Banach and Hilbert Spaces
MAP391/ MAPM91

Lecture Notes 2008 – 2009

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Notations

$\mathbb{N}$ the set positive integers;

$\mathbb{R}$ field of real numbers;

$\mathbb{C}$ field of complex numbers;

Recommended Texts

The following books (amongst many others) could be recommended as a complementary reading for this course:

1. L. Debnath, P. Mikusiński, *Introduction to Hilbert spaces with applications.* Second edition. Academic Press, Inc., San Diego, CA, 1999. xx+551 pp.

2. B. Rynne, M. Youngson, *Linear functional analysis.* Springer Undergraduate Mathematics Series. Springer-Verlag London, Ltd., London, 2000. x+273 pp.

3. V. Hutson, J. Pym, M. Cloud, *Applications of functional analysis and operator theory.* Second edition. Mathematics in Science and Engineering, 200. Elsevier B. V., Amsterdam, 2005. xiv+426 pp.

The text of this Lecture Notes follows closely the content of the first chapters of [1] and the Russian textbook:

V. A. Trenogin, *Functional analysis.* (Russian) "Nauka", Moscow, 1980. 496 pp.

Unfortunately there is no English translation of this book.

Contents

1 Linear vector spaces

1.1 Linear vector spaces

Definition 1.1. By a *linear vector space* over an algebraic field $\mathbb{F}$ we mean a nonempty set $\mathbb{X}$ equipped with two operations:

addition $(x, y) \to x + y$ from $\mathbb{X} \times \mathbb{X}$ into $\mathbb{X}$,

scalar multiplication $(\lambda, x) \to \lambda x$ from $\mathbb{F} \times \mathbb{X}$ into $\mathbb{X}$,

such that the following *axioms* are satisfied for every $x, y, z \in \mathbb{X}$ and $\lambda, \mu \in \mathbb{F}$:

$(A1)$ $x + y = y + x$;

$(A2)$ $(x + y) + z = x + (y + z)$;

$(A3)$ there exists an element $0 \in \mathbb{X}$, called *zero*, such that $x + 0 = x$;

$(A4)$ there exists an element $-x \in \mathbb{X}$, called *opposite of* x, such that $x + (-x) = 0$;

$(A5)$ $\lambda(\mu x) = (\lambda\mu)x$;

$(A6)$ $(\lambda + \mu)x = \lambda x + \mu x$;

$(A7)$ $\lambda(x + y) = \lambda x + \lambda y$;

$(A8)$ $1x = x$, where $1 \in \mathbb{F}$ is the *identity* in $\mathbb{F}$;

Elements of $\mathbb{X}$ are called *vectors* (or sometimes *elements*); elements of $\mathbb{F}$ are called *scalars* (or sometimes *numbers*).

Remark 1.2. In this course we mostly consider the case when $\mathbb{F} = \mathbb{R}$ and sometimes the case $\mathbb{F} = \mathbb{C}$. If $\mathbb{F} = \mathbb{R}$ then $\mathbb{X}$ is called *real vector space* or simply *vector space*. If $\mathbb{F} = \mathbb{C}$ then $\mathbb{X}$ is called *complex vector space*. In what follows, if we do not refer to a specific field then we always assume that $\mathbb{F} = \mathbb{R}$. On the other hand, if $\mathbb{F} = \mathbb{C}$ then we always explicitly mention this.

Exercise 1.3. (i) Prove that for every $x, y \in \mathbb{X}$ there exist a vector $z \in \mathbb{X}$, such that $x + z = y$;

(ii) Prove that $0\,x = 0$ (where on the left 0 is the scalar and on the right 0 is the zero vector).

Remark 1.4. In what follows we use 0 to denote both the scalar 0 and the zero vector, this will not cause any confusion.

Example 1.5. The set $\{0\}$ is a (trivial) linear vector space. The scalar fields $\mathbb{R}$ and $\mathbb{C}$ are the simplest nontrivial linear vector spaces.

Example 1.6. (Classical vector spaces) The spaces

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) \,|\, x_1, \ldots, x_n \in \mathbb{R}\},$$

$$\mathbb{C}^n = \{(x_1, \ldots, x_n) \,|\, x_1, \ldots, x_n \in \mathbb{C}\},$$

are classical examples of linear vector spaces. They are studied in details in *MAC211 Vector Spaces*.

Example 1.7. (Function spaces) Let Ω be an arbitrary nonempty subset of $\mathbb{R}^n$. Denote by $L(\Omega)$ the set of all functions from Ω into $\mathbb{R}$. Let $x, y \in L(\Omega)$ be functions and $\lambda \in \mathbb{R}$ a scalar. Define the *addition* and *scalar multiplication* in the natural pointwise way:

$$(x + y)(t) := x(t) + y(t),$$

$$(\lambda x)(t) := \lambda x(t).$$

Then $L(\Omega)$ becomes a linear vector space. Note that elements (vectors) of $L(\Omega)$ are functions from Ω into $\mathbb{R}$. Linear vector spaces whose elements are functions are often called *function spaces.*

In what follows we mostly consider the cases when Ω is an interval $[a, b]$ of the real line, or $\Omega = \mathbb{R}$. Further examples of function spaces include:

$C(\Omega)$ – the space of all continuous real–valued functions on Ω;

$C^k(\Omega)$ – the space of all k–times continuously differentiable real–valued functions on Ω;

$\mathcal{P}_m(\Omega)$ – the space of all real polynomials on Ω, of degree non greater then m;

$\mathcal{P}(\Omega)$ – the space of all real polynomials on Ω.

Other examples of function spaces will be introduced later.

Exercise 1.8. Verify that the set $L(\Omega)$ with the natural pointwise addition and scalar multiplication indeed satisfies axioms $(A1 - A8)$.

Example 1.9. (Spaces of sequences) Denote by l the set of all (infinite) sequences of real numbers. Let $x = (x_1, x_2, \dots), y = (y_1, y_2, \dots) \in l$ are sequences and $\lambda \in \mathbb{R}$ is a scalar. Define the *addition* and *scalar multiplication* in the natural way:

$$x + y := (x_1 + y_1, x_2 + y_2, \dots),$$

$$\lambda x := (\lambda x_1, \lambda x_2, \dots).$$

Then l becomes a linear vector space. Note that elements (vectors) of l are sequences. Linear vector spaces whose elements are sequences are often called *spaces of sequences.* Further examples of function spaces include:

l_∞ – the space of all bounded sequences of real numbers;

l_c – the space of all convergent sequences of real numbers;

Other examples of spaces of sequences will be introduced later.

Exercise 1.10. Verify that the set $L(\Omega)$ with the natural pointwise addition and scalar multiplication indeed satisfies axioms $(A1 - A8)$.

1.2 Linear combinations and linear dependence

Definition 1.11. Let $\mathbb{X}$ be a linear vector space and $x_1, \dots, x_k \in \mathbb{X}$ vectors. For given scalars $\lambda_1, \dots, \lambda_k \in \mathbb{F}$, the vector $\lambda_1 x_1 + \dots + \lambda_k x_k$ is called *a linear combination* of vectors $x_1, \dots, x_k$. The set of all linear combinations

$$\{\lambda_1 x_1 + \dots + \lambda_k x_k | \lambda_1, \dots \lambda_k \in \mathbb{F}\}$$

is called *(linear) span* of vectors $x_1, \dots, x_k$ and denoted $\operatorname{span}\{x_1, \dots, x_k\}$.

