R-SUPPORTS IN TENSOR TRIANGULATED CATEGORIES

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Abstract. Various authors classified the thick triangulated $\otimes$-subcategories of the category of compact objects for appropriate compactly generated tensor triangulated categories by using supports of objects. In this paper we introduce R-supports for ring objects, showing that these completely determine the thick triangulated $\otimes$-subcategories. R-supports give a general framework for the celebrated classification theorems by Benson-Carlson-Rickard-Friedlander-Pevtsova, Hopkins-Smith and Hopkins-Neeman-Thomason.

Inspired by the celebrated classification theorems of Benson-Carlson-Rickard-Friedlander-Pevtsova, Hopkins-Smith and Hopkins-Neeman-Thomason various authors studied abstract properties of supports in (compactly generated) tensor triangulated categories. Hovey, Palmieri and Strickland [19] used these in abstract stable homotopy theory, Balmer [2, 3] studies supports in essentially small symmetric monoidal triangulated categories. As an application, he reconstructs a noetherian scheme X from its derived category of perfect complexes $D_{per}(X)$. Buan-Krause-Solberg [8] introduced and studied supports in ideal lattices to extend applications to tensor abelian categories as well. Another classification theorem related to the category KK^G has been obtained by Dell'Ambrogio in [9].

It is very common that various authors *a priori* consider support data in a *spectral* topological space for *all* objects of a compactly generated tensor triangulated category. But both cases are *too restrictive* for classification theorems in general. Indeed, if we consider the category of p-local spectra $\mathcal{S}$, then there are many spectra with no $K(n)$-homology for any n, where $K(n)$ stands for the Morava K-theory (e.g. the most important one is the Brown-Comenetz dual of the sphere). However, a well-known theorem by Hopkins-Smith [18] says that there is no *ring spectrum* with trivial $K(n)$-homology for any n. On the other hand, classification results for some tensor abelian categories by Garkusha-Prest [13, 14, 15] (though not directly related to triangulated categories) *a priori* use properties of *non-spectral* spaces – however very close to the latter – such as injective spectra of Grothendieck categories equipped with Gabriel-Ziegler topology. In this paper we use such spaces as well to give some examples.

The main goal of this paper is to introduce and study the notion of an R-support datum $(\mathfrak{X}, \sigma)$ for ring objects in compactly generated tensor triangulated categories. It is essential that all, not necessarily compact, ring objects are used. The stable module category $\mathrm{Stmod}(G)$ over a finite group scheme G, the category of p-local spectra $\mathcal{S}$, the category $D_{\mathrm{Qcoh}}(X)$ of the complexes in the derived category of $\mathcal{O}_X$-modules over a quasi-compact and quasi-separated scheme X having quasi-coherent homology are examples of compactly generated tensor triangulated categories. The fact that $\mathrm{Stmod}(G)$ and $\mathcal{S}$

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have R-supports follows from properties of support varieties of modules and Morava K-theories respectively. One shows in section 2 that $D_{\mathrm{Qcoh}}(X)$ has R-supports as well.

The main result of the paper (Theorem 2.5) says that if a compactly generated tensor triangulated category $\mathcal{S}$ has a R-support datum $(\mathfrak{X}, \sigma)$ then there is a bijection between the set of all open subsets $O \subseteq \mathfrak{X}_\sigma$ and the set of all thick triangulated $\otimes$-subcategories of the triangulated category of compact objects $\mathcal{S}^c$, where $\mathfrak{X}_\sigma$ is a topological space associated to $(\mathfrak{X}, \sigma)$. As an application, we obtain a general framework for classification theorems by Benson-Carlson-Rickard-Friedlander-Pevtsova, Hopkins-Smith and Hopkins-Neeman-Thomason.

Using results from [13, 14, 15], we also show in section 2 that the topological space $\mathfrak{X}_\sigma$ need not be a spectral space in general. We also show that if $\mathcal{S}$ is the category of spectra then the associated topological space $\mathfrak{X}_\sigma$ is homeomorphic to $(\mathsf{Spec}\, V)^*$, where V is a valuation domain.

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1. Introduction

Let $\mathcal{S}$ be a triangulated category with arbitrary coproducts. An object x of $\mathcal{S}$ is said to be *compact* if for every family $\{y_i\}_{i\in I}$ of objects from $\mathcal{S}$ the canonical map

$$\oplus_{i\in I}\mathcal{S}(x, y_i) \longrightarrow \mathcal{S}(x, \coprod_{i\in I} y_i)$$

is an isomorphism. The category $\mathcal{S}$ is *compactly generated* if there exists a set $\mathcal{C}$ of compact objects of $\mathcal{S}$ such that $\mathcal{S}(\mathcal{C}, y) = 0$ (i.e. $\mathcal{S}(c, y) = 0$ for all $c \in \mathcal{C}$) implies $y = 0$ for every object y in $\mathcal{S}$. The triangulated subcategory of $\mathcal{S}$ consisting of compact objects will be denoted by $\mathcal{S}^c$. We observe that $\mathcal{S}$ coincides with the smallest triangulated subcategory closed with respect to coproducts and triangles and containing $\mathcal{S}^c$ (see [25, 2.1]). Also $\mathcal{S}$ is closed under taking direct products.

Definition 1.1. A *compactly generated tensor triangulated category* is a symmetric monoidal compactly generated triangulated category $(\mathcal{S}, \otimes, e)$ such that

- The tensor product is exact in each variable and it preserves coproducts. The Brown representability theorem yields function objects $\mathcal{H}om(x, y)$ satisfying

$$\mathrm{Hom}(x \otimes y, z) = \mathrm{Hom}(x, \mathcal{H}om(y, z)) \quad \text{for all } x, y, z \in \mathcal{S}.$$

- The unit e is in $\mathcal{S}^c$ and all compact objects are *strongly dualizable*, that is, the canonical morphism

$$c^\vee \otimes x \to \mathcal{H}om(c, x), \quad c^\vee := \mathcal{H}om(c, e),$$

is an isomorphism for all $c \in \mathcal{S}^c, x \in \mathcal{S}$.

