe that $s_{n,M_n A}$ is conjugate to $M_n(s_{n,A})$ by a permutation matrix. Proposition 3.1 implies $s_{2,M_{n^2} A} s_{n,M_n A} \simeq s_{2,M_{n^2} A} M_n(s_{n,A})$. Thus,

$$s_{2n,B} f = s_{2,M_n B} s_{n,B} f = M_{2n}(f) s_{2,M_{n^2} A} s_{n,M_n A} \simeq s_{2,M_n B} M_n(f) M_n(s_{n,A}) =$$
$$s_{2,M_n B} M_n(g) M_n(s_{n,A}) \simeq M_{2n}(g) s_{2,M_{n^2} A} s_{n,M_n A} = s_{2n,B} g.$$

It follows that $s_{2n,B} f$ equals $s_{2n,B} g$ in $\mathcal{H}(\mathrm{Alg}_k)$, hence Σ admits a calculus of left fractions. To show that $\mathcal{H}_{mor}(\mathrm{Alg}_k) \cong \mathcal{H}(\mathrm{Alg}_k)[\Sigma^{-1}]$, it is enough to observe that every element of Σ is isomorphic to some $s_{n,A}$ and that every homotopy invariant Morita invariant functor $F : \mathrm{Alg}_k \to \mathcal{C}$ uniquely factors through $\mathcal{H}_{mor}(\mathrm{Alg}_k)$. $\square$

Hom-sets $[A, B]_{mor}$ are naturally equipped with a structure of an abelian monoid, which is described as follows. Let $f, g : A \to B$ be two non-unital algebra homomorphisms and let $f \uplus g : A \to M_2(B)$ be the homomorphism taking $a \in A$ to $\begin{pmatrix} f(a) & 0 \\ 0 & g(a) \end{pmatrix}$. Then $f \uplus g \sim g \uplus f$.

Indeed, it is enough to consider the invertible matrix $T \in GL_2(k[x])$ as above with

$$\partial_x^0(T) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \partial_x^1(T) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then $A \to M_2(B)[x]$, $a \mapsto T \cdot (f \uplus g)(a) \cdot T^{-1}$, yields the desired homotopy. The homotopy of this kind is also referred to as *rotational*. Clearly, $0 \uplus f \sim f \uplus 0 = s_{2,B} \circ f$. Observe that if $f \sim f', g \sim g'$ then $f \uplus g \sim f' \uplus g'$.

The operation $f \uplus g$ is naturally extended to a commutative, associative binary operation on Hom-sets $[A, B]_{mor}$ of $\mathcal{H}_{mor}(\mathrm{Alg}_k)$ making these abelian monoids. To verify this one should use rotational homotopy if necessary. We denote the corresponding group completion by $\{A, B\}_{mor}$. The composition law (5) is "bilinear" in the sense that

$$f \star (g \uplus g') = f \star g \uplus f \star g', \quad (f \uplus f') \star g = f \star g \uplus f' \star g$$

for any $f, f' \in [A, B]_{mor}, g, g' \in [B, C]_{mor}$. To check this, one should use Proposition 3.1 and rotational homotopy.

The composition law (5) is naturally extended to an associative bilinear composition law

$$\{A, B\}_{mor} \times \{B, C\}_{mor} \to \{A, C\}_{mor}.$$

Thus one obtains a category, denoted by $\mathcal{H}^{-1}\mathcal{H}_{mor}(\mathrm{Alg}_k)$, with objects those of Alg_k and morphisms sets $\{A, B\}_{mor}$. The new category is additive with direct product being the usual direct product of algebras.

4. Additive categories of correspondences

There is an important category $\mathcal{HC}or$, closely related to the category $\mathcal{H}^{-1}\mathcal{H}_{mor}(\mathrm{Alg}_k)$, whose objects are the unital algebras Alg_k^u and whose morphisms are "correspondences up to homotopy" $KH(A, B)$ defined by means of bivariant Grothendieck groups $K(A, B)$ in the sense of Kassel [19]. We also refer the reader to [15]. The main result of this section says that $\mathcal{HC}or$ can be regarded as a full subcategory of $\mathcal{H}^{-1}\mathcal{H}_{mor}(\mathrm{Alg}_k)$ by means of a fully faithful functor. It will be used below to get some important computational results. We start with preparations.

Let A, B be two unital algebras and let $K(A, B)$ (respectively $K^{\oplus}(A, B)$) denote the Grothendieck group of the exact category $\mathcal{R}ep(A, B)$ of those A-B-bimodules which are finitely generated projective as right B-modules. The exact structure is given by the short exact sequences (respectively split short exact sequences) of bimodules

$$0 \to P' \to P \to P'' \to 0.$$

Observe that if $k = \mathbb{Z}$ then $\mathcal{R}ep(\mathbb{Z}, B)$ consists of finitely generated projective right B-modules and $K(\mathbb{Z}, B) = K^{\oplus}(\mathbb{Z}, B) = K_0(B)$. If k is a field, then the category $\mathcal{R}ep(A, k)$ is the category of finite dimensional representations $A \to M_p(k)$ of A.

For example, the category $\mathcal{R}ep(\mathbb{Z}[t], B)$ is equivalent to the category of pairs (M, f) where M is a projective right B-module and f is an endomorphism of M. Similarly, the category $\mathcal{R}ep(\mathbb{Z}[t_1, \dots, t_n], B)$ is equivalent to the category of tuples $(M, f_1, \dots, f_n)$ where M is a projective right B-module and $f_1, \dots, f_n$ are commuting endomorphisms of M. The category $\mathcal{R}ep(\mathbb{Z}[t^{\pm}], B)$ is equivalent to the category of pairs (M, f) where M is a projective right B-module and f is an automorphism of M. The category $\mathcal{R}ep(\mathbb{Z}[t_1^{\pm}, \dots, t_n^{\pm}], B)$ is equivalent to the category of tuples $(M, f_1, \dots, f_n)$ where M is a projective right B-module and $f_1, \dots, f_n$ are commuting automorphisms of M.

The groups $K(A, B)$ and $K^{\oplus}(A, B)$ are clearly contravariant in the first argument and covariant in the second.

Given $P \in \mathcal{R}ep(A, B), Q \in \mathcal{R}ep(B, C)$, the object $P \otimes_B Q$ belongs to $\mathcal{R}ep(A, C)$. This induces the corresponding composition products

$$K(A, B) \otimes_{\mathbb{Z}} K(B, C) \to K(A, C), \quad K^{\oplus}(A, B) \otimes_{\mathbb{Z}} K^{\oplus}(B, C) \to K^{\oplus}(A, C).$$

We may regard a class $[P]$ of $K(A, B)$ (respectively $K^{\oplus}(A, B)$) as a (direct sum) Grothendieck group correspondence from A to B.

We define the additive categories of correspondences $\mathcal{C}or$ and $\mathcal{C}or^{\oplus}$; their objects are those of Alg_k^u. An arrow $A \to B$ is an element of $\mathrm{Hom}_{\mathcal{C}or}(A, B) := K(A, B)$ (respectively $\mathrm{Hom}_{\mathcal{C}or^{\oplus}}(A, B) := K^{\oplus}(A, B)$). The composition $[Q] \circ [P]$ of correspondences is defined to be $[P \otimes_B Q]$. The direct sum $A \oplus A'$ of two objects is represented by the direct product $A \times A'$. We define $A \otimes B := A \otimes_k B$ on objects, and extend it to a bilinear function on arrows. It is useful to observe that two Morita equivalent algebras A, B are isomorphic in $\mathcal{C}or$ and $\mathcal{C}or^{\oplus}$. Indeed, there are bimodules ${}_AP_B, {}_BQ_A$ and bimodule isomorphisms $P \otimes_B Q \cong A$, $Q \otimes_A P \cong B$. The correspondences $[P], [Q]$ give isomorphisms between A and B.

Remark. Recall that an algebra R is *flasque* if there is an R-bimodule M, finitely generated projective as a right module, and a bimodule isomorphism $\theta : R \oplus M \cong M$. It easily follows that $[R] = 0$ in $\mathcal{C}or^{\oplus}$ if and only if R is flasque.

We shall also consider another category $\widetilde{\mathcal{R}ep}$. Its objects are those of Alg_k^u. An arrow in $\mathrm{Hom}_{\widetilde{\mathcal{R}ep}}(A, B)$ is given by the isomorphism class $[P]$ of $P \in \mathcal{R}ep(A, B)$. Composition is defined by tensor product as above. Observe that the Hom-sets of $\widetilde{\mathcal{R}ep}$ are abelian monoids. There are natural functors

$$\widetilde{\mathcal{R}ep} \xrightarrow{F} \mathcal{C}or^{\oplus} \xrightarrow{G} \mathcal{C}or.$$

Proposition 4.1. *Let A, B be two unital algebras. Then there is a bijection between:*

(1) *the set of all non-unital homomorphisms $f : A \to B$,*
(2) *the set of all (A, B)-bimodules ${}_AP_B$ such that P_B is an idempotent right ideal of B.*

The bijection is given by the maps

$$g : A \to B \mapsto g(1)B, \quad {}_AP_B \mapsto (A \xrightarrow{f} \mathrm{End}_B P \hookrightarrow B),$$

where f is the unital homomorphism giving a left A-module structure on P.

Proof. Let p be an idempotent of B. Then the right B-module pB is finitely generated projective and

$$\mathrm{End}_B(pB) = \{y \in B \mid y = yp = py\}.$$

Given a non-unital homomorphisms $f : A \to B$, let p be the idempotent $f(1)$. Then f plainly factors through $\mathrm{End}_B(pB)$.

Now let ${}_AP_B$ be an (A,B)-bimodule whose left A-module structure is given by a unital homomorphism $f : A \to \operatorname{End}_B P$ and $P_B = pB$ for some idempotent p. It follows that $p = f(1)p = f(1)$. The desired bijection is now obvious. $\square$

If $P \in \operatorname{Ob}(\mathcal{R}ep(A,B))$, then there is an algebra homomorphism $v : A \to \operatorname{End}_B(P)$. Choose a finitely generated projective B-module Q such that $P \oplus Q \cong B^n$. One obtains a monomorphism $\operatorname{End}_B(P) \rightarrowtail M_n(B)$. Composing with v, we get a homomorphism $u(P) : A \to M_n(B)$ which defines a class in $[A,B]_{mor}$. Let $p = u(P)(1)$; then p is an idempotent and $P \cong \operatorname{Im}(B^n \xrightarrow{p} B^n)$. Suppose $P \cong P'$ and $Q \cong Q'$, then $P \oplus Q \cong P' \oplus Q'$. There is $W \in GL_n(B)$ such that $u(P') = W^{-1}u(P)W$. Note that the homomorphism

$$a \in A \longmapsto \begin{pmatrix} u(P')(a) & 0 \\ 0 & 0 \end{pmatrix} \in M_{2n}(B)$$

is homotopic to the homomorphism

$$a \longmapsto \begin{pmatrix} 0 & 0 \\ 0 & u(P')(a) \end{pmatrix}$$

by the rotational homotopy. Both maps equal $u(P')$ in $[A,B]_{mor}$.

Consider a homomorphism

$$\psi : \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in M_{2n}(B) \longmapsto \begin{pmatrix} W^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} W & 0 \\ 0 & 1 \end{pmatrix} \in M_{2n}(B).$$

Then,

$$\begin{pmatrix} u(P')(a) & 0 \\ 0 & 0 \end{pmatrix} = \psi \begin{pmatrix} u(P)(a) & 0 \\ 0 & 0 \end{pmatrix} \sim \psi \begin{pmatrix} 0 & 0 \\ 0 & u(P)(a) \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ 0 & u(P)(a) \end{pmatrix} \sim \begin{pmatrix} u(P)(a) & 0 \\ 0 & 0 \end{pmatrix}$$

We see that $u(P') = u(P)$ in $[A,B]_{mor}$. If we replace Q with $Q \oplus B$ such that $P \oplus (Q \oplus B) \cong B^{n+1}$ then the homomorphism $A \to M_{n+1}B$ corresponding to this decomposition equals $\iota_n u(P)$, hence equals $u(P)$ in $[A,B]_{mor}$. We see that $u(P) \in [A,B]_{mor}$ does not depend on the isomorphism class of P and the choice of Q.