Definition 1.12. Let $\mathbb{X}$ be a linear vector space. Vectors $x_1, \dots, x_k \in \mathbb{X}$ are called *linearly independent* if for any $\lambda_1, \dots, \lambda_k \in \mathbb{F}$

$$\lambda_1 x_1 + \cdots + \lambda_k x_k = 0 \quad \Longrightarrow \quad \lambda_1 = \cdots = \lambda_k = 0.$$

Otherwise, $x_1, \dots, x_k \in \mathbb{X}$ are called *linearly dependent.*

Example 1.13. Vectors $x_1 = (1,0,0)$, $x_2 = (0,1,0)$, $x_3 = (0,0,1)$ are linearly independent in $\mathbb{R}^3$. Vectors $x_1 = (1,1,0)$, $x_2 = (1,-1,0)$, $x_3 = (1,0,0)$ are linearly dependent in $\mathbb{R}^3$, to see this take $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = -2$.

Example 1.14. Vectors $x_1(t) = \sin^2(t)$, $x_2(t) = \cos^2(t)$, $x_3(t) = 1$ are linearly dependent in $C([0,\pi])$. To see this take $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = -1$.

Exercise 1.15. Vectors $x_1(t) = 1$, $x_2(t) = \cos(2t)$, $x_3(t) = \cos^2(t)$ are linearly dependent in $C([0,\pi])$.

Hint. Use trigonometric identity $\cos(2t) = 2\cos^2(t) - 1$.

Exercise 1.16. Vectors $x_1(t) = 1$, $x_2(t) = t$, $x_3(t) = t^2$ are linearly independent in $\mathcal{P}(\mathbb{R})$.

Hint. Quadratic equation $\lambda_1 + \lambda_2 t + \lambda_3 t^2 = 0$ has no more then two real roots unless $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

1.3 Linear subspaces

Definition 1.17. Let $\mathbb{X}$ be a linear vector space and L a subset of $\mathbb{X}$. L is a *linear subspace* of $\mathbb{X}$ if

$$\text{for every } x, y \in L \text{ and } \lambda \in \mathbb{F} \text{ one has } \lambda x \in L \text{ and } x + y \in L.$$

Remark 1.18. It is easy to see that if $L \subset \mathbb{X}$ is a linear subspace of $\mathbb{X}$ then L is a linear vector space itself, with induced from $\mathbb{X}$ operations of addition and scalar multiplications.

Exercise 1.19. Fix $A \in \mathbb{R}$. Let $M_A := \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = A\}$. For which value(s) of A the set M_A is a linear subspace of $\mathbb{R}^3$?

Example 1.20. The spaces $\mathcal{P}([a,b])$, $C^k([a,b])$, $C([a,b])$ are linear subspaces of $L([a,b])$. In fact, for any integers $k > l \geq 0$ we have the following *sequence of embedded linear subspaces*:

$$\mathcal{P}([a,b]) \subset C^k([a,b]) \subset C^l([a,b]) \subset L([a,b]).$$

Example 1.21. The set $M := \{x \in C([a,b]) \mid x(a) = 0, x(b) = 1\}$ is not a linear subspace of $C([a,b])$. To see this, simply observe that $0 \notin M$, so M is not a linear vector space, although $M \subset C([a,b])$.

Example 1.22. The space l_∞ is a linear subspace of the space l.

Exercise 1.23. Fix $A \in \mathbb{R}$. Let M_A be the set of all sequences of real number which converge to A. Thus $M_A \subset l_\infty$ (explain why!). For which value(s) of A the set M_A is a linear subspace of l_∞ ?

1.4 Dimension

Definition 1.24. Let $\mathbb{X}$ be a linear vector space. We say that X *has dimension* $n \in \mathbb{N}$ and write $\dim X = n$ if:

i) there exists n linearly independent vectors $x_1, \dots, x_n \in \mathbb{X}$;

ii) every $n+1$ vectors $y_1, \dots, y_n, y_{n+1} \in \mathbb{X}$ are linearly dependent.

Remark 1.25. Properties (i) and (ii) together mean that $\mathbb{X} = \operatorname{span}(\{x_1, \dots, x_n\})$.

Example 1.26. The space $\mathbb{R}^n$ has dimension n. Indeed, it is known that $\mathbb{R}^n$ admits a basis $\{e_1, \dots, e_n\}$ so that:

i) vectors $e_1, \dots, e_n$ are linearly independent;

ii) every vector $y \in \mathbb{R}^n$ is represented as a linear combination of basis vectors.

This means that $\dim(\mathbb{R}^n) = n$.

Exercise 1.27. Show that the space $\mathcal{P}_m(\mathbb{R})$ of all real polynomials of degree non greater then m has dimension m.

Definition 1.28. Let $\mathbb{X}$ be a linear vector space. We say that X *has infinite dimension* and write $\dim X = \infty$ if for every $k \in \mathbb{N}$ there exists k linearly independent vectors $x_1, \dots, x_k \in \mathbb{X}$.

Example 1.29. The space l_∞ is infinite dimensional. To see this, define

$$e_1 := (1, 0, 0, 0, 0, \dots),$$
$$e_2 := (0, 1, 0, 0, 0, \dots),$$
$$e_3 := (0, 0, 1, 0, 0, \dots),$$
$$\dots$$

It is easy to see that for every $k \in \mathbb{N}$ vectors $e_1, \dots, e_k \in l_\infty$ are linearly independent.

Exercise 1.30. Prove that the space of all polynomials $\mathcal{P}([a, b])$ is infinite dimensional.

Hint. Show that for all $k \in \mathbb{N}$ vectors

$$x_1(t) = 1, \quad x_2(t) = t, \quad \dots, \quad x_k(t) = t^{k-1},$$

are linearly independent in $\mathcal{P}([a, b])$. To do this, use Fundamental Theorem of Algebra about the roots of polynomials of degree k.

Exercise 1.31. Use previous exercise to conclude that the space $C([a, b])$ and the spaces $C^k([a, b])$ are infinite dimensional.

2 Normed spaces

2.1 Normed spaces

Definition 2.1. Let $\mathbb{X}$ be a linear vector space. A *norm* on $\mathbb{X}$ is a function $\|\cdot\| : \mathbb{X} \to [0, +\infty)$ such that the following *axioms* are satisfied for all $x, y \in \mathbb{X}$ and $\lambda \in \mathbb{F}$:

$(N1)$ $\|x\| = 0$ implies $x = 0$.

$(N2)$ $\|\lambda x\| = |\lambda|\|x\|$;

$(N3)$ $\|x + y\| \leq \|x\| + \|y\|$ (*triangle inequality*).

A *normed space* is a pair $(\mathbb{X}, \|\cdot\|)$, where $\mathbb{X}$ is a linear vector space and $\|\cdot\|$ is a norm on $\mathbb{X}$.