Let c, d be compact objects in $\mathcal{S}$. The following properties are easily verified.

(1) $e^\vee \cong e$ and $c^{\vee\vee} \cong c$;
(2) $\mathrm{Hom}(x \otimes c^\vee, y) \cong \mathrm{Hom}(x, c \otimes y)$, for all x, y in $\mathcal{S}$;
(3) $c^\vee$ and $c \otimes d$ are compact.

These properties are used in the sequel without further comment.

Examples. (1) Let G be a finite group scheme defined over a field k. Thus, G has a commutative coordinate algebra $k[G]$ which is finite dimensional over k and which has a coproduct induced by the group multiplication on G, providing $k[G]$ with the structure of a Hopf algebra over k. We denote by kG the k-linear dual of $k[G]$ and refer to kG as the group algebra of G. Thus, kG is a finite dimensional, co-commutative Hopf algebra over k. By definition, a G-module is a comodule for $k[G]$ (with its coproduct structure) or equivalently a module for kG.

Recall that the stable module category $\mathrm{Stmod}(G)$ is the category whose objects are kG-modules, and whose group of homomorphisms between two kG-modules M, N is given by the following quotient:

$$\mathrm{Hom}_G(M,N)/\{f : M \to N \text{ factoring through some projective}\}.$$

So defined, $(\mathrm{Stmod}(G), \otimes_k, k)$ is a compactly generated tensor triangulated category, where $\mathrm{stmod}(G) := \mathrm{Stmod}(G)^c$ consists of the finite dimensional kG-modules (see [4, 11, 19]).

(2) The category $(\mathcal{S}, \wedge, S^0)$ of p-local spectra with p a prime and S^0 the p-local sphere spectrum is a compactly generated tensor triangulated category (see [19]). The objects of $\mathcal{S}$ are spectra whose homotopy groups are p-local, i.e., $\pi_*(X) = \pi_*(X) \otimes \mathbb{Z}_{(p)}$. The category $\mathcal{F} := \mathcal{S}^c$ consists of the finite p-local spectra.

(3) Recall from [26] that the derived category $D(X)$ with X a scheme of $\mathcal{O}_X$-modules is a closed symmetric monoidal category with the tensor product $\otimes^L_X : D(X) \times D(X) \to D(X)$, the unit $\mathcal{O}_X$ and the function object $R\mathcal{H}om(-,-)$. It is clear that $D(X)$ admits arbitrary coproducts. The tensor product is exact in each variable and it preserves coproducts.

Let $\mathrm{Qcoh}(X)$ denote the category of quasi-coherent sheaves. It is a Grothendieck category by [10]. We denote by $D_{\mathrm{Qcoh}}(X)$ the full triangulated subcategory of $D(X)$ of complexes with quasi-coherent cohomology. Clearly, $D_{\mathrm{Qcoh}}(X)$ is closed under coproducts. It follows from [20, 2.5.8] and [22, p. 36] that $D_{\mathrm{Qcoh}}(X)$ is closed under tensor products.

A complex of $\mathcal{O}_X$-modules is *perfect* if it is locally quasi-isomorphic to a bounded complex of vector bundles and we denote by $D_{per}(X)$ the corresponding full subcategory of $D(X)$. In particular, a perfect complex is in $D_{\mathrm{Qcoh}}(X)$ and if X is quasi-compact then it is in $D^b_{\mathrm{Qcoh}}(X)$.

Let X be a quasi-compact and quasi-separated scheme. Recall that being quasi-separated simply means that the intersection of two quasi-compact open subsets remains quasi-compact. Then $D_{\mathrm{Qcoh}}(X)$ is a compactly generated triangulated category with $(D_{\mathrm{Qcoh}}(X))^c = D_{per}(X)$ [7, 3.1.1]. Moreover, $D_{\mathrm{Qcoh}}(X)$ is generated by a single perfect complex [7, 3.1.1]. We also note that $R\mathcal{H}om(E,F) \in D_{\mathrm{Qcoh}}(X)$ for any $E, F \in D_{\mathrm{Qcoh}}(X)$.

Indeed, $R\mathcal{H}om(E,F) \in D_{\mathrm{Qcoh}}(X)$ for any $E \in D_{per}(X), F \in D_{\mathrm{Qcoh}}(X)$ [28, 2.4.1(c)]. The subcategory

$$\{G \in D_{\mathrm{Qcoh}}(X) \mid R\mathcal{H}om(G,F) \in D_{\mathrm{Qcoh}}(X)\}$$

is triangulated, closed under coproducts and contains all compact objects of $D_{\mathrm{Qcoh}}(X)$. Therefore it is equal to $D_{\mathrm{Qcoh}}(X)$ itself. We conclude that $D_{\mathrm{Qcoh}}(X)$ is a closed symmetric monoidal compactly generated triangulated category.

If X is a quasi-compact semi-separated scheme (being semi-separated means that the intersection of two affine open subsets remains affine) and $D(\mathrm{Qcoh}(X))$ is the derived category of quasi-coherent sheaves then the Bökstedt-Neeman theorem (see [6], [22, p. 34]) says that the canonical triangulated functor

$$D(\mathrm{Qcoh}(X)) \to D_{\mathrm{Qcoh}}(X) \tag{1.1}$$

is an equivalence. By [1] $D(\mathrm{Qcoh}(X))$ is a compactly generated tensor triangulated category, and hence so is $D_{\mathrm{Qcoh}}(X)$. The next result says that $D_{\mathrm{Qcoh}}(X)$ is a compactly generated tensor triangulated category for any quasi-compact and quasi-separated scheme.

Theorem 1.2 (Bökstedt-Neeman). *Let X be a quasi-compact and quasi-separated scheme. Then $(D_{\mathrm{Qcoh}}(X), \otimes^L_X, \mathcal{O}_X)$ is a compactly generated tensor triangulated category.*

Proof. As we have already shown, $D_{\mathrm{Qcoh}}(X)$ is a closed symmetric monoidal compactly generated triangulated category. We need to check that the natural map

$$E^\vee \otimes^L_X F \to R\mathcal{H}om(E, F) \tag{1.2}$$

is an isomorphism for any perfect complex E. This can be checked locally.