There is also an isomorphism of (A,B)-bimodules

$${}_AP_B \cong {}_A(pM_nB \otimes_{M_nB} B^n)_B.$$

We see that each $[P] \in \operatorname{Hom}_{\widetilde{\mathcal{R}ep}}(A,B)$ factors through M_nB for some n. Given $k \geqslant 1$ denote by $M_k(p) \in M_{kn}(B)$ the idempotent matrix which places k copies of p on the diagonal. Then there is an isomorphism of (A,B)-bimodules

$${}_AP_B \cong {}_A(pM_nB \otimes_{M_nB} B^n)_B \cong {}_A(A^k \otimes_{M_kA} M_k(p)M_{kn}B \otimes_{M_{kn}B} B^{kn})_B. \tag{6}$$

Since $A^n_{M_nA} \cong e_{11}M_nA_{M_nA}$ and $A^n_{M_nA} \oplus (1-e_{11})M_nA_{M_nA} \cong M_nA$ where e_{11} is the idempotent matrix with the $(1,1)$-entry equal to 1 and the other entries zero, it follows that $u({}_AA^n_{M_nA}) = s_{n,A}$. On the hand $u({}_{M_nB}B^n_B) = 1_{M_nB}$ as one easily sees.

Proposition 4.2. *There is a natural functor*

$$H : \widetilde{\mathcal{R}ep} \to \mathcal{H}_{mor}(\mathrm{Alg}_k)$$

such that for any $A, B \in \mathrm{Alg}^u_k$ the induced map

$$H : \operatorname{Hom}_{\widetilde{\mathcal{R}ep}}(A,B) \to [A,B]_{mor}$$

is a surjective map of abelian monoids.

Proof. If $P \in \mathrm{Ob}(\mathcal{R}ep(A,B))$, then one sets $H([P]) := u(P) \in [A,B]_{mor}$. We have shown above that $u(P)$ is well-defined. Given $P \in \mathrm{Ob}(\mathcal{R}ep(A,B))$ and $Q \in \mathrm{Ob}(\mathcal{R}ep(B,C))$, there are idempotent matrices $p = u(P)(1) \in M_nB, q = u(Q)(1) \in M_lC$ such that ${}_AP_B \cong {}_A(pM_nB \otimes_{M_nB} B^n)_B$ and ${}_BQ_C \cong {}_B(qM_lC \otimes_{M_lC} B^l)_C$ (see above). It follows from (9) that there is an isomorphism of (A,C)-bimodules

$$P \otimes_B Q \cong pM_nB \otimes_{M_nB} B^n \otimes_B B^n \otimes_{M_nB} M_n(q)M_{nl}C \otimes_{M_{nl}C} C^{nl}.$$

Since ${}_{M_nB}B^n \otimes_B B^n_{M_nB} \cong {}_{M_nB}M_nB_{M_nB}$, one has,

$$P \otimes_B Q \cong pM_nB \otimes_{M_nB} M_n(q)M_{nl}C \otimes_{M_{nl}C} C^{nl} \cong rM_{nl}C \otimes_{M_{nl}C} C^{nl},$$

where $r = u(Q) \star u(P)(1)$. We see that $H([Q] \circ [P]) = u(P \otimes_B Q) = u(Q) \star u(P)$, hence H is a functor.

For any $P', P'' \in \mathrm{Ob}(\mathcal{R}ep(A,B))$ the homomorphism $u(P' \oplus P'') : A \to M_n(B)$ factors as

$$A \to \mathrm{End}_B P' \oplus \mathrm{End}_B P'' \hookrightarrow \mathrm{End}_B(P' \oplus P'') \hookrightarrow M_n(B).$$

Therefore $u(P' \oplus P'') = u(P') \uplus u(P'')$ in $[A,B]_{mor}$. This determines a map of abelian monoids $H : \mathrm{Hom}_{\widetilde{\mathcal{R}ep}}(A,B) \to [A,B]_{mor}$. Let $f : A \to M_nB$ be an element of $[A,B]_{mor}$. Using Proposition 4.1 it follows that $f = H([P])$ with $P = \mathrm{Im}(f(1) : B^n \to B^n)$. Therefore H is surjective. $\square$

Let $J : \mathcal{C}or^{\oplus} \to \mathcal{H}^{-1}\mathcal{H}_{mor}(\mathrm{Alg}_k)$ be the natural functor induced by H. There is a commutative diagram

$$\begin{tikzcd} \widetilde{\mathcal{R}ep} \arrow[r] \arrow[d, "H"'] & \mathcal{C}or^{\oplus} \arrow[d, "J"] \\ \mathcal{H}_{mor}(\mathrm{Alg}_k) \arrow[r] & \mathcal{H}^{-1}\mathcal{H}_{mor}(\mathrm{Alg}_k). \end{tikzcd}$$

The preceding proposition implies the following

Corollary 4.3. *The functor J is additive and for any $A, B \in \mathrm{Alg}_k^u$ the induced map*

$$J : K^{\oplus}(A,B) \to \{A,B\}_{mor}$$

is an epimorphism of abelian groups.

An *elementary homotopy* between correspondences from A to B in $\widetilde{\mathcal{R}ep}$ (respectively $\mathcal{C}or^{\oplus}$ or $\mathcal{C}or$) is an element of $\mathrm{Hom}_{\widetilde{\mathcal{R}ep}}(A,B[x])$ (respectively $\mathrm{Hom}_{\mathcal{C}or^{\oplus}}(A,B[x])$ or $\mathrm{Hom}_{\mathcal{C}or}(A,B[x])$). There are two natural maps

$$d_0, d_1 : \mathrm{Hom}_{\widetilde{\mathcal{R}ep}}(A,B[x]) \to \mathrm{Hom}_{\widetilde{\mathcal{R}ep}}(A,B),$$

induced by $\partial_x^0, \partial_x^1 : B[x] \to B$. We say that two bimodules ${}_AP_B$ and ${}_AQ_B$ are *elementary homotopic* in which case we shall write ${}_AP_B \sim {}_AQ_B$ if there is an elementary homotopy $H \in \mathrm{Hom}_{\widetilde{\mathcal{R}ep}}(A,B[x])$ such that $d_0(H) = P$ and $d_1(H) = Q$. Elementary homotopic bimodules in $\mathrm{Hom}_{\mathcal{C}or^{\oplus}}(A,B)$ or $\mathrm{Hom}_{\mathcal{C}or}(A,B)$ are defined in a similar way. The relation "elementary homotopic" is reflexive and symmetric. One may take the transitive closure of this relation to get an equivalence relation (denoted by the symbol "$\simeq$"). The set of equivalence classes of correspondences $A \to B$ is denoted by $\mathcal{H}\widetilde{\mathcal{R}ep}(A,B)$ (respectively $K^{h,\oplus}(A,B)$ or $K^h(A,B)$). Composition preserves the relation "$\simeq$". Equating homotopic correspondences we get the categories $\mathcal{H}\widetilde{\mathcal{R}ep}$, $\mathcal{H}\mathcal{C}^{\oplus}$, $\mathcal{H}\mathcal{C}or$ respectively. One has natural functors

$$\mathcal{H}\widetilde{\mathcal{R}ep} \xrightarrow{F'} \mathcal{H}\mathcal{C}or^{\oplus} \xrightarrow{G'} \mathcal{H}\mathcal{C}or.$$

Let $\widehat{H} : \mathcal{H}\widetilde{\mathcal{R}ep} \to \mathcal{H}_{mor}(\mathrm{Alg}_k)$ and $\widehat{J} : \mathcal{H}\mathcal{C}^{\oplus} \to \mathcal{H}^{-1}\mathcal{H}_{mor}(\mathrm{Alg}_k)$ be the natural functors induced by H and J respectively. There is a commutative diagram

$$\begin{array}{ccc} \mathcal{H}\widetilde{\mathcal{R}ep} & \longrightarrow & \mathcal{H}\mathcal{C}or^{\oplus} \\ {\scriptstyle \widehat{H}}\downarrow & & \downarrow{\scriptstyle \widehat{J}} \\ \mathcal{H}_{mor}(\mathrm{Alg}_k) & \longrightarrow & \mathcal{H}^{-1}\mathcal{H}_{mor}(\mathrm{Alg}_k). \end{array}$$

Theorem 4.4. *The functors $\widehat{H}, \widehat{J}$ are fully faithful. In particular, for any $A, B \in \mathrm{Alg}_k^u$ the induced maps*

$$\widehat{H} : \mathcal{H}\widetilde{\mathcal{R}ep}(A,B) \to [A,B]_{mor}, \quad \widehat{J} : K^{h,\oplus}(A,B) \to \{A,B\}_{mor}$$

are bijections.

Proof. Let us construct a map

$$I : [A,B]_{mor} \to \mathcal{H}\widetilde{\mathcal{R}ep}(A,B)$$

which is inverse to H. Let $f : A \to M_nB$ be an element of $[A,B]_{mor}$ and let $P = \mathrm{Im}(f(1) : B^n \to B^n)$. We set

$$I(f) := [P].$$

I is consistent with direct limit maps $\iota_n : M_nB \to M_{n+1}B$, because $I(f) \cong I(\iota_n f)$. I is also consistent with homotopy, hence it defines a map. It is directly verified that $I\widehat{H}([P]) = I(u(P)) = [P]$ for any $[P] \in \mathcal{H}\widetilde{\mathcal{R}ep}(A,B)$. Therefore $\widehat{H}$ is injective. It follows from Proposition 4.2 that $\widehat{H}$ is surjective as well, and hence it is bijective. The fact that

$$\widehat{J} : K^{h,\oplus}(A,B) \to \{A,B\}_{mor}$$

is bijective is now obvious. $\square$

Short exact sequences always split up to homotopy. More precisely, start with a short exact sequence $E : 0 \to M' \to M \to M'' \to 0$ of R-modules, and define an $R[x]$-module $\widetilde{M}$ as the pull back in the following diagram.

$$\begin{array}{lccccc} \widetilde{E}: & M'[x] & \rightarrowtail & \widetilde{M} & \twoheadrightarrow & M''[x] \\ & {\scriptstyle 1}\downarrow & & \downarrow & & \downarrow{\scriptstyle x} \\ E[x]: & M'[x] & \rightarrowtail & M[x] & \twoheadrightarrow & M''[x]. \end{array}$$

The short exact sequence $\widetilde{E}$ specializes to E when $x = 1$ and to $0 \to M' \to M' \oplus M'' \to M'' \to 0$ when $x = 0$, and provides the desired homotopy.

Proposition 4.5. *Given two unital algebras $A, B \in \mathrm{Alg}_k^u$, the natural homomorphism $\alpha : K^{h,\oplus}(A,B) \to K^h(A,B)$ is an isomorphism.*

Proof. One has a commutative diagram

$$\begin{tikzcd}
 & & L \arrow[d, tail] \\
\langle \mathcal{E}^{\oplus} \rangle \arrow[r, tail] \arrow[d, tail] & \bigoplus_P \mathbb{Z}[{}_A P_B] \arrow[r, two heads] \arrow[d, "1"'] & K^{\oplus}(A,B) \arrow[d, two heads] \\
\langle \mathcal{E} \rangle \arrow[r, tail] \arrow[d, two heads] & \bigoplus_P \mathbb{Z}[{}_A P_B] \arrow[r, two heads] & K(A,B) \\
L & &
\end{tikzcd}$$

with exact rows and columns and $\langle \mathcal{E} \rangle$ ($\langle \mathcal{E}^{\oplus} \rangle$) standing for the subgroup generated by $[P] - [P'] - [P'']$ for every (split) short exact sequence $P' \rightarrowtail P \twoheadrightarrow P''$. One also has a commutative diagram with exact rows

$$\begin{tikzcd}
K^{\oplus}(A,B[x]) \arrow[r, "\partial_1 - \partial_0"] \arrow[d] & K^{\oplus}(A,B) \arrow[r, two heads] \arrow[d] & K^{h,\oplus}(A,B) \arrow[d, "\alpha"] \\
K(A,B[x]) \arrow[r, "\partial_1 - \partial_0"] & K(A,B) \arrow[r, two heads] & K^h(A,B).
\end{tikzcd}$$

If we showed that the left square is cocartesian, it would follow that α is an isomorphism.