Remark 2.2. For $x \in \mathbb{X}$, the number $\|x\|$ is called *the norm of* $\|x\|$ and has a geometrical meaning of the "length" of the vector x.

Exercise 2.3. Let $\mathbb{X}$ be a normed space with the norm $\|\cdot\|$. Prove that for any $x, y \in \mathbb{X}$ the following useful inequality holds:

$$\big|\|x\| - \|y\|\big| \leq \|x - y\|. \tag{2.1}$$

Example 2.4. The function $\|x\| := \sqrt{|x_1|^2 + \cdots + |x_n|^2}$ defines the *Euclidean norm* on $\mathbb{R}^n$.

Example 2.5. It is possible to define different norms on the same linear vector space. For instance, for $p \in [1, \infty)$ the function

$$\|x\|_p := \sqrt[p]{|x_1|^p + \cdots + |x_n|^p}$$

defines a norm on $\mathbb{R}^n$, which is often called *p-norm.* Indeed, axioms $(N1)$ and $(N2)$ for $\|\cdot\|_p$ are obvious, while the triangle inequality $(N3)$ is nothing else but *Minkowski's inequality*[1], which states that for every $x, y \in \mathbb{R}^n$ one has

$$\left(\sum_{i=1}^n |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^n |y_i|^p\right)^{1/p}. \tag{2.2}$$

Observe, that the Euclidean norm on $\mathbb{R}^n$ is just a particular case of the p-norm with $p = 2$.

Exercise 2.6. Prove that the function

$$\|x\|_\infty := \max\{|x_1|, \ldots, |x_n|\}$$

defines a norm on $\mathbb{R}^n$. This norm is often called *infinity norm.*

Notation 2.7. For $p \in [1, \infty]$, $\mathbb{R}^n_p$ denotes the space $\mathbb{R}^n$ equipped with the norm $\|\cdot\|_p$. However if $p = 2$ then the subscript is often omitted.

[1] Minkowski's inequality was proved in *Analysis 2*.

2.2 Spaces $C([a,b])$ and $C^k([a,b])$

Let $C([a,b])$ be the space of all continuous functions on the closed interval $[a,b]$. The function

$$\|x\|_\infty := \max_{t\in[a,b]} |x(t)| \tag{2.3}$$

defines a norm on the space $C([a,b])$. Indeed, every continuous function on the closed interval $[a,b]$ attains its maximum (this is *Weierstrass Theorem* from Analysis). Therefore, $\|x\|_\infty$ is well defined and nonnegative for every $x \in C([a,b])$. Verification of axioms $(N1)$ and $(N2)$ of the norm is elementary. To verify $(N3)$ observe that

$$|x(t)+y(t)| \le |x(t)| + |y(t)| \le \max_{t\in[a,b]} |x(t)| + \max_{t\in[a,b]} |y(t)| \qquad (t \in [a,b]).$$

Hence

$$\|x+y\|_\infty = \max_{t\in[a,b]} |x(t)+y(t)| \le \max_{t\in[a,b]} |x(t)| + \max_{t\in[a,b]} |y(t)| = \|x\|_\infty + \|y\|_\infty,$$

which proves the triangle inequality $(N3)$.

Notation 2.8. In what follows $C([a,b])$ always denotes the normed space of all continuous functions on the closed interval $[a,b]$, with the norm $\|\cdot\|_\infty$.

For $k \in \mathbb{N}$, let $C^k([a,b])$ be the space of all k–times continuously differentiable functions on the closed interval $[a,b]$. The function

$$\|x\|_{\infty,k} := \max_{t\in[a,b]} |x(t)| + \max_{t\in[a,b]} |x'(t)| + \cdots + \max_{t\in[a,b]} |x^{(k)}(t)| \tag{2.4}$$

defines a norm on $C^k([a,b])$.

Exercise 2.9. Verify, that the function $\|\cdot\|_{\infty,k}$ indeed satisfies axioms of the norm $(N1-N3)$.

Notation 2.10. In what follows $C^k([a,b])$ always denotes the normed space of all k–times continuously differentiable functions on the closed interval $[a,b]$, with the norm $\|\cdot\|_{\infty,k}$.

Exercise 2.11. Let $C(\mathbb{R})$ be the space of all continuous functions on the real line. Explain why

$$\|x\| := \max_{t\in\mathbb{R}} |x(t)|$$

does not define a norm on $C(\mathbb{R})$.

Hint. Is this norm finite for every $x \in C(\mathbb{R})$?

Exercise 2.12. Let $C^1([a,b])$ be the space of all continuously differentiable functions on the interval $[a,b]$. Explain why

$$\|x\| := \max_{t\in\mathbb{R}} |x'(t)|$$

does not define a norm on $C^1([a,b])$.

Hint. Check property $(N1)$ of the norm.

2.3 Spaces l_p

Definition 2.13. Denote by l_p, for $p \geq 1$, the space of all sequences $x = (x_1, x_2, \dots)$ of real numbers, such that

$$\|x\|_p := \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} < \infty.$$

To prove that l_p is a normed space we will need a version of Minkowski's inequality for infinite sequences.

Lemma 2.14. (Minkowski inequality for sequences) *Let $p \geq 1$. For all $x, y \in l_p$ one has*

$$\boxed{\left(\sum_{i=1}^{\infty} |x_i + y_i|^p \right)^{1/p} \leq \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} + \left(\sum_{i=1}^{\infty} |y_i|^p \right)^{1/p}.} \tag{2.5}$$

Proof. Let $x, y \in l_p$ and let $n \in \mathbb{N}$. In the right hand side of the Minkowski's inequality for finite sums (2.2), we can replace n by any integer $m \geq n$, so

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p \right)^{1/p} \leq \left(\sum_{i=1}^{m} |x_i|^p \right)^{1/p} + \left(\sum_{i=1}^{m} |y_i|^p \right)^{1/p}.$$

Passing to the limit as $m \to \infty$ in the above inequality, we conclude that

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p \right)^{1/p} \leq \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} + \left(\sum_{i=1}^{\infty} |y_i|^p \right)^{1/p}.$$

Since n in the left hand side of the above inequality is arbitrary, we conclude that the series in left hand side converges (as the series with nonnegative terms and uniformly bounded partial sums). Passing to the limit as $n \to \infty$ we obtain (2.5). $\square$

Remark 2.15. *Minkowski's inequality for sequences* can be rewritten simply as

$$\boxed{\|x + y\|_p \leq \|x\|_p + \|y\|_p.}$$

Proposition 2.16. $(l_p, \|\cdot\|_p)$ *is a normed space.*

Proof. To prove that $(l_p, \|\cdot\|_p)$ is a normed space we first need to show that l_p is a linear vector space. Since $l_p \subset l$, it is enough to verify that l_p is a linear subspace of l, that is:

$$x, y \in l_p \text{ and } \lambda \in \mathbb{R} \text{ implies that } \lambda x \in l_p \text{ and } x + y \in l_p.$$

Further, we need to show that $\|\cdot\|_p$ satisfies axioms of the norm $(N1 - N3)$.