The map (1.2) is an isomorphism for any affine scheme (we also use the fact that (1.1) is an equivalence). We now use canonical isomorphisms (see [23, p. 12, p. 25])

$$R\mathcal{H}om(E, F)|_U \cong R\mathcal{H}om(E|_U, F|_U)$$

and

$$(E^\vee \otimes^L_X F)|_U \cong E^\vee|_U \otimes^L_U F|_U \cong (E|_U)^\vee \otimes^L_U F|_U$$

to prove that restriction of (1.2) to any affine open subset $U \subset X$ is an isomorphism. Therefore it is locally an isomorphism, and hence an isomorphism in $D_{\mathrm{Qcoh}}(X)$. □

2. R-supports

Definition 2.1. A *weak R-support datum* ("R" for ring) on a compactly generated tensor triangulated category $\mathcal{S}$ is a pair $(\mathfrak{X}, \sigma)$, where $\mathfrak{X}$ is a set and σ is a map which assigns to each ring object $a \in \mathcal{S}$ a subset $\sigma(a) \subseteq \mathfrak{X}$, such that

(1) $\sigma(e) = \mathfrak{X}$ and $\sigma(0) = \emptyset$.
(2) ("Tensor product theorem") $\sigma(a \otimes b) = \sigma(a) \cap \sigma(b)$ for any ring objects a, b with a compact.
(3) If $a \in \mathcal{S}$ is a ring object with $\sigma(a) = \emptyset$, then $a = 0$; that is σ "detects ring objects".

A weak R-support datum $(\mathfrak{X}, \sigma)$ is a *R-support datum* if $\sigma(a)$ is defined for any compact $a \in \mathcal{S}^c$ and the following axioms are true:

(4) $\sigma(a \oplus b) = \sigma(a) \cup \sigma(b)$ for any $a, b \in \mathcal{S}^c$.
(5) $\sigma(\Sigma a) = \sigma(a)$, for $\Sigma : \mathcal{S} \to \mathcal{S}$ the translation (shift, suspension) and $a \in \mathcal{S}^c$.
(6) $\sigma(a) \subset \sigma(b) \cup \sigma(c)$ for any exact triangle $a \to b \to c \to \Sigma a$ in $\mathcal{S}^c$.
(7) $\sigma(a \otimes b) = \sigma(a) \cap \sigma(b)$ for any $a, b \in \mathcal{S}^c$ with b a ring object.

Finally, an R-support datum is a *strict R-support datum* if the following axiom is satisfied:

(7′) $\sigma(a \otimes b) = \sigma(a) \cap \sigma(b)$ for any $a, b \in \mathcal{S}^c$.

Remark 2.2. Our notion for an R-support datum is different from that of a support datum used in [3, 8], because we define the sets $\sigma(a)$ for some non-compact objects.

If $(\mathfrak{X}, \sigma)$ is a weak R-support datum on $\mathcal{S}$, it follows from the axioms that the sets $\sigma(a)$ with $a \in \mathcal{S}^c$ a ring object form a basis of open sets for a topology on X. This topological space will be denoted by $\mathfrak{X}_\sigma$.

Recall from [16] that a topological space is *spectral* if it is T_0, quasi-compact, if the quasi-compact open subsets are closed under finite intersections and form an open basis, and if every non-empty irreducible closed subset has a generic point. Given a spectral topological space, X, Hochster [16] endows the underlying set with a new, "dual", topology, denoted X^*, by taking as open sets those of the form $Y = \bigcup_{i \in \Omega} Y_i$ where Y_i has quasi-compact open complement $X \setminus Y_i$ for all $i \in \Omega$. Then X^* is spectral and $(X^*)^* = X$ (see [16, Prop. 8]). For instance, if $X = \mathsf{Spec}(R)$ with R a commutative ring then the open subsets of $(\mathsf{Spec}(R))^*$ are of the form $\bigcup_{a \in \Omega} V(I_a)$ with each I_a a finitely generated ideal and $V(I_a) = \{P \in \mathsf{Spec}(R) \mid P \supset I_a\}$.

If $(\mathfrak{X}, \sigma)$ is a strict R-support datum on $\mathcal{S}$, then the topological space $\mathfrak{X}_\sigma$ often happens to be spectral. But it is not the case in general. Below we shall construct a strict R-support datum for which $\mathfrak{X}_\sigma$ is is not a T_0-space.

A thick triangulated subcategory $\mathcal{T}$ of $\mathcal{S}^c$ is a $\otimes$-*subcategory* if for every $a \in \mathcal{S}^c$ and every object $t \in \mathcal{T}$, the tensor product $a \otimes t$ also is in $\mathcal{T}$. Note that if e is a generator of $\mathcal{S}$ then every thick triangulated subcategory of $\mathcal{S}^c$ is a $\otimes$-subcategory. Given an object $a \in \mathcal{S}^c$, we denote by $\langle a \rangle$ the thick triangulated $\otimes$-subcategory generated by $\{a \otimes b\}_{b \in \mathcal{S}^c}$. If A is a family of compact objects, then $\bigcup_{a \in A} \langle a \rangle$ will stand for the thick triangulated $\otimes$-subcategory generated by $\{a \otimes b \mid a \in A, b \in \mathcal{S}^c\}$.

Proposition 2.3. (1) *Let $\mathcal{S}$ be a compactly generated tensor triangulated category and let $a \in \mathcal{S}^c$. Then $\langle a \rangle = \langle \mathcal{H}om(a, a) \rangle = \langle a^\vee \rangle$.*

(2) *An R-support datum $(\mathfrak{X}, \sigma)$ on $\mathcal{S}$ is a strict R-support datum if and only if $\sigma(a) = \sigma(\mathcal{H}om(a, a)) = \sigma(a^\vee)$ for any $a \in \mathcal{S}^c$.*

Proof. (1) By [19, A.2.6] a is a direct summand of $a \otimes a \otimes a^\vee \cong a \otimes \mathcal{H}om(a, a)$. Therefore $\langle a \rangle \subseteq \langle a \otimes a^\vee \rangle$. Obviously, $\langle a \otimes a^\vee \rangle \subseteq \langle a \rangle$. Consequently, $\langle a \otimes a^\vee \rangle = \langle a^\vee \rangle$.