Consider a commutative diagram

$$\begin{tikzcd}
\widetilde{L} \arrow[r, tail] \arrow[d, "\beta"'] & K^{\oplus}(A,B[x]) \arrow[r, two heads] \arrow[d, "\partial_1 - \partial_0"'] & K(A,B[x]) \arrow[d, "\partial_1 - \partial_0"] \\
L \arrow[r, tail] & K^{\oplus}(A,B) \arrow[r, two heads] & K(A,B).
\end{tikzcd}$$

The right square is cocartesian if and only if β is an epimorphism. To show that it is an epimorphism, we consider the following commutative diagram.

$$\begin{tikzcd}
\langle \widetilde{\mathcal{E}}^{\oplus} \rangle \arrow[r, tail] \arrow[d, "\partial_1 - \partial_0"'] & \langle \widetilde{\mathcal{E}} \rangle \arrow[r, two heads] \arrow[d, "\partial_1 - \partial_0"'] & \widetilde{L} \arrow[d, "\beta"] \\
\langle \mathcal{E}^{\oplus} \rangle \arrow[r, tail] & \langle \mathcal{E} \rangle \arrow[r, two heads] & L
\end{tikzcd}$$

Consider a generator $e = [P] - [P'] - [P'']$ of $\langle \mathcal{E} \rangle$ represented by a short exact sequence $E : 0 \to P' \to P \to P'' \to 0$ whose class in L is denoted by $[e]$. Let us construct a short exact sequence $\widetilde{E} : 0 \to P'[x] \to \widetilde{P} \to P''[x] \to 0$ as above. Let $\tilde{e} = [\widetilde{P}] - [P'[x]] - [P''[x]] \in \langle \widetilde{\mathcal{E}} \rangle$; then $\beta([\tilde{e}]) = [e]$, and hence $\beta : \widetilde{L} \to L$ is an epimorphism. $\square$

Corollary 4.6. *The natural functor $G' : \mathcal{HC}or^{\oplus} \to \mathcal{HC}or$ is an isomorphism of categories.*

Let L be the inverse functor to G'. We are now in a position to state the main result of the section.

Theorem 4.7. *The functor*

$$\widehat{J} \circ L : \mathcal{HC}or \to \mathcal{H}^{-1}\mathcal{H}_{mor}(\mathrm{Alg}_k)$$

is full and faithful. In particular, the map of abelian groups

$$\widehat{J} \circ L : K^h(A, B) \to \{A, B\}_{mor}$$

is an isomorphism for all $A, B \in \mathrm{Alg}_k^u$.

Proof. This follows from Theorem 4.4 and Corollary 4.6. □

Given an algebra $A \in \mathrm{Alg}_k$ we set,

$$[K_0](A) := \mathrm{Coker}(K_0(A[x]) \xrightarrow{\partial_1 - \partial_0} K_0(A)).$$

Corollary 4.8. *For any unital algebra* $A \in \mathrm{Alg}_k^u$ *there is a natural isomorphism of abelian groups*

$$\{k, A\}_{mor} \cong [K_0](A),$$

functorial in A.

Proof. Let A be a k-algebra, $\ell : \mathbb{Z} \to k$ the structure map. Composition with ℓ induces a bijection

$$\mathrm{Hom}_{\mathrm{Alg}_k}(k, A) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{Z}}}(\mathbb{Z}, A).$$

This induces a bijection of abelian monoids

$$\mathrm{Hom}_{\mathcal{H}_{mor}(\mathrm{Alg}_k)}(k, A) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{H}_{mor}(\mathrm{Alg}_{\mathbb{Z}})}(\mathbb{Z}, A).$$

Thus one obtains a bijection of their group completions

$$\{k, A\}_{mor} = \mathrm{Hom}_{\mathcal{H}^{-1}\mathcal{H}_{mor}(\mathrm{Alg}_k)}(k, A) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{H}^{-1}\mathcal{H}_{mor}(\mathrm{Alg}_{\mathbb{Z}})}(\mathbb{Z}, A).$$

By Theorem 4.7 the right hand side is isomorphic to $[K_0](A)$. □

5. The category kh

For the main computational result of the paper we have to introduce a new category kh whose objects are those of Alg_k and morphisms are defined by means of Hom-sets of $\mathcal{H}^{-1}\mathcal{H}_{mor}(\mathrm{Alg}_k)$. We start with preparations.

Let $B \in \mathrm{Alg}_k$ and let $B_+ \in \mathrm{Alg}_k^u$ be the unital k-algebra which is $B \oplus k$ as a group and

$$(x, n)(y, m) = (xy + mx + ny, nm).$$

The map $A \mapsto A_+$ determines a functor $\mathrm{Alg}_k \to \mathrm{Alg}_k^u$. We put $\varepsilon : B_+ \to k$ to be the augmentation $\varepsilon(x, n) = n$ and $\iota : k \to B$ to be the natural inclusion. Note that if B happens to be a unital k-algebra then the map $\eta : B_+ \to B \times k$, $(x, n) \mapsto (x + n \cdot 1, n)$, is an isomorphism of k-algebras. We do not know whether the short exact sequence of algebras

$$B \xrightarrow{i} B_+ \xrightarrow{\varepsilon} k \qquad (7)$$

is split exact in $\mathcal{H}^{-1}\mathcal{H}_{mor}(\mathrm{Alg}_k)$. However, it is the case for unital algebras.

Lemma 5.1. *The sequence* (7) *is split exact in* $\mathcal{H}^{-1}\mathcal{H}_{mor}(\mathrm{Alg}_k)$ *for any unital algebra* B.

Proof. There is a commutative diagram in Alg_k^u

$$\begin{tikzcd} B \arrow[r, "i"] \arrow[d, equal] & B_+ \arrow[r, "\varepsilon"] \arrow[d, "\eta"] & k \arrow[d, equal] \\ B \arrow[r, "j"] & B \times k \arrow[r, "p"] & k \end{tikzcd}$$

with j, p natural inclusion and projection respectively, η the isomorphism defined above. The lower sequence of the diagram is split exact in $\mathcal{H}^{-1}\mathcal{H}_{mor}(\mathrm{Alg}_k)$, because $B \times k$ is a coproduct of B and k, hence so is the upper one. $\square$

Given $A, B \in \mathrm{Alg}_k$ one sets

$$kh(A,B) := \mathrm{Ker}(\{A, B_+\}_{mor} \xrightarrow{\varepsilon_*} \{A, k\}_{mor}).$$

We have,

$$\{A, B_+\}_{mor} = kh(A,B) \oplus \{A, k\}_{mor}.$$

Corollary 5.2. *For any unital algebra B and any $A \in \mathrm{Alg}_k$ there is a natural isomorphism $kh(A,B) \cong \{A,B\}_{mor}$.*

Let $\mathcal{H}^{-1}\mathcal{H}_{mor}(\mathrm{Alg}_k^u)$ be the full subcategory of $\mathcal{H}^{-1}\mathcal{H}_{mor}(\mathrm{Alg}_k)$ whose objects are those of Alg_k^u. Let $\mathcal{A}$ be the idempotent completion of the additive category $\mathcal{H}^{-1}\mathcal{H}_{mor}(\mathrm{Alg}_k^u)$. Namely we introduce new objects denoted by pA whenever $A \in \mathrm{Alg}_k^u$ and $p : A \to A$ is an idempotent, i.e., $p = p^2$. For instance, if $f : k \to k[t^{\pm}]$ is the natural inclusion and $g : k[t^{\pm}] \to k$ is the natural augmentation, then $p = fg$ is an idempotent. We define $\mathcal{A}(pA, qB) := q\{A,B\}_{mor}p$, and with this definition composition is nothing new. We define $pA \otimes qB := (p \otimes q)(A \otimes_k B)$.

We identify $A \in \mathrm{Alg}_k^u$ with $1A$, and prove easily that $A = pA \oplus \bar{p}A$, where $\bar{p} := 1 - p$. Any functor F from the old category to an idempotent complete category can be extended to $\mathcal{A}$ by defining $F(pA) := F(p)F(A)$.

Lemma 5.3. *Given two homomorphisms $f, g : A \to B$ of k-algebras such that $f(a)g(a') = g(a)f(a') = 0$ for all $a, a' \in A$, the map $f + g$ is an algebra homomorphism and the following relation holds in $\mathcal{H}^{-1}\mathcal{H}_{mor}(\mathrm{Alg}_k)$:*

$$[f] + [g] = [f+g],$$

where $[f]$ stands for the class of f in $\mathcal{H}^{-1}\mathcal{H}_{mor}(\mathrm{Alg}_k)$.

Proof. It is enough to observe that $A \times A$ is a direct sum in $\mathcal{H}^{-1}\mathcal{H}_{mor}(\mathrm{Alg}_k)$ of two copies of A and that $H : A \times A \to B$, $(a, a') \mapsto f(a) + g(a')$ is a k-algebra homomorphism whose restriction to each direct factor is f or g respectively. $\square$

Proposition 5.4. *For any $A \in \mathrm{Alg}_k$ and $B \in \mathrm{Alg}_k^u$ there is a split short exact sequence of abelian groups*

$$\{k, B\}_{mor} \xrightarrow{\varepsilon^*} \{A_+, B\}_{mor} \xrightarrow{\pi} \{A, B\}_{mor}, \tag{8}$$

where π is defined by restriction and ε^* is induced by the augmentation $\varepsilon : A_+ \to k$, $(a, n) \mapsto n$.

Proof. Clearly, ε^* is a split monomorphism. Every homomorphism $f : A \to B$ can be extended to a unital homomorphism $\widetilde{f} : A_+ \to B$ by the rule: $(a,n) \mapsto f(a) + n \cdot 1_B$. Notice that if f is homotopic to g then so are $\widetilde{f}$ and $\widetilde{g}$. Also $\widetilde{f} \uplus \widetilde{g} = \widetilde{f \uplus g}$. We have that every element $f : A \to M_nB$ of $[A,B]_{mor}$ is the image of $\widetilde{f} : A_+ \to M_nB$ under the natural map $[A_+,B]_{mor} \to [A,B]_{mor}$ of abelian monoids. Therefore π is an epimorphism.

Given an idempotent matrix $e \in M_nB$ let $\ell_e : A_+ \to M_nB$ be the homomorphism $(a,n) \mapsto n \cdot e$. Suppose $f : A_+ \to M_nB$ is a k-algebra homomorphism. One sets $\bar{f} := \widetilde{\pi(f)}$ and $e_f := f(0,1)$. Then e_f is an idempotent matrix and

$$\bar{f}(a,n) = f(a,n) + \ell_{1-e_f}(a,n)$$

for all $(a,n) \in A_+$.

Since $f(a_1,n_1)\ell_{1-e_f}(a_2,n_2) = f(a_1,n_1)\cdot e_f\cdot(1-e_f)\cdot\ell_{1-e_f}(a_2,n_2) = 0$ and $\ell_{1-e_f}(a_2,n_2)f(a_1,n_1) = \ell_{1-e_f}(a_2,n_2)\cdot(1-e_f)\cdot e_f\cdot f(a_1,n_1) = 0$, it follows from the preceding lemma that

$$[\bar{f}] = [f] + [\ell_{1-e_f}]. \tag{9}$$

Similarly,

$$[\ell_1] = [\ell_{e_f}] + [\ell_{1-e_f}]. \tag{10}$$

Observe that $\bar{0} = \ell_1$.

Suppose $f, g : A_+ \to M_nB$ be such that $[\pi(f)] = [\pi(g)]$. We may choose n big enough to find a homomorphism $h : A \to M_nB$ such that $\pi(f) \uplus h$ is homotopic to $\pi(g) \uplus h$. Then $\bar{f} \uplus \widetilde{h}$ is homotopic to $\bar{g} \uplus \widetilde{h}$, hence $[\bar{f}] = [\bar{g}]$.