Assume that $x, y \in l_p$ and $\lambda \in \mathbb{R}$. We first observe that

$$\|\lambda x\|_p = |\lambda|\|x\|_p < \infty,$$

which shows that $\lambda x \in l_p$ and also proves $(N2)$. Further, from Minkowski's inequality (2.5) we conclude that

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p,$$

which shows that $x + y \in l_p$ and also proves $(N3)$. Finally, axiom $(N1)$ of the norm is obvious. This completes the proof. $\square$

Definition 2.17. Denote by l_∞ the normed space of all bounded sequences $x = (x_1, x_2, \dots)$ of real numbers, with the norm

$$\|x\|_\infty := \max_{i \in \mathbb{N}} |x_i|.$$

Exercise 2.18. Verify that $(l_\infty, \|\cdot\|_\infty)$ is a normed space.

Notation 2.19. To simplify the notations, in what follows l_p always denotes the normed space $(l_p, \|\cdot\|_p)$ $(p \in [1, \infty])$.

Exercise 2.20. Show that if $p \neq q$ then $l_p \neq l_q$.

Hint. Consider the sequence $x_k = 1/k^a$ $(k \in \mathbb{N})$. For which values of $a \geq 0$ the sequence (x_k) belongs to l_p ?

Exercise* 2.21. Let $p, q \in [1, \infty]$ and $p < q$. Prove that

$$l_p \subset l_q.$$

Definition 2.22. Let $p \in [1, \infty]$. The *conjugate* to p is the number p' such that

$$\boxed{\frac{1}{p} + \frac{1}{p'} = 1.} \tag{2.6}$$

We agree that if $p = 1$ then $p' = \infty$, and if $p = \infty$ then $p' = 1$.

Lemma 2.23. (Hölder inequality for sequences[2]) *Let $p \in [1, +\infty]$. Let $x \in l_p$ and $y \in l_{p'}$, where p' is the conjugate to p. Then*

$$\boxed{\sum_{i=1}^{\infty} |x_i y_i| \leq \|x\|_p \|y\|_{p'}.} \tag{2.7}$$

Exercise 2.24. Derive Hölder's inequality for sequences from Hölder's inequality for finite sums in a way, similar to the proof of Lemma 2.14.

[2]Hölder's inequality for finite sums was proved in *Analysis 2*

2.4 Spaces $\mathscr{L}_p([a,b])$

Definition 2.25. Denote by $\mathscr{L}_p([a,b])$, for $p \geq 1$, the space of all continuous functions $x : [a,b] \to \mathbb{R}$ with the norm

$$\|x\|_p := \left(\int_a^b |x(t)|^p \, dt \right)^{1/p} < \infty,$$

where the integral in the right hand side is understood in the Riemann sense.

To prove that $\mathscr{L}_p([a,b])$ is a normed space we will need a version of Minkowski's inequality for Riemann integrals.

Lemma 2.26. (Minkowski inequality for Riemann integrals[3]) *Let $p \geq 1$. For all continuous function $x, y : [a,b] \to \mathbb{R}$ one has*

$$\boxed{\left(\int_a^b |x(t) + y(t)|^p \, dt \right)^{1/p} \leq \left(\int_a^b |x(t)|^p \, dt \right)^{1/p} + \left(\int_a^b |y(t)|^p \, dt \right)^{1/p}.} \tag{2.8}$$

Remark 2.27. *Integral Minkowski's inequality* can be rewritten as

$$\boxed{\|x + y\|_p \leq \|x\|_p + \|y\|_p.}$$

Proposition 2.28. $(\mathscr{L}_p([a,b]), \|\cdot\|_p)$ *is a normed space.*

Proof. To verify that $\|\cdot\|_p$ is indeed a norm observe that if $x(t)$ is continuous function on $[a,b]$ then $|x(t)|^p$ is also continuous, and hence, Riemann integrable on $[a,b]$. Therefore, $\|x\|_p$ is well defined, moreover it is nonnegative for every $x \in C([a,b])$. Verification of axioms $(N1)$ and $(N2)$ is elementary and is left as an exercise, while the triangle inequality $(N3)$ is nothing else but *Minkowski's inequality for integrals* □

Remark 2.29. Observe that $\mathscr{L}_p([a,b])$ contains the same collection of function as $C([a,b])$, however the norms in $\mathscr{L}_p([a,b])$ and $C([a,b])$ are different.

Remark 2.30. The space $C([a,b])$ with the usual norm $\|\cdot\|_\infty$ can be interpreted as $\mathscr{L}_\infty([a,b])$ if we consider the maximum norm $\|\cdot\|_\infty$ as the "limit" of integral p-norms when $p \to \infty$. Such interpretation will be further justified later in this course.

Lemma 2.31. (Hölder inequality for integrals[4]) *Let $p > 1$. Let $x \in \mathscr{L}_p([a,b])$ and $y \in \mathscr{L}_{p'}([a,b])$, where p' is the conjugate to p. Then*

$$\boxed{\int_a^b |x(t)y(t)| \, dt \leq \|x\|_p \|y\|_{p'}.} \tag{2.9}$$

Exercise* 2.32. Let $p, q \in [1, \infty]$ and $p < q$. Prove that

$$\mathscr{L}_q([a,b]) \subset \mathscr{L}_p([a,b]).$$

Hint. Use Hölder's inequality for integrals.

[3] Minkowski's inequality for integrals was proved in *Analysis 2*
[4] Hölder's inequality for integrals was proved in *Analysis 2*

3 Analysis in Normed spaces

3.1 Convergence in normed spaces

Definition 3.1. Let $(X, \|\cdot\|)$ be a normed space and $(x_n) \subset \mathbb{X}$ be a sequence of vectors in $\mathbb{X}$. We say that the sequence (x_n) *converges* to a vector $x \in \mathbb{X}$ and write

$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \to x$$

if

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

In other words, this means that for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that for all integers $n \geq N$ one has

$$\|x_n - x\| < \varepsilon.$$

Remark 3.2. The limit of a convergent sequence is unique. Indeed, assume that (x_n) converges to $x \in \mathbb{X}$ and $y \in \mathbb{X}$. Then using the triangle inequality we obtain

$$\|x - y\| = \|(x - x_n) + (x_n - y)\| \leq \|x_n - x\| + \|x_n - y\| \to 0.$$

Since the right hand side gets arbitrary small, we conclude that $\|x - y\| = 0$ and hence $x = y$.

Exercise 3.3. Let $(X, \|\cdot\|)$ be a normed space, $(x_n), (y_n) \subset \mathbb{X}$ sequences of vectors in $\mathbb{X}$ and $(\lambda_n) \subset \mathbb{F}$ a sequence of scalars. Prove the following *properties of convergent sequences*:

(*i*) if $x_n \to x$ and $\lambda_n \to \lambda$ then $\lambda_n x_n \to \lambda x$;

(*ii*) if $x_n \to x$ and $y_n \to y$ then $x_n + y_n \to x + y$.

(*iii*) if $x_n \to x$ then $\|x_n\| \to \|x\|$

Hint. To prove (*iii*), use (2.1).