(2) Suppose $(\mathfrak{X}, \sigma)$ is a strict R-support datum. Since a is a direct summand of $a \otimes \mathcal{H}om(a, a)$ then $\sigma(a) \subset \sigma(a \otimes \mathcal{H}om(a, a)) = \sigma(a) \cap \sigma(\mathcal{H}om(a, a)) \subset \sigma(\mathcal{H}om(a, a))$. On the other hand, $\sigma(\mathcal{H}om(a, a)) = \sigma(a \otimes a^\vee) = \sigma(a) \cap \sigma(a^\vee) \subset \sigma(a)$, and hence $\sigma(a) = \sigma(\mathcal{H}om(a, a))$. We also have $\sigma(a^\vee) = \sigma(\mathcal{H}om(a^\vee, a^\vee)) = \sigma(a^\vee \otimes a) = \sigma(\mathcal{H}om(a, a)) = \sigma(a)$.

Assume now that $\sigma(a) = \sigma(\mathcal{H}om(a, a)) = \sigma(a^\vee)$ for any $a \in \mathcal{S}^c$. Let $b \in \mathcal{S}^c$ then $\sigma(a) \cap \sigma(b) = \sigma(\mathcal{H}om(a, a)) \cap \sigma(\mathcal{H}om(b, b)) = \sigma(\mathcal{H}om(a, a) \otimes \mathcal{H}om(b, b)) = \sigma(a \otimes a^\vee \otimes b \otimes b^\vee)$. Since

$$\mathcal{H}om(a, \mathcal{H}om(b, c)) \cong \mathcal{H}om(a \otimes b, c)$$

in $\mathcal{S}^c$ [19, A.2.3] then

$$(a \otimes b)^\vee \cong \mathcal{H}om(a, b^\vee) \cong a^\vee \otimes b^\vee.$$

Therefore $\sigma(a \otimes a^\vee \otimes b \otimes b^\vee) = \sigma((a \otimes b) \otimes (a \otimes b)^\vee) = \sigma(\mathcal{H}om(a \otimes b, a \otimes b)) = \sigma(a \otimes b)$. □

Corollary 2.4. *Let $(\mathfrak{X}, \sigma)$ be a strict R-support datum on a compactly generated tensor triangulated category $\mathcal{S}$. Then the subsets $\sigma(a), a \in \mathcal{S}^c$, form a basis of open sets for the topological space $\mathfrak{X}_\sigma$.*

Examples. (1) Let G be a finite group scheme over a field k. If M is a kG-module, then the cohomology of G with coefficients in M is

$$H^*(G, M) = \mathrm{Ext}^*_G(k, M).$$

If $p = 2$, then $H^*(G,k)$ is itself a commutative k-algebra. If $p > 2$, then the even dimensional cohomology $H^\bullet(G,k)$ is a commutative k-algebra. We denote by

$$H^\bullet(G,k) = \begin{cases} H^*(G,k), & \text{if } p = 2 \\ H^{ev}(G,k), & \text{if } p > 2 \end{cases}$$

Extending results of Benson, Carlson, and Rickard [4] to finite group schemes, Friedlander and Pevtsova [11] define a strict R-support datum $(\mathfrak{X},\sigma)$ on $\mathrm{Stmod}(G)$. One sets $\mathfrak{X} = (\mathsf{Proj}(H^\bullet(G,k)))^*$, the dual space to the projective support variety of the finite group scheme G over the field k. For $M \in \mathrm{stmod}(G)$, $\sigma(M)$ corresponds naturally to the cohomological support variety $V_G(M) := \mathsf{ann}_{H^\bullet(G,k)}(\mathrm{Ext}^*_G(M,M))$ of M. It is easy to show that every basic open subset in $(\mathsf{Proj}(H^\bullet(G,k)))^*$ is of the form $\sigma(M)$ with $M \in \mathrm{stmod}(G)$. However, $\sigma(M)$ does not have an evident cohomological interpretation for infinite dimensional kG-modules. One has $\sigma(M \otimes_k N) = \sigma(M) \cap \sigma(N)$ for any $M, N \in \mathrm{Stmod}(G)$ [11, 5.2]. Moreover, $\sigma(M) = 0$ implies $M = 0$ in $\mathrm{Stmod}(G)$ [11, 5.5].

(2) Let $\mathcal{S}$ be the category of p-local spectra. For each $n \geqslant 1$ there is a spectrum $K(n)$ called the *n-th Morava K-theory* whose coefficient ring $K(n)_*$ is isomorphic to $\mathbb{F}_p[v_n, v_n^{-1}]$ with $|v_n| = 2(p^n - 1)$. We also set $K(0)$ to be the rational Eilenberg-Mac Lane spectrum $H\mathbb{Q}$ and $K(\infty)$ the mod-p Eilenberg-Mac Lane spectrum $H\mathbb{F}_p$. These theories have the following properties (see [18]).

- For every spectrum X, $K(n) \wedge X$ has the homotopy type of a wedge of suspensions of $K(n)$.
- Künneth isomorphism: $K(n)_*(X \wedge Y) \cong K(n)_*X \otimes_{K(n)_*} K(n)_*Y$. In particular $K(n)_*(X \wedge Y) = 0$ if and only if either $K(n)_*X = 0$ or $K(n)_*Y = 0$.
- If $X \neq 0$ and finite, then for all $n \gg 0$, $K(n)_*X \neq 0$.
- For each n, $K(n+1)_*X = 0$ implies $K(n)_*X = 0$.
- (Nilpotence theorem) Morava K-theories detect ring spectra: if R is a non-trivial ring spectrum, then there exists a n ($0 \leqslant n \leqslant \infty$) such that $K(n)_*R \neq 0$.