Using relations (9) and (10) we obtain

$$[f] - [g] = [\ell_{e_f}] - [\ell_{e_g}] = \varepsilon^*\iota^*([f] - [g]).$$

We conclude that (8) is a split short exact sequence of abelian groups, and therefore $\{A,B\}_{mor} = \mathrm{Coker}(\{k,B\}_{mor} \to \{A_+,B\}_{mor})$. $\square$

Given $A \in \mathrm{Alg}_k$ let p_A be the idempotent $(1-\iota\varepsilon) : A_+ \to A_+$. Clearly, $(1-p_A)A_+ \cong k$. We get an isomorphism in $\mathcal{A}$

$$A_+ \cong p_AA_+ \oplus k.$$

By Proposition 5.4 and Corollary 5.2 there are natural isomorphisms for any unital algebra B

$$\{A,B\}_{mor} \cong kh(A,B) \cong \mathcal{A}(p_AA_+,B). \tag{11}$$

Given $B \in \mathrm{Alg}_k$ there is a commutative diagram of abelian groups

$$\begin{tikzcd}
kh(k,B) \arrow[r, tail] \arrow[d, tail] & \{k,B_+\}_{mor} \arrow[r, two heads] \arrow[d, tail] & \{k,k\}_{mor} \arrow[d, tail] \\
kh(A_+,B) \arrow[r, tail] \arrow[d, two heads] & \{A_+,B_+\}_{mor} \arrow[r, two heads] \arrow[d, two heads] & \{A_+,k\}_{mor} \arrow[d, two heads] \\
kh(A,B) \arrow[r, tail] & \{A,B_+\}_{mor} \arrow[r, two heads] & \{A,k\}_{mor}.
\end{tikzcd}$$

The rows are split exact by definition. By Proposition 5.4 the right two columns are split exact as well. Using 3×3-lemma for exact categories [11, 2.11] the left column is split exact. It follows from (11) that there is a natural isomorphism

$$\varphi_{AB} : kh(A,B) \xrightarrow{\cong} \mathcal{A}(p_AA_+, p_BB_+).$$

The composition law in $\mathcal{A}$

$$\mathcal{A}(p_A A_+, p_B B_+) \times \mathcal{A}(p_B B_+, p_C C_+) \to \mathcal{A}(p_A A_+, p_C C_+)$$

induces a composition law

$$kh(A,B) \times kh(B,C) \to kh(A,C), \quad (f,g) \mapsto g \circ f := \varphi_{AC}^{-1}(\varphi_{BC}(g)\varphi_{AB}(f)).$$

This determines an additive category, denoted by kh, whose objects are those of Alg_k and morphisms are given by the abelian groups $kh(A,B)$. Also, we obtain a natural functor

$$\varphi : kh \to \mathcal{A}$$

taking an algebra $A \in \mathrm{Alg}_k$ to $p_A A_+$ and a morphism $f \in kh(A,B)$ to $\varphi_{AB}(f) \in \mathcal{A}(p_A A_+, p_B B_+)$.

We have thus proved the following statement.

Proposition 5.5. *The functor*

$$\varphi : kh \to \mathcal{A}, \quad A \to p_A A_+,$$

is full and faithful.

We can think of morphisms in kh as correspondences between non-unital algebras. We finish the section by proving the following

Theorem 5.6. *For any algebra $A \in \mathrm{Alg}_k$ there is a natural isomorphism of abelian groups*

$$kh(k,A) \cong [K_0](A),$$

functorial in A.

Proof. By definition $kh(k,A) = \mathrm{Ker}(\{k, A_+\}_{mor} \to \{k,k\}_{mor})$. There is a commutative diagram of abelian groups with exact rows and columns

$$\begin{tikzcd}
L \arrow[r, tail] \arrow[d, tail] & L' \arrow[r, "\alpha'"] \arrow[d, tail, "i"] & L'' \arrow[d, tail] \\
K_0(A[x]) \arrow[r, tail] \arrow[d, "\partial_x^1 - \partial_x^0"'] & K_0((A_+)[x]) \arrow[r, two heads, "\alpha"] \arrow[d, "\partial_x^1 - \partial_x^0"] & K_0(k[x]) \arrow[d, "\partial_x^1 - \partial_x^0"] \\
K_0(A) \arrow[r, tail] \arrow[d] & K_0(A_+) \arrow[r, two heads] \arrow[d] & K_0(k) \arrow[d] \\
{[K_0](A)} \arrow[r] & {[K_0](A_+)} \arrow[r, two heads] & {[K_0](k)}
\end{tikzcd}$$

The middle two rows are split exact. The map $\alpha = K_0(\varepsilon[x])$ with $\varepsilon : A_+ \to k$ the natural projection splits. Suppose $\beta = K_0(\iota[x])$ with $\iota : k \to A_+$ the natural inclusion; then $\alpha\beta = 1$. We have $((\partial_x^1 - \partial_x^0) \circ \beta)|_{L''} = 0$, hence there is $\beta' : L'' \to L'$ such that $i\beta' = \beta|_{L''}$. One easily sees that $\alpha'\beta' = 1$.

The Snake Lemma implies $[K_0](A) = \mathrm{Ker}([K_0](A_+) \to [K_0](k))$. Now our assertion follows from Corollary 4.8. $\square$

The proof of the preceding theorem and the fact that K_0 takes split exact sequences in Alg_k to split exact sequences [4, 2.4.3] show that the following statement is true.

Corollary 5.7. *Suppose a sequence $I \rightarrowtail R \twoheadrightarrow R/I$ is split exact in* Alg_k *(i.e., I is an ideal in an algebra R, and there is a splitting homomorphism $R/I \to R$), then*

$$kh(k,I) \rightarrowtail kh(k,R) \twoheadrightarrow kh(k,R/I)$$

is a split exact sequence of abelian groups.

6. The triangulated category $D_{mor}(\Re, \mathfrak{F})$

In this section we construct explicitly a universal $\mathfrak{F}$-excisive, homotopy invariant *and* Morita invariant homology theory $\Re \to D_{mor}(\Re, \mathfrak{F})$. More precisely, we define the category $D_{mor}(\Re, \mathfrak{F})$ as follows. Its objects are those of $\Re$ and the set of morphisms between two algebras $A, B \in \Re$ is defined as the colimit of the sequence of abelian groups

$$D(\Re, \mathfrak{F})(A, B) \to D(\Re, \mathfrak{F})(A, M_2B) \to D(\Re, \mathfrak{F})(A, M_3B) \to \cdots$$

It turns out that $D_{mor}(\Re, \mathfrak{F})$ is a triangulated category and, moreover, every homotopy invariant, Morita invariant, excisive homology theory $X : \Re \to \mathcal{T}$ factors through $D_{mor}(\Re, \mathfrak{F})$ (Theorem 6.5).

Throughout this section $\Re$ is an admissible category of k-algebras with $M_nA \in \Re$ for any $A \in \Re$ and $n \geqslant 1$. We assume $\mathfrak{F}$ to be a saturated family of fibrations satisfying $M_n(f) \in \mathfrak{F}$ for any $f \in \mathfrak{F}$. For instance, this is the case when $\mathfrak{F}$ is either $\mathfrak{F}_{\mathrm{spl}}$ or $\mathfrak{F}_{\mathrm{surj}}$. We note that $M_n(A) \cong A \otimes_k M_n(k) \in \Re$ for any $n \geqslant 1$ and $A \in \Re$. The proof of Corollary 2.8 shows that $M_n(f) = M_n(k) \otimes f \in \mathcal{W}_\triangle$ for any $n \geqslant 1$ and $f \in \mathcal{W}_\triangle$. Note that if B^I is a path object for an algebra $B \in \Re$ then $M_n(B^I)$ is a path algebra for M_nB.

We denote by $\mathcal{H}_{mor}\Re$ (respectively $\mathcal{H}^{-1}\mathcal{H}_{mor}\Re$) the full subcategory of $\mathcal{H}_{mor}(\mathrm{Alg}_k)$ (respectively $\mathcal{H}^{-1}\mathcal{H}_{mor}(\mathrm{Alg}_k)$) whose objects are those of $\Re$. Let

$$\Gamma : \mathcal{H}\Re \to \mathcal{H}_{mor}\Re$$

be the canonical functor. We set

$$\Sigma_{mor} = \{\Gamma(f) \mid f \in \mathcal{W}_\triangle\}.$$

Proposition 6.1. *Σ_{mor} admits a calculus of right fractions in $\mathcal{H}_{mor}\Re$.*

Proof. Clearly, the identity of each object and the composition of two elements of Σ_{mor} belong to Σ_{mor}. Consider a diagram

$$A \xrightarrow{f} M_nB \xleftarrow{s_{n,B}\sigma} D$$

with $\sigma : D \to B$ from $\mathcal{W}_\triangle$. One has $s_{n,B}\sigma = M_n(\sigma)s_{n,D}$. We can construct a commutative diagram in $\mathcal{H}_{mor}\Re$

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "\sigma'"'] & M_nD \arrow[d, "M_n(\sigma)"] \\ A \arrow[r, "f"] & M_nB \end{tikzcd}$$

with $\sigma' \in \mathcal{W}_\triangle$ and A' a limit in $\Re$ for the diagram

$$\begin{tikzcd}[column sep=small] A \arrow[dr, "f"] & & M_n(B^I) \arrow[dl, "M_n(d_0)"'] \arrow[dr, "M_n(d_1)"] & & M_nD \arrow[dl, "M_n(\sigma)"'] \\ & M_nB & & M_nB & \end{tikzcd}$$

To show that Σ_{mor} admits a calculus of right fractions, it remains to verify that if $f, g : A \to M_nB$ are morphisms in $\Re$ and $\sigma : B \to M_lC$ from $\mathcal{W}_\triangle$ is such that $M_n(\sigma)f = M_n(\sigma)g$ then there is $\sigma' \in \Sigma$ such that $f\sigma' = g\sigma'$. This follows from Proposition 2.1 and the fact that $M_n(\sigma) \in \mathcal{W}_\triangle$. $\square$

We denote by $D^-_{mor}(\Re, \mathfrak{F})$ the category $\mathcal{H}_{mor}\Re[\Sigma^{-1}_{mor}]$. It follows from Theorem 3.2 and Proposition 6.1 that the category is obtained from $\Re$ by inverting the maps from $\mathcal{W}_\triangle \cup \{s_{n,A} \mid n \in \mathbb{N}, A \in \Re\}$. Maps between $A, B \in \Re$ are defined as

$$\varinjlim_n D^-(\Re, \mathfrak{F})(A, M_n(B)) = \varinjlim_{A' \to A \in \mathcal{W}_\triangle} \varinjlim_n [A', M_n(B)] = \varinjlim_{A' \to A \in \mathcal{W}_\triangle} [A', B]_{mor}.$$

The composition of two maps $A \xleftarrow{s} A' \xrightarrow{f} M_n(B)$ and $B \xleftarrow{t} B' \xrightarrow{g} M_l(C)$ in $D^-_{mor}(\Re, \mathfrak{F})$ is a common denominator

$$\begin{tikzcd}[column sep=small] & & A'' \arrow[dl, "u"'] \arrow[dr, "v"] & & \\ & A' \arrow[dl, "s"'] \arrow[dr, "f"] & & M_n(B') \arrow[dl, "M_n(t)"'] \arrow[dr, "M_n(g)"] & \\ A & & M_n(B) & & M_{nl}(C) \end{tikzcd}$$

with $M_n(t) \in \mathcal{W}_\triangle$ and A'' defined as above.

Recall that $[A', B]_{mor}$ is an abelian monoid with respect to the binary operation $f \uplus g : A' \to M_2M_nB \cong M_{2n}B$, $f, g : A' \to M_nB$. For any $A'' \to A' \in \mathcal{W}_\triangle$ the induced map

$$[A', B]_{mor} \to [A'', B]_{mor}$$

is a morphism of abelian monoids. $\Omega^n A$ with $n \geqslant 2$ is an abelian group object of $D^-(\Re, \mathfrak{F})$ (see [10, 6.5]). Let us discuss consistence of the binary operations $\uplus$ and that coming from the abelian group structure for $\Omega^n A$.

The binary operation $\uplus$ is naturally extended to a binary operation

$$\uplus : D^-_{mor}(\Re, \mathfrak{F})(A, B) \times D^-_{mor}(\Re, \mathfrak{F})(A, B) \to D^-_{mor}(\Re, \mathfrak{F})(A, B)$$

as follows. Any two maps $\varphi = [A \xleftarrow{s} A' \xrightarrow{f} M_n(B)]$ and $\psi = [B \xleftarrow{t} B' \xrightarrow{g} M_l(C)]$ in $D^-_{mor}(\Re, \mathfrak{F})$ can be replaced with equivalent two maps $A \xleftarrow{s'} D \xrightarrow{f'} M_{n+l}(B)$ and $B \xleftarrow{s'} D \xrightarrow{g'} M_{n+l}(C)$. We set $\varphi \uplus \psi = [B \xleftarrow{s'} D \xrightarrow{f' \uplus g'} M_{n+l}(C)]$.

On the other hand, for any $n \geqslant 2$ there is a binary operation

$$+ : D^-_{mor}(\Re, \mathfrak{F})(A, \Omega^n B) \times D^-_{mor}(\Re, \mathfrak{F})(A, \Omega^n B) \to D^-_{mor}(\Re, \mathfrak{F})(A, \Omega^n B)$$

induced by the abelian group structure on $\Omega^n B$.