Example 3.4. (Convergence in $C([a,b])$) According to Definition 3.1, a sequence $(x_n) \subset C([a,b])$ converges to $x \in C([a,b])$ if

$$\|x_n - x\|_\infty \to 0.$$

This means that for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that for all integers $n \geq N$ one has

$$\|x_n - x\|_\infty = \max_{t\in[a,b]} |x_n(t) - x(t)| < \varepsilon.$$

Thus *convergence in $C([a,b])$ coincides with the uniform convergence* which is studied Analysis. For this reason, the norm $\|x\|_\infty = \max_{t\in[a,b]} |x(t)|$ is often called *uniform convergence* norm.

Exercise 3.5. Let $x_n(t) := nt^n$. Show that:

(*i*) The sequence (x_n) converges to $x(t) =$ in $C([0, 1/2])$;

(*ii*) The sequence (x_n) has no limit in $C([0, 1])$.

Hint. To prove (*ii*), use Exercise 3.3 (*iii*).

Example 3.6. (Relation between convergences in $C([a, b])$ and $\mathscr{L}_p([a, b])$) Consider a sequence of continuous functions $x_n : [a, b] \to \mathbb{R}$ $(n \in \mathbb{N})$. Thus $(x_n) \subset C([a, b])$ and $(x_n) \subset \mathscr{L}_p([a, b])$. Assume that $x_n \to x$ in $C([a, b])$. Then

$$\begin{aligned} \|x_n - x\|_p &= \left(\int_a^b |x_n(t) - x(t)|^p \, dt\right)^{1/p} \leq \left(\max_{t\in[a,b]} |x_n(t) - x(t)|^p (b - a)\right)^{1/p} \\ &= (b - a)^{1/p} \|x_n - x\|_\infty \to 0. \end{aligned}$$

Thus, we proved that *if a sequence* (x_n) *converges to* x *in* $C([a, b])$ *then for every* $p \in [1, \infty)$ *the sequence* (x_n) *converges to the same limit* x *in* $\mathscr{L}_p([a, b])$.

Exercise 3.7. Consider the sequence

$$x_n(t) := \begin{cases} \sin(nt), & t \in [0, \pi/n]; \\ 0, & t \in [\pi/n, \pi]. \end{cases} \tag{3.1}$$

Clearly $(x_n) \subset C([0, \pi])$ and $(x_n) \subset \mathscr{L}_p([0, \pi])$ for any $p \geq 1$. Prove that $x_n \to 0$ in $\mathscr{L}_p([0, \pi])$ for any $p \geq 1$, but (x_n) has no limit in $C([0, \pi])$. This shows, in particular, that convergence in $\mathscr{L}_p([0, \pi])$ does not imply converges in $C([0, \pi])$.

Exercise* 3.8. Let $p > q \geq 1$. Construct an example of a sequence (x_n) which converges to zero in $\mathscr{L}_p([0, \pi])$ but does not have a limit in $\mathscr{L}_q([0, \pi])$.

Hint. Consider sequences of the form $y_n(t) := A_n x_n(t)$, where $x_n(t)$ are functions from (3.1) and $A_n > 0$, then determine appropriate coefficients A_k.

Exercise 3.9.** (Relation between convergences in l_1 and l_∞) Consider a sequence $(x_n) \subset l_1$. Prove that if (x_n) converges to x in l_1 then (x_n) converges to the same limit x in l_∞.

Construct an example of a sequence (x_n) which converges to zero in l_∞ but does not have a limit in l_1.

3.2 Cauchy sequences

Definition 3.10. Let $(X, \|\cdot\|)$ be a normed space and $(x_n) \subset \mathbb{X}$ be a sequence of vectors in $\mathbb{X}$. We say that (x_n) is a *Cauchy sequence* if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon) \in \mathbb{N}$ such that for all positive integers $n \geq N$ and $m \geq N$ one has

$$\|x_n - x_m\| < \varepsilon.$$

Exercise 3.11. Show that if $(x_n) \subset X$ is a Cauchy sequence then the sequence of norms $(\|x_n\|) \subset \mathbb{R}$ is a Cauchy sequence of real numbers. Give an example, showing that the opposite is not true.

Theorem 3.12. *Let $(X, \|\cdot\|)$ be a normed space and $(x_n) \subset \mathbb{X}$ be a sequence. If the sequence (x_n) converges then (x_n) is a Cauchy sequence.*

Proof. Let $x := \lim_{n\to\infty} x_n$. Fix $\varepsilon > 0$. Then, according to Definition 3.1,

$$(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})[(n \geq N) \Rightarrow (\|x_n - x\| < \frac{\varepsilon}{2})].$$

Let $m, n \geq N$. By the triangle inequality $(N3)$ we obtain

$$\|x_n - x_m\| \leq \|x_n - x\| + \|x_m - x\| < \varepsilon.$$

Therefore (x_n) is a Cauchy sequence. □

Exercise 3.13. Show that the sequence $x_n(t) := \sin(nt)$ $(n \in \mathbb{N})$ is not a Cauchy sequence in $C([0, \pi])$. Conclude from this that (x_n) has no limit in $C([0, \pi])$. (This shows, in particular, that the converse to Exercise 3.3 *(iii)* is not true.)

3.3 Bounded sets

Definition 3.14. Let $(X, \|\cdot\|)$ be a normed space. We define:

$B_R(x) := \{y \in \mathbb{X} \,|\, \|y - x\| < R\}$ – *open ball* of radius $R > 0$, centered at $x \in \mathbb{X}$;
$\bar{B}_R(x) := \{y \in \mathbb{X} \,|\, \|y - x\| \leq R\}$ – *closed ball* of radius $R > 0$, centered at $x \in \mathbb{X}$;
$S_R(x) := \{y \in \mathbb{X} \,|\, \|y - x\| = R\}$ – *sphere* of radius $R > 0$, centered at $x \in \mathbb{X}$.

Exercise 3.15. Draw a picture of $B_1(0)$, $\bar{B}_1(0)$, $S_1(0)$ in: $(\mathbb{R}^2, \|\cdot\|_2)$; $(\mathbb{R}^2, \|\cdot\|_1)$; $(\mathbb{R}^2, \|\cdot\|_\infty)$.

Exercise 3.16. Try to illustrate how $B_1(0)$ looks like in: $C([0, 1])$; $\mathscr{L}_p([0, 1])$.

Definition 3.17. Let $(X, \|\cdot\|)$ be a normed space and $M \subset \mathbb{X}$ a nonempty subset of $\mathbb{X}$. We say that the set M is *bounded* in $\mathbb{X}$ if there exists $R > 0$ such that $M \subset B_R(0)$. In other words, this means that there exists $R > 0$ such that $\|x\| < R$, for every $x \in M$. If a set M is not bounded then it is said to be *unbounded*.