Let $\mathfrak{X}$ be the set $\mathbb{Z}_+ \cup \{\infty\}$. To any spectrum $X \in \mathcal{S}$ one associates a subset

$$\sigma(X) = \{n \in \mathfrak{X} \mid K(n)_*X \neq 0\}.$$

The properties above imply that $(\mathfrak{X},\sigma)$ is a support datum on $\mathcal{S}$. Given a finite spectrum $F \in \mathcal{F}$ one has $\sigma(F) = O_n := \{n, n+1, \ldots\} \cup \{\infty\}$, where $n = \min\{s \mid K(s)_*F \neq 0\}$. It follows from Mitchell's theorem [21] that for any $n \geqslant 0$ there exists a finite spectrum $F_n \in \mathcal{F}$ such that $\sigma(F) = O_n$. It is easy to see that the lattice of open sets for the topological space $\mathfrak{X}_\sigma$ is a totally ordered set and looks as follows:

$$\sigma(0) = \emptyset = O_\infty \subsetneq \cdots \subsetneq O_{n+1} \subsetneq O_n \subsetneq O_{n-1} \subsetneq \cdots \subsetneq O_1 \subsetneq O_0 = \mathfrak{X}.$$

Clearly, the space $\mathfrak{X}_\sigma$ is spectral. Below we shall show that $\mathfrak{X}_\sigma$ is homeomorphic to $(\mathsf{Spec}(V))^*$ for some valuation domain V.

(3) Let X be a quasi-compact and quasi-separated scheme. For any $F \in D_{\mathrm{Qcoh}}(X)$ we denote by

$$\sigma(F) := \{x \in X \mid F \otimes^L_X k(x) \neq 0\},$$

where $k(x)$ is the residue field at x. If $F \in D_{per}(X)$ then $\sigma(F) = \mathsf{supph}_X(F) := \bigcup_{n\in\mathbb{Z}} \mathsf{Supp}(H^n(F))$ is the union of the supports in the classic sense of the cohomology sheaves of F [27, 3.3].

Let $x \in X$ and let U be an affine neighborhood of x. It follows from [6, 2.17] that $(F \otimes_X^L k(x))|_U \cong F|_U \otimes_U^L k(x)$ is a direct sum of suspensions of $k(x)$ for any $F \in D_{\mathrm{Qcoh}}(X)$. Therefore $\sigma(F \otimes_X^L G) = \sigma(F) \cap \sigma(G)$ for any $F, G \in D_{\mathrm{Qcoh}}(X)$.

Suppose $\sigma(F) = 0$ with $F \in D_{\mathrm{Qcoh}}(X)$ a ring object. Let $\iota : \mathcal{O}_X \to F$ be the unit map. Then by the Nilpotence Theorem [27, 3.6] there is $n \geqslant 1$ such that $\otimes^n \iota = 0$ in $D_{\mathrm{Qcoh}}(X)$. It follows that $1_F = 0$, hence $F = 0$. We conclude that σ detects ring objects.

Let $Y \subseteq X$ be a closed subspace such that $X \setminus Y$ is quasi-compact. Then there exists a perfect complex F such that $\sigma(F) = Y$ [27, 3.4]. On the other hand, $\sigma(E)$, $E \in D_{per}(X)$, is closed in X and $X \setminus \sigma(E)$ is quasi-compact [27, 3.3]. Let $\mathfrak{X} := X^*$; then $(\mathfrak{X}, \sigma)$ is plainly a strict R-support datum on $D_{\mathrm{Qcoh}}(X)$.

Theorem 2.5 (Classification)**.** *Let $(\mathfrak{X}, \sigma)$ be a weak R-support datum on a compactly generated tensor triangulated category $\mathcal{S}$. Consider the maps*

$$O \overset{\varphi}{\longmapsto} \mathcal{T}_O = \bigcup \{\langle a \rangle \mid a \in \mathcal{S}^c \text{ is a ring object with } \sigma(a) \subseteq O\}$$

and

$$\mathcal{T} \overset{\psi}{\longmapsto} O_{\mathcal{T}} = \bigcup \{\sigma(a) \mid a \in \mathcal{T} \text{ is a ring object}\}.$$

Then $\varphi\psi = \mathrm{id}$ or, equivalently, $\mathcal{T} = \bigcup\{\langle a \rangle \mid a \in \mathcal{S}^c$ is a ring object with $\sigma(a) \subseteq O_{\mathcal{T}}\}$, that is $(\mathfrak{X}, \sigma)$ determines thick triangulated $\otimes$-subcategories.

If $(\mathfrak{X}, \sigma)$ is an R-support datum then the topological space $\mathfrak{X}_\sigma$ is quasi-compact, the quasi-compact open subsets are closed under finite intersections and form an open basis, and the maps φ, ψ induce bijections between

1. *the set of all open subsets $O \subseteq \mathfrak{X}_\sigma$,*
2. *the set of all thick triangulated $\otimes$-subcategories of $\mathcal{S}^c$.*

Proof. Let $\mathcal{T}$ be a thick triangulated $\otimes$-subcategory of $\mathcal{S}^c$. By Proposition 2.3 and the fact that $\mathcal{H}om(a, a)$ is canonically a ring object we have

$$\mathcal{T} = \bigcup_{a \in \mathcal{T}} \langle a \rangle = \bigcup_{a \in \mathcal{T}} \langle \mathcal{H}om(a, a) \rangle = \bigcup \{\langle a \rangle \mid a \in \mathcal{T} \text{ is a ring object}\}.$$

It immediately follows that $\mathcal{T} \subset \mathcal{T}_{O_{\mathcal{T}}}$.