Proposition 6.2. *$f + g = f \uplus g$ for any $f, g \in D^-_{mor}(\Re, \mathfrak{F})(A, \Omega^n B)$ and any $n \geqslant 2$.*

Proof. Since

$$D^-_{mor}(\Re,\mathfrak{F})(A,\Omega^n B\times\Omega^n B)=D^-_{mor}(\Re,\mathfrak{F})(A,\Omega^n B)\oplus D^-_{mor}(\Re,\mathfrak{F})(A,\Omega^n B)$$

and restrictions of both $f+g$ and $f\uplus g$ to each summand coincide, our assertion follows from the fact two summation maps from a direct sum of abelian groups coincide if and only if so do their restrictions to each summand. $\square$

The following result gives a relation with correspondences.

Corollary 6.3. $D^-_{mor}(\Re,\mathfrak{F})(A,\Omega^n B)=\varinjlim_{A'\to A\in\mathcal{W}_\triangle}\{A',\Omega^n B\}_{mor},\quad n\geqslant 2.$

Proof. Use the fact that the group completion of an abelian monoid operation commutes with direct limits and that the left hand side is an abelian group. $\square$

The loop functor $\Omega: D^-(\Re,\mathfrak{F})\to D^-(\Re,\mathfrak{F})$ can plainly be extended to $D^-_{mor}(\Re,\mathfrak{F})$. Let $\Gamma: D^-(\Re,\mathfrak{F})\to D^-_{mor}(\Re,\mathfrak{F})$ be the canonical functor. We say that a sequence in $D^-_{mor}(\Re,\mathfrak{F})$

$$\Omega A\to C\to B\to A$$

is a left triangle if it is isomorphic to

$$\Omega\Gamma A'\to\Gamma C'\to\Gamma B'\to\Gamma A'$$

for some left triangle $\Omega A'\to C'\to B'\to A'$ in $D^-(\Re,\mathfrak{F})$. We denote by $\mathcal{L}tr_{mor}(\Re,\mathfrak{F})$ the category of left triangles in $D^-_{mor}(\Re,\mathfrak{F})$.

Theorem 6.4. *$\mathcal{L}tr_{mor}(\Re,\mathfrak{F})$ is a left triangulation of the (possibly "large") category $D^-_{mor}(\Re,\mathfrak{F})$, i.e. it is closed under isomorphisms and enjoys the axioms (LT1)-(LT4) of Theorem 2.3.*

Proof. (LT1). For any $A\in\Re$ the left triangle $0\xrightarrow{0}A\xrightarrow{1_A}A\xrightarrow{0}0$ belongs to $\mathcal{L}tr_{mor}(\Re,\mathfrak{F})$. Let $h:B\to C$ be any morphism in $D^-_{mor}(\Re,\mathfrak{F})$ represented by $B\xleftarrow{t}B'\xrightarrow{g}M_l(C)$. Then h is isomorphic to the map g. By Theorem 2.3 there is a left triangle in $D^-(\Re,\mathfrak{F})$ of the form $\Omega M_lC\xrightarrow{f}A\xrightarrow{u}B'\xrightarrow{g}M_lC$. We conclude that h can be embedded into a left triangle from $\mathcal{L}tr_{mor}(\Re,\mathfrak{F})$.

(LT2). Any left triangle $\Omega C\xrightarrow{f}A\xrightarrow{g}B\xrightarrow{h}C$ in $\mathcal{L}tr_{mor}(\Re,\mathfrak{F})$ is by definition isomorphic to a sequence

$$\Omega\Gamma A'\to\Gamma C'\to\Gamma B'\xrightarrow{\Gamma h'}\Gamma A'$$

for some left triangle $\Omega A'\to C'\to B'\to A'$ in $D^-(\Re,\mathfrak{F})$. By Theorem 2.3 $\Omega B'\xrightarrow{-\Omega h'}\Omega C'\to A'\to B'$ is a left triangle in $D^-(\Re,\mathfrak{F})$. It is obvious that $\Omega B\xrightarrow{-\Omega h}\Omega C\to A\to B$ is isomorphic to $\Omega\Gamma B'\xrightarrow{-\Omega\Gamma h'}\Omega\Gamma C'\to\Gamma A'\to\Gamma B'$, and hence is a left triangle in $D^-_{mor}(\Re,\mathfrak{F})$.

(LT3). Suppose we are given two left triangles $\Omega B\xrightarrow{\gamma}F\xrightarrow{\beta}A\xrightarrow{\alpha}B$ and $\Omega B'\xrightarrow{\gamma'}F'\xrightarrow{\beta'}A'\xrightarrow{\alpha'}B'$ and two morphisms $\varphi:A\to A'$ and $\psi:B\to B'$ in $D^-_{mor}(\Re,\mathfrak{F})$ with $\psi\alpha=\alpha'\varphi$. We claim that there exists a morphism $\chi:F\to F'$ such that the triple (χ,φ,ψ) is a morphism from the first triangle to the second.

Without loss of generality we can assume that the first left triangle is the sequence $\Omega B\xrightarrow{j}P(g)\xrightarrow{g_1}A\xrightarrow{g}B$ and the second one is $\Omega M_nB'\xrightarrow{j'}P(M_ng')\xrightarrow{M_ng_1'}M_nA'\xrightarrow{M_ng'}$

M_nB'. Moreover, ψ is represented by $B \xleftarrow{s} U \xrightarrow{u} M_nB'$ and φ represented by $A \xleftarrow{t} V \xrightarrow{v} M_nA'$.

By [10, p. 586] there is a commutative diagram in $\Re$

$$\begin{tikzcd}
\Omega B \arrow[r, "j"] & P(g) \arrow[r, "g_1"] & A \arrow[r, "g"] & B \\
\Omega B'' \arrow[u, "\Omega\delta"] \arrow[r, "j''"] \arrow[d, "\Omega z"] & P(g'') \arrow[u] \arrow[r, "g_1''"] \arrow[d] & A'' \arrow[u, "\alpha"] \arrow[r, "g''"] \arrow[d, "\tau"] & B'' \arrow[u, "\delta"] \arrow[d, "z"] \\
\Omega M_nB' \arrow[r, "M_nj'"] & P(M_ng') \arrow[r, "M_ng_1'"] & M_nA' \arrow[r, "M_ng'"] & M_nB'
\end{tikzcd}$$

such that $\psi = z\delta^{-1}$ and $\varphi = \tau\alpha^{-1}$. The desired triple (χ, φ, ψ) is constructed.

(LT4). Since every morphism in $D^-_{mor}(\Re, \mathfrak{F})$ is of the form $p \circ i \circ s^{-1}$ with p a fibration and i, s weak equivalences, it follows that two composable morphisms h, k fit into a commutative diagram in $D^-_{mor}(\Re, \mathfrak{F})$

$$\begin{tikzcd}
B \arrow[r, "h"] \arrow[d, "\cong"'] & C \arrow[r, "k"] \arrow[d, "\cong"'] & D \arrow[d, "s_{nl,D}"] \\
B' \arrow[r, two heads, "p"] & M_nC' \arrow[r, two heads, "M_nq"] & M_{nl}D
\end{tikzcd}$$

with the vertical maps isomorphisms and p, q fibrations in $\Re$. It is routine to verify that (LT4) follows from the following fact: any two fibrations $B \xrightarrow{h} M_nC \xrightarrow{M_nk} M_{nl}D$ of $\Re$ can be fitted into a commutative diagram in $D^-_{mor}(\Re, \mathfrak{F})$

$$\begin{tikzcd}
 & \Omega E \arrow[d, "f\circ\Omega\ell"] & & \\
\Omega M_nC \arrow[r, "f"] \arrow[d, "\Omega M_nk"] & A \arrow[r, "g"] \arrow[d, "\alpha"] & B \arrow[r, "h"] \arrow[d, "1_B"'] & M_nC \arrow[d, "M_nk"] \\
\Omega M_{nl}D \arrow[r, "v"] \arrow[d, "1"] & F \arrow[r, "m"] \arrow[d, "\beta"] & B \arrow[r, "M_n(k)h"] \arrow[d, "h"'] & M_{nl}D \arrow[d, "1"] \\
\Omega M_{nl}D \arrow[r, "u"] & E \arrow[r, "\ell"] & M_nC \arrow[r, "M_nk"] & M_{nl}D
\end{tikzcd}$$

in which the rows are standard left triangles and the second column on the left is a left triangle. The proof of this fact literally repeats the proof of a similar assertion in [10, pp. 587-588]. $\square$

One can stabilize the loop endofunctor Ω on $D^-_{mor}(\Re, \mathfrak{F})$ (see paragraph 2.2) to get a new category $D_{mor}(\Re, \mathfrak{F})$ which is clearly triangulated.

We are now in a position to prove the main result of the section.

Theorem 6.5. *Let $X : \Re \to \mathcal{T}$ be an excisive, homotopy invariant, Morita invariant homology theory. Then the following statements are true:*

(1) *there is a unique functor $\bar{X} : D^-_{mor}(\Re, \mathfrak{F}) \to \mathcal{T}$ such that $X = \bar{X} \circ j$ with $j : \Re \to D^-_{mor}(\Re, \mathfrak{F})$ the canonical functor. Moreover, $\bar{X}$ takes left triangles in $D^-_{mor}(\Re, \mathfrak{F})$ to triangles in $\mathcal{T}$;*

(2) *there is a unique functor* $\widetilde{X} : D_{mor}(\Re, \mathfrak{F}) \to \mathcal{T}$ *such that* $\bar{X} = \widetilde{X} \circ S$ *with* $S : D^-_{mor}(\Re, \mathfrak{F}) \to D_{mor}(\Re, \mathfrak{F})$ *the canonical stabilization functor. Moreover,* $\widetilde{X}$ *is triangulated, i.e. it takes triangles in* $D_{mor}(\Re, \mathfrak{F})$ *to triangles in* $\mathcal{T}$*.*

Proof. (1) By Theorem 2.6 X factors through $D^-(\Re, \mathfrak{F})$. Since X is Morita invariant it factors through $D^-_{mor}(\Re, \mathfrak{F})$. It is plainly triangulated.

(2) The proof is similar to that of Theorem 2.6(2). □

Let $\mathcal{W}_{mor}$ denote the class of those homomorphisms f for which $X(f)$ is an isomorphism for any excisive, homotopy invariant, Morita invariant homology theory X. It can be shown similar to Lemma 2.5 that the triple $(\Re, \mathfrak{F}, \mathcal{W}_{mor})$ is a Brown category.

Theorem 6.6. *If* $D_{mor}(\Re, \mathfrak{F})$ *is a category with small Hom-sets, then there is a natural triangulated equivalence of the triangulated categories* $D_{mor}(\Re, \mathfrak{F})$ *and* $D(\Re, \mathfrak{F}, \mathcal{W}_{mor})$*.*

Proof. One can prove similar to the preceding theorem that there is a unique triangulated functor

$$D^-_{mor}(\Re, \mathfrak{F}) \to D(\Re, \mathfrak{F}, \mathcal{W}_{mor}).$$

It is uniquely extended to a triangulated functor

$$\Upsilon : D_{mor}(\Re, \mathfrak{F}) \to D(\Re, \mathfrak{F}, \mathcal{W}_{mor}).$$

On the other hand, one can prove similar to Theorem 2.6 that the category $D(\Re, \mathfrak{F}, \mathcal{W}_{mor})$ is universal for excisive, homotopy invariant, Morita invariant homology theories. Since $D_{mor}(\Re, \mathfrak{F})$ is a category with small Hom-sets, then $\Re \to D_{mor}(\Re, \mathfrak{F})$ is an excisive, homotopy invariant, Morita invariant homology theory. Therefore there is a unique triangulated functor

$$\Xi : D(\Re, \mathfrak{F}, \mathcal{W}_{mor}) \to D_{mor}(\Re, \mathfrak{F}).$$

We conclude that Ξ, Υ are mutually inverse. □

Proposition 6.7. *If* $\mathfrak{F}$ *is a tensor closed collection of fibrations then the tensor product* $\otimes_k$ *of algebras induces a tensor product on* $D_{mor}(\Re, \mathfrak{F})$ *making it a symmetric monoidal category. Moreover, the tensor product is exact in either variable, i.e. it takes triangles to triangles.*

Proof. The proof is similar to that of Corollary 2.8. □

7. Morita stable algebraic KK- and E-theories

Let us fix an underlying category $\mathcal{U}$, which can be a full subcategory of either the category of sets $Sets$ or $\operatorname{Mod} k$. The category $\mathcal{U}$ will depend on $\mathfrak{F}$. Namely, we shall assume that $\mathcal{U} \subseteq Sets$ if $\mathfrak{F} = \mathfrak{F}_{\mathrm{surj}}$ and $\mathcal{U} \subseteq \operatorname{Mod} k$ if $\mathfrak{F} = \mathfrak{F}_{\mathrm{spl}}$. We also assume that $\Re$ is a T-closed admissible category of k-algebras with $M_n A \in \Re$ for any $A \in \Re$ and $n \geqslant 1$. Clearly, $M_n(f) \in \mathfrak{F}$ for any $f \in \mathfrak{F}$.