Theorem 3.18. *Let $(X, \|\cdot\|)$ be a normed space and $(x_n) \subset \mathbb{X}$ be a sequence. If the sequence (x_n) converges then (x_n) is bounded.*

Proof. Let $x := \lim_{n\to\infty} x_n$. By the triangle inequality $(N3)$ we have

$$\|x_n\| = \|(x_n - x) + x\| \leq \|x_n - x\| + \|x\|.$$

Note that the sequence of real numbers $(\|x_n - x\|)$ converges to zero. It is known from Analysis that every convergent sequence of real numbers is bounded. Hence the sequence of real numbers $(\|x_n - x\|)$ is bounded, that is there exists $C > 0$ such that

$$(\forall n \in \mathbb{N})[\|x_n - x\| < C].$$

Therefore by the triangle inequality

$$(\forall n \in \mathbb{N})[\|x_n\| \leq \|x_n - x\| + \|x\| \leq C + \|x\|],$$

which means that $(x_n) \subset B_{C+\|x\|}(0)$ and hence (x_n) is bounded. □

Example 3.19. The converse to Theorem 3.18 in generally is not true in normed spaces. To see this consider the sequence $x_n(t) := \sin(nt)$ $(n \in \mathbb{N})$ in $C([0, \pi])$. The sequence (x_n) has no limit in $C([0, \pi])$. However, $\|x_n\|_\infty = 1$, so (x_n) is bounded in $C([0, \pi])$!

3.4 Open and closed sets

Definition 3.20. Let $(X, \| \cdot \|)$ be a normed space and $M \subset \mathbb{X}$ a nonempty subset of $\mathbb{X}$. We say that the set M is *open* in $\mathbb{X}$ if for every $x \in M$ there exists $R > 0$ such that $B_R(x) \subset M$. A subset M is *closed* if its complement is open, that is if $\mathbb{X} \setminus M$ is open.

Remark 3.21. There are sets which are neither open nor closed. For instance, the interval $[0, 1) \subset \mathbb{R}$ is neither open nor closed in $\mathbb{R}$!

Example 3.22. Open ball $B_R(x)$ in a normed space X is an open set. Closed ball $\bar{B}_R(x)$ and sphere $S_R(x)$ are closed sets.

Example 3.23. For a given function $x \in C([a, b])$, the following set are open in $C([a, b])$:

$$\{y \in C([a, b]) \,|\, y(t) < x(t) \quad \text{for all} \quad t \in [a, b]\},$$

$$\{y \in C([a, b]) \,|\, y(t) > x(t) \quad \text{for all} \quad t \in [a, b]\},$$

$$\{y \in C([a, b]) \,|\, |y(t)| < x(t) \quad \text{for all} \quad t \in [a, b]\},$$

$$\{y \in C([a, b]) \,|\, |y(t)| > x(t) \quad \text{for all} \quad t \in [a, b]\}.$$

The following sets are closed in $C([a, b])$:

$$\{y \in C([a, b]) \,|\, y(t) \leq x(t) \quad \text{for all} \quad t \in [a, b]\},$$

$$\{y \in C([a, b]) \,|\, y(t) \geq x(t) \quad \text{for all} \quad t \in [a, b]\},$$

$$\{y \in C([a, b]) \,|\, |y(t)| \leq x(t) \quad \text{for all} \quad t \in [a, b]\},$$

$$\{y \in C([a, b]) \,|\, |y(t)| \geq x(t) \quad \text{for all} \quad t \in [a, b]\}.$$

Theorem 3.24. (Properties of open and closed sets)

(*i*) *The union of any collection of open sets is open.*

(*ii*) *The intersection of a finite number of open sets is open.*

(*iii*) *The union of a finite number of closed sets is closed.*

(*iv*) *The intersection of any collection of closed sets is closed.*

(*v*) *The empty set and the whole space are both open and closed.*

Proof. Prove these properties as an exercise (see also *Analysis 3*). □

Theorem 3.25. *Let $(X, \|\cdot\|)$ be a normed space and $M \subset \mathbb{X}$ a nonempty subset of $\mathbb{X}$. M is closed if and only if every convergent sequence of elements in M has its limit in M, that is*

$$(x_n) \subset M \quad \textit{and} \quad x_n \to x \quad \textit{implies} \quad x \in M.$$

Proof. Let M be a closed subset of $\mathbb{X}$ and $(x_n) \subset \mathbb{X}$ a sequence in $\mathbb{X}$ such that $x_n \to x$. Suppose that $x \notin M$. Since M is closed, $\mathbb{X} \setminus M$ is open. Then there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset \mathbb{X} \setminus M$. On the other hand, since $x_n \to x$ we have $\|x_n - x\| < \varepsilon$ for all large $n \in \mathbb{N}$. But $(x_n) \subset M$! This contradictions shows that $x \in M$.

Now suppose that

$$(x_n) \subset M \quad \text{and} \quad x_n \to x \quad \text{implies} \quad x \in M.$$

If M is not closed then $\mathbb{X} \setminus M$ is not open. Thus there exists $x \in \mathbb{X} \setminus M$ such that every ball $B_\varepsilon(x)$ contains elements of M. Thus, we can find $(x_n) \subset M$ such that $x_n \in B_{1/n}(x)$. But then $x_n \to x$ and according to our assumption, $x \in M$. Therefore M must be a closed set! □

3.5 Closure. Dense sets

Definition 3.26. Let $(X, \|\cdot\|)$ be a normed space and $M \subset \mathbb{X}$ a nonempty subset of $\mathbb{X}$. The *closure* of the set M is the of limits of all convergent sequences of elements of M, i.e.

$$clM := \{x \in \mathbb{X} \,|\, \text{there exists } (x_n) \subset M \text{ such that } x_n \to x\}.$$

Remark 3.27. It is clear that if a set M is closed then $clM = M$. One can show that the closure of a set M is the "smallest" closed set which contains M.

Example 3.28. Let $\mathbb{X} = \mathbb{R}$. Then $cl(0,1) = [0,1]$ and $cl\mathbb{Q} = \mathbb{R}$.

Definition 3.29. Let $(X, \|\cdot\|)$ be a normed space and $M \subset \mathbb{X}$ a nonempty subset of $\mathbb{X}$. The set M is *dense* in $\mathbb{X}$ if $clM = \mathbb{X}$.

Example 3.30. The set of rationals $\mathbb{Q}$ is dense in $\mathbb{R}$.

Example 3.31. (Weierstrass Theorem) The Weierstrass theorem says that every continuous function on an interval $[a, b]$ can be uniformly approximated by polynomials. This can be interpreted as follows: *the space of polynomials $\mathcal{P}([a, b])$ is dense in $C([a, b])$.*

3.6 Compact sets

Definition 3.32. Let $(X, \|\cdot\|)$ be a normed space and $M \subset \mathbb{X}$ a nonempty subset of $\mathbb{X}$. The set M is *compact* if every sequence $(x_n) \subseteq M$ contain a subsequence (x_{n_k}) which converges in M, that is $x_{n_k} \to x \in M$.

Example 3.33. Every bounded and closed subset of $\mathbb{R}^n$ is compact. In particular, the closed unit ball in $\mathbb{R}^N$ is compact.

Theorem 3.34. *Let $(X, \|\cdot\|)$ be a normed space and $M \subset \mathbb{X}$ a nonempty subset of $\mathbb{X}$. If M is compact then M is bounded and closed.*

Proof. Suppose $(x_n) \subseteq M$ and $x_n \to x$. Since M is compact, (x_n) contain a convergent subsequence x_{n_k} such that $x_{n_k} \to y \in M$. On the other hand, we have $x_n \to x$. Thus $x = y$ and $x \in M$. By Theorem 3.25 we conclude that M is closed.