Let $b \in \mathcal{T}_{O_{\mathcal{T}}}$ be a ring object with $\emptyset \neq \sigma(b) \subset O_{\mathcal{T}}$. Let $L_{\mathcal{T}}$ be the localization functor on $\mathcal{S}$ associated with the finite localizing subcategory generated by $\mathcal{T}$. Then $b \otimes L_{\mathcal{T}}(e) \cong L_{\mathcal{T}}(b)$ [19, 3.3.1] and $L_{\mathcal{T}}(e)$ is a commutative ring object in $\mathcal{S}$ [19, 3.1.8]. Therefore $b \otimes L_{\mathcal{T}}(e)$ is a ring object. We have that $L_{\mathcal{T}}(b) \neq 0$ if and only if $\sigma(b \otimes L_{\mathcal{T}}(e)) = \sigma(b) \cap \sigma(L_{\mathcal{T}}(e)) \neq \emptyset$. If it is the case then there is a ring object $a \in \mathcal{T}$ such that $\sigma(a) \cap \sigma(b \otimes L_{\mathcal{T}}(e)) \neq \emptyset$, because

$$\emptyset \neq \sigma(b \otimes L_{\mathcal{T}}(e)) \subset \sigma(b) \subset O_{\mathcal{T}} = \bigcup \{\sigma(a) \mid a \in \mathcal{T} \text{ is a ring object}\}.$$

It follows that $\sigma(a) \cap \sigma(b \otimes L_{\mathcal{T}}(e)) = \sigma(a \otimes b \otimes L_{\mathcal{T}}(e)) \neq \emptyset$, and hence $a \otimes b \otimes L_{\mathcal{T}}(e) \neq 0$. On the other hand, $a \otimes b \otimes L_{\mathcal{T}}(e) \cong b \otimes a \otimes L_{\mathcal{T}}(e) = 0$ since $a \otimes L_{\mathcal{T}}(e) = 0$, a contradiction. Thus $\mathcal{T} = \mathcal{T}_{O_{\mathcal{T}}}$ or, equivalently, $\varphi\psi = \mathrm{id}$.

Now suppose $(\mathfrak{X}, \sigma)$ is an R-support datum. We want to show that $\psi\varphi = \mathrm{id}$. Given a ring object $a \in \mathcal{S}^c$, it follows from the axioms that $b \in \langle a \rangle$ implies $\sigma(b) \subset \sigma(a)$. Therefore $O_{\langle a \rangle} = \sigma(a)$. Since $\varphi\psi = \mathrm{id}$ then

$$\langle a \rangle = \mathcal{T}_{O_{\langle a \rangle}} = \mathcal{T}_{\sigma(a)} = \bigcup \{\langle b \rangle \mid b \in \mathcal{S}^c \text{ is a ring object with } \sigma(b) \subseteq \sigma(a)\},$$

and hence $\psi\varphi(\sigma(a)) = \psi(\langle a\rangle) = \sigma(a)$.

Every open subset O of $\mathfrak{X}_\sigma$ is, by definition, a union $\bigcup \sigma(a)$ with each $a \in \mathcal{S}^c$ a ring object. Clearly,

$$O = \bigcup\{\sigma(a) \mid a \in \mathcal{S}^c \text{ is a ring object with } \sigma(a) \subseteq O\}.$$

Suppose $b \in \mathcal{T}_O = \bigcup\{\langle a\rangle \mid a \in \mathcal{S}^c \text{ is a ring object with } \sigma(a) \subseteq O\}$ is a ring object. Then there are finitely many ring objects $a_1, a_2, \dots, a_n \in \mathcal{T}_O$ such that $\bigcup_{i=1}^n \sigma(a_i) \subset O$ and $b \in \bigcup_{i=1}^n \langle a_i\rangle$. It follows that $\sigma(b) \subset \bigcup_{i=1}^n \sigma(a_i) \subset O$. We see that $\psi\varphi = \mathrm{id}$.

Let us show now that each basic open set $\sigma(a)$ with $a \in \mathcal{S}^c$ a ring object is quasi-compact. Suppose $\sigma(a) \subset \bigcup_{i\in I} \sigma(b_i)$. Then

$$\langle a\rangle = \varphi(\sigma(a)) \subset \bigcup_{i\in I}\langle b_i\rangle = \varphi(\bigcup_{i\in I}\sigma(b_i)).$$

Then there are $i_1, i_2, \dots, i_n \in I$ such that $\langle a\rangle \subset \bigcup_{s=1}^n \langle b_{i_s}\rangle$. It follows that $\sigma(a) = \psi(\langle a\rangle) \subset \bigcup_{s=1}^n \psi(\langle b_{i_s}\rangle) = \bigcup_{s=1}^n \sigma(b_{i_s})$, hence $\sigma(a)$ is quasi-compact. Then $X = \sigma(e)$ is quasi-compact as well. Since $\sigma(a)\cap\sigma(b) = \sigma(a\otimes b)$ for any two compact ring objects a and b, we conclude that the quasi-compact open subsets are closed under finite intersections. □

Corollary 2.6. *Let $(\mathfrak{X}, \sigma)$ be a strict R-support datum on a compactly generated tensor triangulated category $\mathcal{S}$. Then the maps*

$$O \overset{\varphi}{\longmapsto} \mathcal{T}_O = \bigcup\{a \in \mathcal{S}^c \mid \sigma(a) \subseteq O\}, \qquad \mathcal{T} \overset{\psi}{\longmapsto} O_{\mathcal{T}} = \bigcup\{\sigma(a) \mid a \in \mathcal{T}\}$$

induce bijections between

(1) *the set of all open subsets $O \subseteq \mathfrak{X}_\sigma$,*
(2) *the set of all thick triangulated $\otimes$-subcategories of $\mathcal{S}^c$.*

Proof. This is a consequence of Proposition 2.3, Theorem 2.5 and Corollary 2.4. □

We derive from the preceding corollary and the corresponding examples considered above the celebrated classification theorems by Benson-Carlson-Rickard-Friedlander-Pevtsova, Hopkins-Smith and Hopkins-Neeman-Thomason.

Corollary 2.7 ([5, 11, 18, 17, 24, 27])**.** (1) *Let G be a finite group scheme over a field k. Then the assignments*

$$\mathcal{T} \longmapsto Y(\mathcal{T}) = \bigcup_{M\in\mathcal{T}} V_G(M), \quad Y \longmapsto \mathcal{T}(Y) = \{M \in \mathrm{stmod}(G) \mid V_G(M) \subseteq Y\}$$

induce a bijection between the open subsets of $(\mathsf{Proj}(H^\bullet(G,k)))^*$ and the thick triangulated $\otimes$-subcategories of $\mathrm{stmod}(G)$.