Definition. (1) The *Morita stable algebraic* KK*-theory* for $\Re$ is the triangulated category $D_{mor}(\Re, \mathfrak{F}_{\mathrm{spl}})$. The *Morita stable algebraic* KK*-groups* are, by definition,

$$\mathfrak{K}\mathfrak{K}^{mor}_n(A, B) := D_{mor}(\Re, \mathfrak{F}_{\mathrm{spl}})(A, \Omega^n B), \quad n \in \mathbb{Z},\ A, B \in \Re.$$

(2) The *Morita stable algebraic* E*-theory* for $\Re$ is the triangulated category $D_{mor}(\Re, \mathfrak{F}_{\mathrm{surj}})$. The *Morita stable algebraic* E*-groups* are, by definition,

$$\mathfrak{E}^{mor}_n(A, B) := D_{mor}(\Re, \mathfrak{F}_{\mathrm{surj}})(A, \Omega^n B), \quad n \in \mathbb{Z},\ A, B \in \Re.$$

In the next two sections we study universal bivariant excisive, homotopy invariant *and* M_∞-invariant homology theories.

8. The category $\mathcal{H}_{st}(\mathrm{Alg}_k)$

Given an algebra $A \in \mathrm{Alg}_k$ we write $M_\infty A = \cup_n M_n A$. One can identify $M_\infty A$ with infinite matrices having finitely many non-zero entries. Note that $M_\infty A \cong M_\infty k \otimes A$. One has a natural homomorphism $\iota_{\infty,A} : A \to M_\infty k \otimes A$, defined similar to $s_{n,A}$, and a sequence of maps

$$A \xrightarrow{\iota_{\infty,A}} M_\infty k \otimes A \xrightarrow{\iota_{\infty,M_\infty k\otimes A}} M_\infty k \otimes M_\infty k \otimes A \xrightarrow{\iota_{\infty,M_\infty^{\otimes 2}k\otimes A}} \cdots \xrightarrow{\iota_{\infty,M_\infty^{\otimes n-1}k\otimes A}} M_\infty^{\otimes n} k \otimes A.$$

Denote its composition by $\iota_{\infty,A}^n$. We set

$$S = \{1_A, \iota_{\infty,A}^n \mid A \in \mathrm{Alg}_k, n \in \mathbb{N}\}.$$

Let $\mathcal{H}_{st}(\mathrm{Alg}_k)$ be the category $\mathcal{H}(\mathrm{Alg}_k)[S^{-1}]$.

Definition. A functor $X : \mathrm{Alg}_k \to \mathcal{C}$ from Alg_k to a category $\mathcal{C}$ is M_∞-*invariant* or *stable* if $X(\iota_{\infty,A}) : X(A) \to X(M_\infty k \otimes A)$ is an isomorphism.

Theorem 8.1. *S admits a calculus of left fractions in $\mathcal{H}(\mathrm{Alg}_k)$ and every stable homotopy invariant functor $X : \mathrm{Alg}_k \to \mathcal{C}$ factors through $\mathcal{H}_{st}(\mathrm{Alg}_k)$.*

Proof. Clearly, the identity map of each object and the composition of two elements of S belong to S. Each diagram

$$M_\infty^{\otimes n} k \otimes A \xleftarrow{\iota_{\infty,A}^n} A \xrightarrow{f} B$$

can obviously be completed to a commutative square

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\iota_{\infty,A}^n"'] & B \arrow[d, "\iota_{\infty,B}^n"] \\
M_\infty^{\otimes n} k \otimes A \arrow[r, "f'"] & M_\infty^{\otimes n} k \otimes B.
\end{tikzcd}$$

To show that S admits a calculus of left fractions, it remains to verify that if f, g are morphisms in $\mathcal{H}(\mathrm{Alg}_k)$ and $\sigma \in S$ is such that $f\sigma = g\sigma$ then there is $\sigma' \in S$ such that $\sigma' f = \sigma' g$. Let us show first if $f, g : M_\infty k \otimes A \to B$ are two homomorphisms with $f\iota_{\infty,A} = g\iota_{\infty,A}$ then $\iota_{\infty,B}^n f = \iota_{\infty,B}^n g$ for some n.

Consider the diagram

$$\begin{tikzcd}
A \arrow[r, "\iota_{\infty,A}"] & M_\infty k \otimes A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, shift right, "1\otimes\iota_{\infty,A}"'] \arrow[d, shift left, "\iota_{\infty,M_\infty k\otimes A}"] & B \arrow[d, "\iota_{\infty,B}"] \\
& M_\infty^{\otimes 2} k \otimes A \arrow[r, shift left, "1\otimes f"] \arrow[r, shift right, "1\otimes g"'] \arrow[d, "s_2"'] & M_\infty k \otimes B \arrow[d, "s_2"] \\
& M_2(M_\infty^{\otimes 2} k \otimes A) \arrow[r, shift left, "1\otimes f"] \arrow[r, shift right, "1\otimes g"'] & M_2(M_\infty k \otimes B).
\end{tikzcd}$$

The algebra $M_\infty^{\otimes 2}k$ can be regarded as a subalgebra of $\mathrm{End}_k(k^{(\mathbb{N})}\otimes k^{(\mathbb{N})})$. Let $\{e_i\}_{i\in\mathbb{N}}$ denote the standard basis of $k^{(\mathbb{N})}$. Then $\{e_i\otimes e_j\}_{i,j\in\mathbb{N}}$ is a basis of $k^{(\mathbb{N})}\otimes k^{(\mathbb{N})}$. There is a permutation matrix $W\in C$ such that $1\otimes\iota_{\infty,A}=W^{-1}\iota_{\infty,M_\infty k\otimes A}W$. More precisely, W swaps $e_1\otimes e_i, e_i\otimes e_1$, $i\geqslant 1$, and leaves the other basis elements unchanged. Using Proposition 3.1 one has,

$$s_2\iota_{\infty,B}f=(1\otimes f)s_2\iota_{\infty,M_\infty k\otimes A}\simeq s_2(1\otimes f)(1\otimes\iota_{\infty,A})=$$
$$s_2(1\otimes g)(1\otimes\iota_{\infty,A})\simeq(1\otimes g)s_2\iota_{\infty,M_\infty k\otimes A}=s_2\iota_{\infty,B}g.$$

Composing s_2 with the natural embedding of $M_2(M_\infty k\otimes B)$ into $M_\infty^{\otimes 2}k\otimes B$, it follows that $\iota^2_{\infty,B}f$ equals $\iota^2_{\infty,B}g$ in $\mathcal{H}(\mathrm{Alg}_k)$.

Let $n>1$ and let $f\iota^n_{\infty,A}=g\iota^n_{\infty,A}$. Using induction in n we can find a $\sigma_1\in S$ such that $\sigma_1 f\iota_{\infty,M_\infty^{\otimes n-1}k\otimes A}=\sigma_1 g\iota_{\infty,M_\infty^{\otimes n-1}k\otimes A}$. By above there is a $\sigma_2\in S$ such that $\sigma_2\sigma_1 f=\sigma_2\sigma_1 g$. Hence S admits a calculus of left fractions.

To show that every stable homotopy invariant functor $X:\mathrm{Alg}_k\to\mathcal{C}$ factors through $\mathcal{H}_{st}(\mathrm{Alg}_k)$, it is enough to observe that X takes every element of S to an isomorphism. □

It follows from the preceding theorem that morphisms in $\mathcal{H}_{st}(\mathrm{Alg}_k)$ are given by the sets

$$[A,B]_{st}:=\varinjlim_n[A,M_\infty^{\otimes n}k\otimes B].$$

Note that the arrows $s_{n,A}:A\to M_nA$ become invertible in $\mathcal{H}_{st}(\mathrm{Alg}_k)$. Therefore the canonical functor $\Re\to\mathcal{H}_{st}(\mathrm{Alg}_k)$ factors through $\mathcal{H}_{mor}(\mathrm{Alg}_k)$.

9. The triangulated category $D_{st}(\Re,\mathfrak{F})$

In this section we construct explicitly a universal $\mathfrak{F}$-excisive, homotopy invariant *and* stable homology theory. Throughout this section $\Re$ is an admissible category of k-algebras with $M_\infty A\in\Re$ for any $A\in\Re$. We assume $\mathfrak{F}$ to be a saturated family of fibrations satisfying $M_\infty k\otimes f\in\mathfrak{F}$ for any $f\in\mathfrak{F}$. For instance, this is the case when $\mathfrak{F}$ is either $\mathfrak{F}_{\mathrm{spl}}$ or $\mathfrak{F}_{\mathrm{surj}}$. The proof of Corollary 2.8 shows that $M_\infty k\otimes f\in\mathcal{W}_\triangle$ for any $f\in\mathcal{W}_\triangle$. Note that if B^I is a path object for an algebra $B\in\Re$ then $M_\infty k\otimes(B^I)$ is a path algebra for $M_\infty k\otimes B$.

We denote by $\mathcal{H}_{st}\Re$ the full subcategory of $\mathcal{H}_{st}(\mathrm{Alg}_k)$ whose objects are those of $\Re$. Let

$$\Gamma_{st}:\mathcal{H}\Re\to\mathcal{H}_{st}\Re$$

be the canonical functor. We set

$$\Sigma_{st}=\{\Gamma_{st}(f)\mid f\in\mathcal{W}_\triangle\}.$$

Proposition 9.1. *Σ_{st} admits a calculus of right fractions in $\mathcal{H}_{st}\Re$.*

Proof. The proof is like that of Proposition 6.1. □

We denote by $D^-_{st}(\Re,\mathfrak{F})$ the category $\mathcal{H}_{st}\Re[\Sigma_{st}^{-1}]$. It follows from Theorem 8.1 and Proposition 9.1 that the category is obtained from $\Re$ by inverting the maps from $\mathcal{W}_\triangle\cup(S\cap\mathrm{Mor}\,\Re)$. Maps between $A,B\in\Re$ are defined as

$$\varinjlim_n D^-(\Re,\mathfrak{F})(A,M_\infty^{\otimes n}k\otimes B)=\varinjlim_{A'\to A\in\mathcal{W}_\triangle}\varinjlim_n[A',M_\infty^{\otimes n}k\otimes B]=\varinjlim_{A'\to A\in\mathcal{W}_\triangle}[A',B]_{st}.$$

Composition of two maps $A \xleftarrow{s} A' \xrightarrow{f} M_\infty^{\otimes n}k \otimes B$ and $B \xleftarrow{t} B' \xrightarrow{g} M_\infty^{\otimes l}k \otimes C$ in $D_{st}^-(\Re, \mathfrak{F})$ is a common denominator

$$\begin{tikzcd}
 & & A'' \arrow[dl, "u"'] \arrow[dr, "v"] & & \\
 & A' \arrow[dl, "s"'] \arrow[dr, "f"] & & M_\infty^{\otimes n}k \otimes B' \arrow[dl, "M_\infty^{\otimes n}k\otimes t"'] \arrow[dr, "M_\infty^{\otimes n}k\otimes g"] & \\
A & & M_\infty^{\otimes n}k \otimes B & & M_\infty^{\otimes (n+l)}k \otimes C
\end{tikzcd}$$

with $M_\infty^{\otimes n}k \otimes t \in \mathcal{W}_\triangle$ and A'' a limit in $\Re$ for the diagram

$$\begin{tikzcd}
A \arrow[dr, "f"] & & M_\infty^{\otimes n}k \otimes B^I \arrow[dl, "M_\infty^{\otimes n}k\otimes d_0"'] \arrow[dr, "M_\infty^{\otimes n}k\otimes d_1"] & & M_\infty^{\otimes n}k \otimes B' \arrow[dl, "M_\infty^{\otimes n}k\otimes t"'] \\
 & M_\infty^{\otimes n}k \otimes B & & M_\infty^{\otimes n}k \otimes B &
\end{tikzcd}$$

The loop functor $\Omega : D^-(\Re, \mathfrak{F}) \to D^-(\Re, \mathfrak{F})$ can plainly be extended to $D_{st}^-(\Re, \mathfrak{F})$. Let $\Gamma_{st} : D^-(\Re, \mathfrak{F}) \to D_{st}^-(\Re, \mathfrak{F})$ be the canonical functor. We say that a sequence in $D_{st}^-(\Re, \mathfrak{F})$

$$\Omega A \to C \to B \to A$$

is a left triangle if it is isomorphic to

$$\Omega\Gamma_{st}A' \to \Gamma_{st}C' \to \Gamma_{st}B' \to \Gamma_{st}A'$$

for some left triangle $\Omega A' \to C' \to B' \to A'$ in $D^-(\Re, \mathfrak{F})$. We denote by $\mathcal{L}tr_{st}(\Re, \mathfrak{F})$ the category of left triangles in $D_{st}^-(\Re, \mathfrak{F})$.