Assume that M is unbounded. Then for every $m \in \mathbb{N}$ there exists $x_m \in M$ such that $\|x_m\| \geq m$. In such a way, we constructed a sequence $(x_m) \subset M$ which is unbounded. Clearly, (x_m) can not contain a convergent subsequence (cf. Theorem 3.18). Hence M is not compact. □

Example 3.35. (The unit ball in $C([0,1])$ is non compact) Consider the sequence $x_n(t) := t^n$ $(n \in \mathbb{N})$. Clearly $(x_n) \subset C([0,1])$ and $\|x_n\|_\infty = 1$, so (x_n) belongs to the closed unit ball $B_1(0) \subset C([0,1])$. Since convergence in $C([0,1])$ coincides with the uniform convergence, we conclude that the sequence (x_n) does not have a convergent subsequence in $C([0,1])$. Therefore, the unit ball in $C([0,1])$ is not compact.

Theorem 3.36. *A normed space $\mathbb{X}$ is infinite dimensional if and only if the closed unit ball in $\mathbb{X}$ is not compact.*

Proof. A special case of this theorem will be proved later. For the full proof see, e.g. [1,2]. □

Equivalent norms[5]

Definition 3.37. Let $(\mathbb{X}, \|\cdot\|_1)$ and $(\mathbb{X}, \|\cdot\|_2)$ be normed spaces. The norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are called *equivalent* if there exist $\alpha, \beta \in \mathbb{R}$, $\beta \geq \alpha > 0$, such that for all $x \in \mathbb{X}$ one has

$$\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_2. \tag{3.2}$$

Exercise 3.38. Show that the following norms on $\mathbb{R}^2$ are equivalent:

$$\|x\|_1 = |x_1| + |x_2|, \quad \|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2}, \quad \|x\|_\infty = \max\{|x_1|, |x_2|\} \quad (x = (x_1, x_2) \in \mathbb{R}^2).$$

[5] Optional

Theorem 3.39. *Let* $(\mathbb{X}, \|\cdot\|_1)$ *and* $(\mathbb{X}, \|\cdot\|_2)$ *be normed spaces. The norms* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *on* $\mathbb{X}$ *are* equivalent *if and only if for any sequence* $(x_n) \subset \mathbb{X}$ *and any* $x \in \mathbb{X}$ *one has*

$$\|x_n - x\|_1 \to 0 \quad \Longleftrightarrow \quad \|x_n - x\|_2 \to 0.$$

Proof. See [2,Theorem 1.3.11]. □

Remark 3.40. The above theorem tells that two norms on the same linear vector space are equivalent if they define the same convergence, and therefor the same classes of convergent sequences, Cauchy sequences, open and closed sets, compact sets etc. Effectively, this means that Analysis in normed spaces with equivalent norms is essentially identical.

Example 3.41. Exercises 3.7 and 3.8 show that the norms of $C([a,b])$ and $\mathscr{L}_p([a,b])$, and of $\mathscr{L}_p([a,b])$ and $\mathscr{L}_q([a,b])$ with $p \neq q$, are *not* equivalent.

Theorem 3.42. *Let* $\mathbb{X}$ *be a finite dimensional linear vector space. Then any two norms on* $\mathbb{X}$ *are equivalent.*

Proof. See [2,Theorem 1.3.13]. □

Remark 3.43. The above theorem tells that Analysis in finite dimensional spaces essentially does not depend on the choice of a particular norm.

4 Banach spaces

4.1 Completeness property and Banach spaces

Recall that every convergent sequence in a normed space is a Cauchy sequence, see Theorem 3.12. The converse is not true, in general.

Example 4.1. Let $\mathcal{P}([0,1])$ be the normed space of polynomials on $[0,1]$ with the norm $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$. Define

$$x_n(t) := 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!},$$

for $n \in \mathbb{N}$. Then (x_n) is a Cauchy sequence in $\mathcal{P}([0,1])$, but it does not converge in $\mathcal{P}([0,1])$ because its pointwise limit is the exponential function $x(t) = e^t$, which not a polynomial.

Definition 4.2. Let $(\mathbb{X}, \|\cdot\|)$ be a normed space. We say that $\mathbb{X}$ is *complete* if every Cauchy sequence $(x_n) \subset \mathbb{X}$ converges to the limit in $\mathbb{X}$. A complete normed space is called *Banach space.*

Remark 4.3. According to the above example, the space $\mathcal{P}([0,1])$ is not complete.

Example 4.4. ($\mathbb{R}^n_p$ IS A BANACH SPACE) It is proved in Analysis that a sequence in $\mathbb{R}^n$ converges if and only if it is a Cauchy sequence. This means that the spaces $\mathbb{R}^n_p$ are Banach spaces, for all $p \in [1,\infty]$.

Theorem 4.5. *l_p is a Banach space.*

Proof. We give the proof for $1 \le p < \infty$. Let (x_n) be a Cauchy sequence in l_p, and denote

$$x_n = (\chi_{n,1}, \chi_{n,2}, \dots).$$

Then for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all integers $m, n > N$ one has

$$\sum_{k=1}^{\infty} |\chi_{m,k} - \chi_{n,k}|^p < \varepsilon^p. \tag{4.1}$$

This implies that for every fixed $k \in \mathbb{N}$ and for every $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for all integers $m, n > N_0$ one has

$$|\chi_{m,k} - \chi_{n,k}| < \varepsilon.$$

But this means that for every fixed $k \in \mathbb{N}$ the sequence of real numbers $(\chi_{n,k})$ is a Cauchy sequence in $\mathbb{R}$ and thus converges to a limit in $\mathbb{R}$. Denote

$$\chi_k := \lim \chi_{n,k} \quad \text{and} \quad \chi := (\chi_1, \chi_2, \dots).$$

We are going to show that $\chi \in l_p$ and that x_n converges to χ in l_p.

Indeed, by letting $m \to \infty$ in (4.1), for every integer $n \geq N_0$ we obtain

$$\sum_{k=1}^{\infty} |\chi_k - \chi_{n,k}|^p < \varepsilon^p. \tag{4.2}$$

Since

$$\sum_{k=1}^{\infty} |\chi_{N_0,k}|^p < \infty,$$

by Minkowski's inequality we have

$$\begin{aligned}
\left(\sum_{k=1}^{\infty} |\chi_k|^p\right)^{1/p} &= \left(\sum_{k=1}^{\infty} (|\chi_k| - |\chi_{N_0,k}| + |\chi_{N_0,k}|)^p\right)^{1/p} \\
&\leq \left(\sum_{k=1}^{\infty} (|\chi_k| - |\chi_{N_0,k}|)^p\right)^{1/p} + \left(\sum_{k=1}^{\infty} |\chi_{N_0,k}|^p\right)^{1/p} \\
&\leq \left(\sum_{k=1}^{\infty} |\chi_k - \chi_{N_0,k}|^p\right)^{1/p} + \left(\sum_{k=1}^{\infty} |\chi_{N_0,k}|^p\right)^{1/p} < \infty.
\end{aligned}$$

This proves that $\chi \in l_p$. Moreover, since $\varepsilon > 0$ is arbitrary small, (4.2) implies that

$$\lim \|\chi - \chi_n\|_p = \lim \left(\sum_{k=1}^{\infty} |\chi_k - \chi_{N_0,k}|^p\right)^{1/p} = 0,$$

which means that $\chi_n \to \chi$ in l_p. □

Exercise 4.6. Prove that l_∞ is a Banach space.