(2) *Let $\mathcal{C}_0 := \mathcal{F}$ be the category of finite p-local spectra, and for $n \geqslant 1$, let $\mathcal{C}_n := \{X \in \mathcal{F} \mid K(n-1)_*X = 0\}$, and finally let $\mathcal{C}_\infty$ denote the subcategory of contractible spectra. Then a subcategory $\mathcal{C}$ of $\mathcal{F}$ is thick triangulated if and only if $\mathcal{C} = \mathcal{C}_n$ for some n. Further these subcategories form a decreasing filtration of $\mathcal{F}$:*

$$\mathcal{C}_\infty \subsetneq \cdots \subsetneq \mathcal{C}_{n+1} \subsetneq \mathcal{C}_n \subsetneq \mathcal{C}_{n-1} \subsetneq \cdots \subsetneq \mathcal{C}_1 \subsetneq \mathcal{C}_0.$$

(3) *Let X be a quasi-compact and quasi-separated scheme. Then the assignments*

$$\mathcal{T} \longmapsto Y(\mathcal{T}) = \bigcup_{E\in\mathcal{T}} \mathsf{supph}_X(E), \quad Y \longmapsto \mathcal{T}(Y) = \{E \in D_{per}(X) \mid \mathsf{supph}_X(E) \subseteq Y\}$$

induce a bijection between the open subsets of X^* and the thick triangulated $\otimes$-subcategories of $D_{per}(X)$.

A strict R-support datum on a compactly generated tensor triangulated category is not a spectral space in general. As an example, let us consider the category $D_{\mathrm{Qcoh}}(X)$, where X is quasi-compact and quasi-separated. We denote by $\mathsf{Sp}(X)$ the set of isomorphism classes of indecomposable injective objects in the Grothendieck category of quasi-coherent sheaves $\mathrm{Qcoh}(X)$. Given $E \in \mathrm{Qcoh}(X)$ let $P(E)$ be the point corresponding to E (see [13, 15]). If $X = \mathsf{Spec}(R)$ is affine then $P(E)$ is the sum of annihilator ideals in R of non-zero elements of E. Given $G \in D_{\mathrm{Qcoh}}(X)$ one sets

$$\sigma(G) = \{E \in \mathsf{Inj}(X) \mid G \otimes^L_X k(P(E)) \neq 0\}.$$

It follows from [13, 15] and arguments above that $(\mathsf{Inj}(X), \sigma)$ is a strict R-support datum for $D_{\mathrm{Qcoh}}(X)$. However the space $\mathsf{Inj}(X)$ is not T_0 in general [14].

To conclude, consider the category of p-local spectra $\mathcal{S}$. Let us show, as promised, that $\mathfrak{X}_\sigma$ is homeomorphic to $(\mathsf{Spec}(V))^*$ for some valuation domain V, where $\mathcal{X} = \mathbb{Z}_+ \cup \{\infty\}$, $\sigma(F) = O_n := \{n, n+1, \dots\} \cup \{\infty\}$ for any $F \in \mathcal{F}$, $n = \min\{s \mid K(s)_* F \neq 0\}$.

Looking at the lattice of open sets in $\mathfrak{X}_\sigma$, a ring theorist immediately recognizes the lattice of prime ideals of a valuation domain in it. G. Puninski has pointed out to me how to construct such a ring.

Let α be the ordinal $\aleph_0$ and let V be any commutative valuation domain with value group isomorphic to $\Gamma = \oplus_{n\in\alpha}\mathbb{Z}$, an α-indexed direct sum of copies of $\mathbb{Z}$. The order on Γ is defined as follows: $(a_n)_{n\in\alpha} > (b_n)_{n\in\alpha}$ if $a_i > b_i$ for some i and $a_k = b_k$ for every $k < i$.

Let ∞ be a symbol regarded as larger than any element of Γ. We set $\infty + n = \infty$ for all $n \in \alpha$. Let k denote the field of quotients of V. Then there is a valuation

$$v : k \to \Gamma \cup \{\infty\}$$

such that $V = \{x \in k \mid v(x) \geqslant 0\}$ [12, p. 11].

By a filter $\mathfrak{F}$ in $\Gamma^+ := \{x \in \Gamma \mid x \geqslant 0\}$ is meant a non-empty proper subset $\mathfrak{F}$ of Γ^+ such that

$$x \in \mathfrak{F} \text{ and } x \leqslant y \in \Gamma^+ \text{ imply } y \in \mathfrak{F}.$$

$\mathfrak{F}$ is a *prime filter* if $x, y \in \Gamma^+ \setminus \mathfrak{F}$ implies $x + y \in \Gamma^+ \setminus \mathfrak{F}$, and a *principal filter* if, for some $x \in \Gamma^+$, $\mathfrak{F} = \{y \in \Gamma^+ \mid y \geqslant x\}$.

It follows from [12, I.3.2] that the correspondences

$$I \mapsto v(I) \quad \text{and} \quad \mathfrak{F} \mapsto I(\mathfrak{F}) = \{x \in V \mid v(x) \in \mathfrak{F}\}$$

define a bijection between the set of non-zero ideals of V and the set of filters in Γ^+. In particular, prime (principle) ideals correspond to prime (principal) filters.

Theorem 2.8. *The topological space $\mathfrak{X}_\sigma$ is naturally homeomorphic to $(\mathsf{Spec}(V))^*$.*

Proof. We denote by

$$\mathfrak{F}_m = \{x \in \Gamma^+ \mid \min(\mathsf{Supp}(x)) \leqslant m\},$$

where $\mathsf{Supp}(x = (x_n)_{n\in\alpha}) = \{n \in \alpha \mid x_n \neq 0\}$. It follows that $x_{\min(\mathsf{Supp}(x))} > 0$ for any $x \in \mathfrak{F}_m$. Obviously, $\mathfrak{F}_m \subsetneq \mathfrak{F}_{m+1}$ for any $m \in \alpha$. Let $\mathfrak{F}_\infty := \bigcup_m \mathfrak{F}_m = \{x \in \Gamma^+ \mid v(x) > 0\}$; then the prime ideal P_∞ corresponding to this filter is maximal. We claim that a filter $\mathfrak{F} \neq \mathfrak{F}_\infty$ is prime if and only if it is equal to $\mathfrak{F}_m$ for some m.