Theorem 9.2. *$\mathcal{L}tr_{st}(\Re, \mathfrak{F})$ is a left triangulation of the (possibly "large") category $D_{st}^-(\Re, \mathfrak{F})$, i.e. it is closed under isomorphisms and enjoys the axioms (LT1)-(LT4) of Theorem 2.3.*

Proof. The proof is like that of Theorem 6.4. □

One can stabilize the loop endofunctor Ω on $D_{st}^-(\Re, \mathfrak{F})$ (see paragraph 2.2) to get a new category $D_{st}(\Re, \mathfrak{F})$ which is clearly triangulated.

We are now in a position to prove the main result of the section.

Theorem 9.3. *Let $X : \Re \to \mathcal{T}$ be an excisive, homotopy invariant, stable homology theory. Then the following statements are true:*

(1) there is a unique functor $\bar{X} : D_{st}^-(\Re, \mathfrak{F}) \to \mathcal{T}$ such that $X = \bar{X} \circ j$ with $j : \Re \to D_{st}^-(\Re, \mathfrak{F})$ the canonical functor. Moreover, $\bar{X}$ takes left triangles in $D_{st}^-(\Re, \mathfrak{F})$ to triangles in $\mathcal{T}$;

(2) there is a unique functor $\widetilde{X} : D_{st}(\Re, \mathfrak{F}) \to \mathcal{T}$ such that $\bar{X} = \widetilde{X} \circ S$ with $S : D_{st}^-(\Re, \mathfrak{F}) \to D_{st}(\Re, \mathfrak{F})$ the canonical stabilization functor. Moreover, $\widetilde{X}$ is triangulated, i.e. it takes triangles in $D_{st}(\Re, \mathfrak{F})$ to triangles in $\mathcal{T}$.

Proof. The proof is like that of Theorem 6.5. □

Cortiñas–Thom [5] constructed a universal excisive, homotopy invariant, stable homology theory $\mathrm{Alg}_k \to kk$ depending on $\mathfrak{F}_{\mathrm{spl}}$ or $\mathfrak{F}_{\mathrm{surj}}$.

Corollary 9.4. *Let $\mathfrak{F}$ be either $\mathfrak{F}_{\mathrm{spl}}$ or $\mathfrak{F}_{\mathrm{surj}}$. Then there is a natural triangle equivalence of the triangulated categories $D_{st}(\mathrm{Alg}_k, \mathfrak{F})$ and kk.*

Proof. If we use the preceding theorem, then the proof is like that of [10, 7.4]. $\square$

Let $\mathcal{W}_\infty$ denote the class of those homomorphisms f for which $X(f)$ is an isomorphism for any excisive, homotopy invariant, stable homology theory X. It can be shown similar to Lemma 2.5 that the triple $(\Re, \mathfrak{F}, \mathcal{W}_\infty)$ is a Brown category. Denote by $D_\infty^-(\Re, \mathfrak{F})$ and $D_\infty(\Re, \mathfrak{F})$ the categories $D^-(\Re, \mathfrak{F}, \mathcal{W}_\infty)$ and $D(\Re, \mathfrak{F}, \mathcal{W}_\infty)$ respectively.

Theorem 9.5. *If $D_{st}(\Re, \mathfrak{F})$ is a category with small Hom-sets, then there is a natural triangulated equivalence of the triangulated categories $D_{st}(\Re, \mathfrak{F})$ and $D_\infty(\Re, \mathfrak{F})$.*

Proof. The proof is like that of Theorem 6.6. $\square$

Proposition 9.6. *If $\mathfrak{F}$ is a tensor closed collection of fibrations then the tensor product $\otimes_k$ of algebras induces a tensor product on $D_{st}(\Re, \mathfrak{F})$ making it a symmetric monoidal category. Moreover, the tensor product is exact in either variable, i.e. it takes triangles to triangles.*

Proof. The proof is similar to that of Corollary 2.8. $\square$

10. Comparison with kh

In this section we prove the main computational result of the paper which is a generalization of a similar result by Cortiñas–Thom [5].

Let ΓA, $A \in \mathrm{Alg}_k$, be the algebra of $\mathbb{N} \times \mathbb{N}$-matrices which satisfy the following two properties.

(i) The set $\{a_{ij} \mid i, j \in \mathbb{N}\}$ is finite.
(ii) There exists a natural number $N \in \mathbb{N}$ such that each row and each column has at most N nonzero entries.

$M_\infty A \subset \Gamma A$ is an ideal. We put

$$\Sigma A = \Gamma A / M_\infty A.$$

By [5] there are natural algebra isomorphisms

$$\Gamma A \cong \Gamma k \otimes A, \quad \Sigma A \cong \Sigma k \otimes A.$$

We call the short exact sequence

$$M_\infty A \rightarrowtail \Gamma A \overset{\gamma}{\twoheadrightarrow} \Sigma A$$

the *cone extension.* By [5] $\Gamma A \twoheadrightarrow \Sigma A \in \mathfrak{F}_{\mathrm{spl}}$.

Definition. An admissible category of algebras $\Re$ is Γ-*closed* if $\Gamma A \in \Re$ for any $A \in \Re$. A class of fibration $\mathfrak{F}$ in a Γ-closed admissible category of algebras $\Re$ is Γ-*saturated* if $\Gamma A \twoheadrightarrow \Sigma A \in \mathfrak{F}$ for any $A \in \Re$.

Suppose $\Re$ is Γ-closed then $\mathfrak{F}_{\mathrm{spl}}, \mathfrak{F}_{\mathrm{surj}}$ are Γ-saturated. Every proper class ω in the category of k-modules also gives rise to a Γ-saturated class of fibrations $\mathfrak{F}_\omega$ (see p. 6).

Given $A \in \mathrm{Alg}_k$ consider the commutative diagram as follows:

$$
\begin{tikzcd}
 & \Omega\Sigma A \arrow[r, equal] \arrow[d, tail, "j"'] & \Omega\Sigma A \arrow[d, tail] \\
M_\infty A \arrow[r, tail, "i"] \arrow[d, equal] & P(\gamma) \arrow[r, two heads] \arrow[d, "g"'] & E(\Sigma A) \arrow[d, "\partial_x^1"] \\
M_\infty A \arrow[r, tail] & \Gamma A \arrow[r, two heads, "\gamma"] & \Sigma A
\end{tikzcd}
$$

Note that g is a fibre product of a GL-fibration. One has a commutative diagram with exact rows and columns

$$
\begin{tikzcd}
 & & & KV_1(\Gamma A)=0 \arrow[r, equal] \arrow[d] & KV_1(\Sigma A) \arrow[d] \\
 & & & K_0(\Omega\Sigma A) \arrow[r, equal] \arrow[d, "j"', "\cong"] & K_0(\Omega\Sigma A) \arrow[d] \\
 & K_1(E(\Sigma A)) \arrow[r] \arrow[d, "\alpha"'] & K_0(A) \arrow[r, "i"] \arrow[d, equal] & K_0(P(\gamma)) \arrow[r] \arrow[d] & K_0(E(\Sigma A)) \arrow[d] \\
K_1(\Gamma A) \arrow[r, "\nu"] & K_1(\Sigma A) \arrow[r, "\delta"] \arrow[d, two heads, "p"'] & K_0(A) \arrow[r] \arrow[dl, dotted, "\exists!\beta"] & K_0(\Gamma A)=0 \arrow[r, "\gamma"] & K_0(\Sigma A) \\
 & KV_1(\Sigma A) \arrow[d, tail, "\iota"'] & & & \\
 & K_0(\Omega\Sigma A) & & &
\end{tikzcd}
$$

By [4, 5] p is a cokernel of α, ιp equals the boundary map $\partial : K_1(\Sigma A) \to K_0(\Omega\Sigma A)$, $K_0(\Gamma A) = KV_1(\Gamma A) = 0$ and $p\nu = 0$. It follows that δ is a cokernel of ν, and hence there is a unique β such that $\beta\delta = p$.

Lemma 10.1. *The homomorphism $j^{-1}i : K_0(A) \to K_0(\Omega\Sigma A)$ equals $\iota\beta$.*

Proof. $j^{-1}i = \iota\beta$ if and only if $\partial = \iota\beta\delta == j^{-1}i\delta$ if and only if $j\partial = i\delta$. This follows from commutativity of the following diagram induced by Mayer–Vietoris sequences

$$
\begin{tikzcd}
K_1(\Sigma A) \arrow[r, "\delta"] \arrow[d, equal] & K_0(A) \arrow[r] \arrow[d, "i"] & K_0(\Gamma A) \arrow[r, "\gamma"] \arrow[d] & K_0(\Sigma A) \arrow[d, equal] \\
K_1(\Sigma A) \arrow[r] & K_0(P(\gamma)) \arrow[r] & K_0(\Gamma A) \oplus K_0(E(\Sigma A)) \arrow[r] & K_0(\Sigma A) \\
K_1(\Sigma A) \arrow[r, "\partial"] \arrow[u, equal] & K_0(\Omega\Sigma A) \arrow[r] \arrow[u, "j"] & K_0(E(\Sigma A)) \arrow[r] \arrow[u] & K_0(\Sigma A) \arrow[u, equal]
\end{tikzcd}
$$

□

So we obtain an infinite sequence of natural maps

$$K_0(A) \xrightarrow{j^{-1}i} K_0(\Sigma\Omega A) \xrightarrow{j^{-1}i} K_0(\Sigma^2\Omega^2 A) \to \cdots \tag{12}$$

It induces an infinite sequence of natural maps

$$[K_0](A) \xrightarrow{j^{-1}i} [K_0](\Sigma\Omega A) \xrightarrow{j^{-1}i} [K_0](\Sigma^2\Omega^2 A) \to \cdots \tag{13}$$

Using Theorem 5.6 the latter sequence yields a sequence

$$kh(k, A)(A) \xrightarrow{j^{-1}i} kh(k, \Sigma\Omega A) \xrightarrow{j^{-1}i} kh(k, \Sigma^2\Omega^2 A) \to \cdots \tag{14}$$

Theorem 10.2. *For any $A \in \mathrm{Alg}_k$ there is a natural isomorphism of abelian groups*

$$KH_0(A) \cong \mathrm{colim}_n\, kh(k, \Sigma^n\Omega^n A),$$

where $KH_0(A)$ is the zeroth homotopy K-theory group in the sense of Weibel [24].

Proof. By [5, 8.1.1] $KH_0(A)$ is the colimit of the sequence

$$K_0(A) \xrightarrow{\iota\beta} K_0(\Sigma\Omega A) \xrightarrow{\iota\beta} K_0(\Sigma^2\Omega^2 A) \to \cdots$$

By Lemma 10.1 it is the colimit of sequence (12). Since $KH_0(A)$ is homotopy invariant then $KH_0(A)$ is the colimit of sequence (13). It remains to apply Theorem 5.6. □

Denote by $\mathcal{W}'_\infty$ the class of morphisms in $\Re$ which become invertible in $D_{st}(\Re, \mathfrak{F})$. One has,

$$\mathcal{W}_\triangle \cup \{\iota_{\infty,A} : A \to M_\infty(k) \otimes A\}_{A\in\mathrm{Alg}_k} \subset \mathcal{W}'_\infty \subset \mathcal{W}_\infty.$$

It can be shown similar to Lemma 2.5 that the triple $(\Re, \mathfrak{F}, \mathcal{W}'_\infty)$ is a Brown category. There is a natural functor

$$F : D^-(\Re, \mathfrak{F}, \mathcal{W}'_\infty) \to D_{st}(\Re, \mathfrak{F}).$$

Proposition 10.3. *If $\Re$ is Γ-closed and $\mathfrak{F}$ is Γ-saturated, then $D^-_\infty(\Re, \mathfrak{F}, \mathcal{W}'_\infty)$ is a triangulated category and the functor*

$$F : D^-(\Re, \mathfrak{F}, \mathcal{W}'_\infty) \to D_{st}(\Re, \mathfrak{F})$$

is a triangle equivalence of triangulated categories.