There exists $m \geqslant 1$ such that $\mathfrak{F}_m \cap \mathfrak{F} \neq \emptyset$. Let us show that $\mathfrak{F} \supset \mathfrak{F}_m$. We first assume that $m = 1$. Therefore there is $x \in \mathfrak{F} \cap \mathfrak{F}_1$. Suppose $(1, 0, 0, \ldots) \notin \mathfrak{F}$. If $x_1 = 1$ then the first non-zero component $x_k, k > 1$, must be positive, because otherwise $x < (1, 0, 0, \ldots) \in \mathfrak{F}$. Hence $x = (1, 0, 0, \ldots) + (0, \ldots, x_k, x_{k+1}, \ldots) \in \mathfrak{F}$ implies $(0, \ldots, x_k, x_{k+1}, \ldots) \in \mathfrak{F}$. But $(0, \ldots, x_k, x_{k+1}, \ldots) < (1, 0, 0, \ldots) \in \mathfrak{F}$, a contradiction. If $x_1 > 1$ then $x = (1, 0, 0, \ldots)+(x_1-1, x_2, x_3, \ldots) \in \mathfrak{F}$ implies $(x_1-1, x_2, x_3, \ldots) \in \mathfrak{F}$. Continuing this procedure $x_1 - 1$ times we get $(1, x_2, x_3, \ldots) \in \mathfrak{F}$, and hence $(1, 0, 0, \ldots) \in \mathfrak{F}$ by above, again a contradiction. We conclude that $(1, 0, 0, \ldots)$ belongs to $\mathfrak{F}$.

Now let $x \in \mathfrak{F}_1$; then $x_1 > 0$. Let x_k be the first positive component of x such that $k \geqslant 2$. Then $(1, 0, 0, \ldots) < (2x_1, \ldots, 2x_{k-1}, 0, \ldots) \in \mathfrak{F}$. Therefore $(2x_1, \ldots, 2x_{k-1}, 0, \ldots) < 2x$ and $2x = x + x$ imply $x \in \mathfrak{F}$ since $\mathfrak{F}$ is prime, and hence $\mathfrak{F} \supset \mathfrak{F}_1$. If $m > 1$ and $\mathfrak{F} \cap \mathfrak{F}_m \neq \emptyset$ then the fact that $\mathfrak{F} \supset \mathfrak{F}_m$ is proved similar to the case $m = 1$.

By assumption, $\mathfrak{F} \neq \mathfrak{F}_\infty$, and therefore $\mathfrak{F} \cap (\mathfrak{F}_{m+1} \setminus \mathfrak{F}_m) = \emptyset$ for some m. Suppose m is minimal such. We see that an element $x \in \mathfrak{F}$ if and only if its first positive component x_k must be such that $k \leqslant m$. Thus $\mathfrak{F} \subset \mathfrak{F}_m$, and hence $\mathfrak{F} = \mathfrak{F}_m$.

Therefore the chain of the filters in Γ^+

$$\mathfrak{F}_\infty \supsetneq \cdots \supsetneq \mathfrak{F}_{n+1} \supsetneq \mathfrak{F}_n \supsetneq \cdots \supsetneq \mathfrak{F}_1$$

gives rise to a chain of the proper non-zero prime ideals in V

$$P_\infty \supsetneq \cdots \supsetneq P_{n+1} \supsetneq P_n \supsetneq \cdots \supsetneq P_1.$$

The trivial ideal is prime as well and therefore one obtains that the set of all proper prime ideals in V is well-ordered and looks as follows:

$$P_\infty \supsetneq \cdots \supsetneq P_{n+1} \supsetneq P_n \supsetneq \cdots \supsetneq P_1 \supsetneq P_0 = 0.$$

We observe that for any $0 \leqslant n < \infty$, $P_n = \sqrt{a_n}$ with $a_0 := 0$ and $a_{n \geqslant 1}$ the element whose n-th component is 1 and all other components are zero.

Suppose $V(P_\infty) = \{P_\infty\}$ is open in $(\mathsf{Spec}\, V)^*$; then $V(P_\infty) = \bigcup_\lambda V(I_\lambda)$ with each I_λ finitely generated. Since P_∞ is the largest proper ideal each $V(I_\lambda)$, if non-empty, equals $\{P_\infty\}$. Therefore $P_\infty = \sqrt{I_\lambda}$ for some λ. But the prime radical of every finitely generated ideal in V is prime (since V is a valuation ring) and different from P_∞. To see the latter, we have, since I_λ is finitely generated, that all elements of I_λ have value $\geqslant (a'_n)_n$ for some $(a'_n)_n \in \Gamma^+$ with $a'_n = 0$ for all $n \leqslant N$ for some fixed N. (Recall that every finitely generated ideal in a valuation ring must be principal [12, I.1.6].) It follows that there is a prime ideal properly between I_λ and P_∞. This gives a contradiction. Therefore $V(P_\infty) = \{P_\infty\}$ can not be open in $(\mathsf{Spec}\, V)^*$.

It follows that $V(a_n) = \{P_n, P_{n+1}, \ldots\} \cup \{P_\infty\}$ for any $n \geqslant 0$ and the lattice of open subsets of $(\mathsf{Spec}\, V)^*$ looks as follows:

$$\emptyset = V(1) \subsetneq \cdots \subsetneq V(a_{n+1}) \subsetneq V(a_n) \subsetneq V(a_{n-1}) \subsetneq \cdots \subsetneq V(a_1) \subsetneq V(a_0) = (\mathsf{Spec}\, V)^*.$$

Now the desired homeomorphism is plainly given by the maps $n \in \mathfrak{X}_\sigma \leftrightarrows P_n \in (\mathsf{Spec}\, V)^*$. □

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