Proof. Since $\mathfrak{F}$ is Γ-saturated, then we have a left triangle corresponding to the cone extension

$$\Sigma\Omega A \cong \Omega\Sigma A \xrightarrow{\partial} M_\infty A \to \Gamma A \to \Sigma A.$$

By [4, 2.3.1] ΓA is zero in $D_{st}(\Re, \mathfrak{F})$, and hence $0 \to \Gamma A \in \mathcal{W}'_\infty$. It follows that ΓA is zero in $D^-(\Re, \mathfrak{F}, \mathcal{W}'_\infty)$. Since $\partial = i^{-1}j$ and i are isomorphisms in $D_{st}(\Re, \mathfrak{F})$, then so is j. We see that ∂ is an isomorphism in $D^-(\Re, \mathfrak{F}, \mathcal{W}'_\infty)$. The isomorphism is easily seen to be functorial in A. Since $\iota_{\infty,A} : A \to M_\infty k \otimes A \in \mathcal{W}'_\infty$ we obtain an isomorphism of endofunctors

$$\tau : \Sigma\Omega \xrightarrow{\cong} \Omega\Sigma \xrightarrow{\iota_\infty^{-1}\circ\partial} \mathrm{id}\,.$$

It follows that $\Omega : D^-(\Re, \mathfrak{F}, \mathcal{W}'_\infty) \to D^-(\Re, \mathfrak{F}, \mathcal{W}'_\infty)$ is an autoequivalence. Therefore $D^-(\Re, \mathfrak{F}, \mathcal{W}'_\infty)$ is triangulated and the natural functor $D^-_{st}(\Re, \mathfrak{F}) \to D^-(\Re, \mathfrak{F}, \mathcal{W}'_\infty)$ can be extended to a triangulated functor

$$G : D_{st}(\Re, \mathfrak{F}) \to D^-(\Re, \mathfrak{F}, \mathcal{W}'_\infty).$$

One easily checks that F, G are mutually inverse equivalences. □

Corollary 10.4. *If $\Re$ is Γ-closed and $\mathfrak{F}$ is Γ-saturated, then*

$$D^-(\Re,\mathfrak{F},\mathcal{W}'_\infty)(A,B) = \varinjlim_{A'\to A\in\mathcal{W}'_\infty}\{A',B\}_{mor}$$

for all $A, B \in \Re$.

Proof. By the preceding proposition $D^-(\Re,\mathfrak{F},\mathcal{W}'_\infty)(A,B)$ is an abelian group. Now our proof is similar to that of Corollary 6.3. □

Proposition 10.5. *Let $\Re$ be Γ-closed and let $\mathfrak{F}$ be Γ-saturated. Suppose $B \in \Re$ is such that $B_+ \in \Re$ and $\varepsilon : B_+ \to k \in \mathfrak{F}$. Then*

$$D^-(\Re,\mathfrak{F},\mathcal{W}'_\infty)(A,B) = \varinjlim_{A'\to A\in\mathcal{W}'_\infty} kh(A',B)$$

for any $A \in \Re$.

Proof. By definition, $kh(A,B) = \operatorname{Ker}(\{A,B_+\}_{mor} \xrightarrow{\varepsilon} \{A,k\}_{mor})$. Taking a colimit over the arrows $A' \to A \in \mathcal{W}'_\infty$, one has a split exact sequence

$$\varinjlim_{A'\to A\in\mathcal{W}'_\infty} kh(A',B) \rightarrowtail \varinjlim_{A'\to A\in\mathcal{W}'_\infty}\{A',B_+\}_{mor} \twoheadrightarrow \varinjlim_{A'\to A\in\mathcal{W}'_\infty}\{A',k\}_{mor}.$$

By the preceding proposition $D^-(\Re,\mathfrak{F},\mathcal{W}'_\infty)$ is triangulated and by our assumption $\varepsilon : B_+ \to k \in \mathfrak{F}$. Hence one has a (split) triangle in $D^-(\Re,\mathfrak{F},\mathcal{W}'_\infty)$

$$\Omega k \to B \to B_+ \to k.$$

Therefore one has a split exact sequence

$$D^-(\Re,\mathfrak{F},\mathcal{W}'_\infty)(A,B) \rightarrowtail D^-(\Re,\mathfrak{F},\mathcal{W}'_\infty)(A,B_+) \twoheadrightarrow D^-(\Re,\mathfrak{F},\mathcal{W}'_\infty)(A,k).$$

Our assertion now follows from the preceding corollary. □

We are now in a position to prove the main computational result of the paper.

Theorem 10.6 (Cortiñas–Thom)**.** *Let $\Re$ be Γ-closed and let $\mathfrak{F}$ be Γ-saturated. Suppose $B_+ \in \Re$ and $\varepsilon : B_+ \to k \in \mathfrak{F}$ for any $B \in \Re$. Then there is an isomorphism of $\mathbb{Z}$-graded abelian groups for any $A \in \Re$*

$$\bigoplus_{n\in\mathbb{Z}} KH_n(A) \cong \bigoplus_{n\in\mathbb{Z}} D_{st}(\Re,\mathfrak{F})(k,\Omega^n A),$$

where the left hand side is homotopy K-theory in the sense of Weibel [24]*. Furthermore, this isomorphism is functorial in A.*

Proof. Let $KH(A)$ be any functorial (non-connective) homotopy K-theory spectrum. Then

$$KH(-) : \Re \to \operatorname{Ho}(Sp), \quad A \mapsto KH(A),$$

determines an excisive, homotopy invariant, stable homology theory with values in the homotopy category of spectra (see [24]). It follows that there is an isomorphism in $\operatorname{Ho}(Sp)$

$$\Omega^n KH(A) \cong KH(\Omega^n A), \quad A \in \Re, n \in \mathbb{Z}.$$

Thus,

$$KH_0(-) = \pi_0(KH(-)) : \Re \to \mathrm{Ab}$$

takes maps from $\mathcal{W}'_\infty$ to isomorphisms and $KH_n(A) \cong KH_0(\Omega^n A)$ for all $A \in \Re, n \in \mathbb{Z}$. So it is enough to prove the theorem for $n = 0$.

By Theorem 10.2 there is a natural isomorphism of abelian groups

$$KH_0(A) \cong \operatorname{colim}_n kh(k, \Sigma^n\Omega^n A),$$

functorial in A. By Proposition 10.5

$$D^-(\Re, \mathfrak{F}, \mathcal{W}'_\infty)(A, B) = \varinjlim_{A' \to A \in \mathcal{W}'_\infty} kh(A', B).$$

So one gets a natural map

$$\alpha : KH_0(A) \to \operatorname{colim}_n D^-(\Re, \mathfrak{F}, \mathcal{W}'_\infty)(k, \Sigma^n\Omega^n A) \xrightarrow{\cong} D^-(\Re, \mathfrak{F}, \mathcal{W}'_\infty)(k, A),$$

where the colimit maps on the right are induced by the isomorphism of endofunctors $\tau : \Sigma\Omega \xrightarrow{\cong} \mathrm{id}$ (see the proof of Proposition10.5).

Denote by β the composite of canonical maps

$$D^-(\Re, \mathfrak{F}, \mathcal{W}'_\infty)(k, A) \to \operatorname{Hom}_{\mathbb{Z}}(KH_0(k), KH_0(A)) \xrightarrow{\ell^*} \operatorname{Hom}_{\mathbb{Z}}(KH_0(\mathbb{Z}), KH_0(A)) \cong KH_0(A),$$

where $\ell : \mathbb{Z} \to k$ is the canonical map and the left arrow sends a diagram $k \xleftarrow{s} A' \xrightarrow{f} A$, $s \in \mathcal{W}'_\infty$, to $KH_0(f) \circ KH_0(s)^{-1}$. If A is unital and $E \in M_\infty(A)$ an idempotent, then the composite $\beta\alpha$ sends the class of E to its image in $KH_0(A)$. By construction of α, this is enough to prove that $\beta\alpha$ is the identity. To complete the proof we shall show that α is surjective.

Suppose $k \xleftarrow{s} A' \xrightarrow{f} A$, $s \in \mathcal{W}'_\infty$, is in $D^-(\Re, \mathfrak{F}, \mathcal{W}'_\infty)(k, A)$. Since $KH_0(s)$ is an isomorphism, there are $n \geqslant 0$ and a homomorphism $t : k \to \Sigma^n\Omega^n A'$ such that $\Sigma^n\Omega^n(s) \circ t : k \to \Sigma^n\Omega^n A$ is the image of $1 : k \to k$ in $kh(k, \Sigma^n\Omega^n k)$ under the map (14). Observe that $\tau^n \circ \Sigma^n\Omega^n(s) \circ t : k \to k$ equals 1 in $D^-(\Re, \mathfrak{F}, \mathcal{W}'_\infty)$. It follows that $t \in \mathcal{W}'_\infty$ because $\Sigma^n\Omega^n(s) \in \mathcal{W}'_\infty$.

We claim that $k \xleftarrow{s} A' \xrightarrow{f} A$ equals $\alpha(\Sigma^n\Omega^n(f) \circ t) = \tau^n \circ \Sigma^n\Omega^n(f) \circ t$. But this follows from commutativity of the diagram

$$\begin{tikzcd}
 & k \arrow[d, "t"] & \\
\Sigma^n\Omega^n(k) \arrow[d, "\tau^n"'] & \Sigma^n\Omega^n(A') \arrow[l, "\Sigma^n\Omega^n(s)"'] \arrow[r, "\Sigma^n\Omega^n(f)"] \arrow[d, "\tau^n"'] & \Sigma^n\Omega^n(A) \arrow[d, "\tau^n"] \\
k & A' \arrow[l, "s"'] \arrow[r, "f"] & A
\end{tikzcd}$$

The theorem is proved. □

11. Stable algebraic KK- and E-theories

We finish the paper by introducing stable algebraic KK- and E-theories. Throughout this section we assume fixed an underlying category $\mathcal{U}$, which can be a full subcategory of either the category of sets $Sets$ or $\operatorname{Mod} k$. The category $\mathcal{U}$ will depend on $\mathfrak{F}$. Namely, we shall assume that $\mathcal{U} \subseteq Sets$ if $\mathfrak{F} = \mathfrak{F}_{\mathrm{surj}}$ and $\mathcal{U} \subseteq \operatorname{Mod} k$ if $\mathfrak{F} = \mathfrak{F}_{\mathrm{spl}}$. We also assume that an admissible category of k-algebras $\Re$ is both T-closed and Γ-closed. We also assume $B_+ \in \Re$ for any $B \in \Re$. Then $M_\infty A \in \Re$ for any $A \in \Re$ and $M_\infty(f) \in \mathfrak{F}$ for any $f \in \mathfrak{F}$. Also, $\mathfrak{F}$ is Γ-saturated.

Definition. (1) The *stable algebraic KK-theory* for $\Re$ is the triangulated category $D_{st}(\Re, \mathfrak{F}_{\mathrm{spl}})$. The *stable algebraic KK-groups* are, by definition,

$$\mathfrak{K}\mathfrak{K}_n^{st}(A,B) := D_{st}(\Re, \mathfrak{F}_{\mathrm{spl}})(A, \Omega^n B), \quad n \in \mathbb{Z},\ A, B \in \Re.$$

(2) The *stable algebraic E-theory* for $\Re$ is the triangulated category $D_{st}(\Re, \mathfrak{F}_{\mathrm{surj}})$. The *stable algebraic E-groups* are, by definition,

$$\mathfrak{E}_n^{st}(A,B) := D_{st}(\Re, \mathfrak{F}_{\mathrm{surj}})(A, \Omega^n B), \quad n \in \mathbb{Z},\ A, B \in \Re.$$

Theorem 11.1 (Cortiñas–Thom)**.** *There are isomorphisms of $\mathbb{Z}$-graded abelian groups for any $A \in \Re$*

$$\bigoplus_{n\in\mathbb{Z}} KH_n(A) \cong \bigoplus_{n\in\mathbb{Z}} \mathfrak{K}\mathfrak{K}_n^{st}(k, A)$$

and

$$\bigoplus_{n\in\mathbb{Z}} KH_n(A) \cong \bigoplus_{n\in\mathbb{Z}} \mathfrak{E}_n^{st}(k, A),$$

functorial in A, where the left hand side is homotopy K-theory in the sense of Weibel [24]*.*

Proof. This a consequence of Theorem 10.6. □

Corollary 11.2 (Cortiñas–Thom [5])**.** *There is a natural isomorphism*

$$kk_*(k, A) \cong KH_*(A).$$

Proof. This a consequence of the preceding theorem and Corollary 9.4. □

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