Theorem 4.7. *$C([a,b])$ is a Banach space.*

Proof. Let (x_n) be a Cauchy sequence in $C([a,b])$. Then for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all integers $m, n \geq N$ one has

$$\|x_n - x_m\|_\infty < \varepsilon,$$

or, in in other words, for all $t \in [a,b]$ one has

$$|x_n(t) - x_m(t)| < \varepsilon. \tag{4.3}$$

This, in particular, implies that for every fixed $t \in [a,b]$ the sequence $(x_n(t))$ is a Cauchy sequence of real numbers and therefore $(x_n(t))$ converges in $\mathbb{R}$. Denote the limit of $(x_n(t))$ by $x(t)$, so

$$x(t) := \lim x_n(t) \qquad (t \in [a,b]).$$

We are going to show that $x \in C([a,b])$ and that x_n converges to χ in $C([a,b])$.

Indeed, by letting $m \to \infty$ in (4.3), for all integers $n \geq N$ and all $t \in [a,b]$ we obtain

$$|x_n(t) - x(t)| < \varepsilon. \tag{4.4}$$

Let $t_0 \in [a,b]$. Since x_N is continuous on $[a,b]$, there exists $\delta > 0$ such that for all $s \in [a,b]$ such that $|s - t_0| < \delta$ one has

$$|x_N(s) - x_N(t_0)| < \varepsilon.$$

Then

$$|x(s) - x(t_0)| \leq \underbrace{|x(s) - x_N(s)|}_{<\varepsilon} + \underbrace{|x_N(s) - x_N(t_0)|}_{<\varepsilon} + \underbrace{|x_N(t_0) - x_N(t_0)|}_{<\varepsilon} < 3\varepsilon,$$

which proves continuity of x. Finally, (4.4) implies that for all integers $n \geq N$ we have

$$\|x_n - x\|_\infty \leq \varepsilon,$$

which means that $x_n \to x$ in $C([a,b])$. □

Theorem 4.8. $\mathscr{L}_p([a,b])$ *is not complete.*

Proof. We present the proof for $p = 2$. Consider the sequence of continuous functions $(x_n) \subset \mathscr{L}_2([-1,1])$, defined by

$$x_n(t) := \begin{cases} -1, & t \in [-1, -1/n], \\ nt & t \in [-1/n, 1/n], \\ 1, & t \in [1/n, 1]. \end{cases}$$

It is clear that $|x_n(t)| \leq 1$ for all $t \in [-1,1]$ and hence $|x_n(t) - x_m(t)| \leq 2$ for all $t \in [-1,1]$. Therefore, we compute

$$\|x_m - x_m\|^2 = \int_{-1}^{1} |x_m(t) - x_n(t)|^2 \, dt \leq 4 \int_{-1/n}^{1/n} dt = \frac{8}{n} \to 0.$$

Hence, (x_n) is a Cauchy sequence in $\mathscr{L}_2([-1,1])$. However, the "limit" function

$$x(t) := \begin{cases} -1, & t \in [-1, 0), \\ 0 & t = 0, \\ 1, & t \in (0, 1]. \end{cases}$$

is discontinuous and does not belong to the space $\mathscr{L}_2([-1,1])$. □

Exercise 4.9. Show that $\mathscr{L}_p([a,b])$ is incomplete for any $p \in [1, \infty)$.

4.2 Series in Banach spaces

Definition 4.10. Let $(\mathbb{X}, \|\cdot\|)$ be a normed space and $(x_k) \subset \mathbb{X}$ be a sequence. A series

$$\sum_{k=1}^{\infty} x_k$$

is called *convergent* in $\mathbb{X}$ if the sequence of partial sums

$$S_n := x_1 + x_2 + \cdots + x_n$$

converges in $\mathbb{X}$, that is there exists $x \in \mathbb{X}$ such that

$$\|S_n - x\| \to 0.$$

In this case we write

$$\sum_{k=1}^{\infty} x_k = x.$$

If the series of norms of the sequence (x_n) converges, that is

$$\sum_{k=1}^{\infty} \|x_k\| < \infty$$

then the series $\sum_{k=1}^{\infty} x_k$ is called *absolutely convergent* in $\mathbb{X}$.

Theorem 4.11. *A normed space $\mathbb{X}$ is complete if and only if every absolute convergent series in $\mathbb{X}$ converges in $\mathbb{X}$.*

Proof. Suppose $\mathbb{X}$ is complete. Let $(x_n) \subset \mathbb{X}$ and $\sum_{k=1}^{\infty} \|x_n\| < \infty$. We will show that

$$S_n := x_1 + \cdots + x_n$$

is a Cauchy sequence in $\mathbb{X}$. Indeed, let $\varepsilon > 0$ and $N \in \mathbb{N}$ be such that

$$\sum_{k=N+1}^{\infty} \|x_n\| < \varepsilon.$$

Then for every integer $m > n > N$, by the triangle inequality we have

$$\|S_m - S_n\| = \|x_{n+1} + \cdots + x_m\| \leq \sum_{k=n+1}^{m} \|x_k\| \leq \sum_{k=N+1}^{\infty} \|x_k\| < \varepsilon.$$

This proves that (S_n) is a Cauchy sequence in $\mathbb{X}$. Since $\mathbb{X}$ is complete, (S_n) converges in $\mathbb{X}$. This means that the series $\sum_{k=1}^{\infty} x_n$ converges in $\mathbb{X}$.

Assume now that $\mathbb{X}$ is a normed space in which every absolutely convergent series converges. We will show that $\mathbb{X}$ is complete. Indeed, let (x_n) be a Cauchy sequence in $\mathbb{X}$. Then for every $k \in \mathbb{N}$ there exists a $N_k \in \mathbb{N}$ such that for all integers $m, n \geq N_k$ one has

$$\|x_m - x_n\| < 2^{-k},$$

and moreover, without loss of generality we may assume that the sequence N_k is strictly increasing. Note that the series

$$\sum_{k=1}^{\infty}(x_{N_{k+1}} - x_{N_k})$$

are absolutely convergent, as

$$\sum_{k=1}^{\infty} \|x_{N_{k+1}} - x_{N_k}\| < \sum_{k=1}^{\infty} 2^{-k} < \infty.$$

Therefore the series $\sum_{k=1}^{\infty}(x_{N_{k+1}} - x_{N_k})$ converges in $\mathbb{X}$, and hence the sequence

$$x_{N_k} = x_{N_1} + (x_{N_2} - x_{N_1}) + \cdots + (x_{N_k} - x_{N_{k-1}})$$

converges to an element $x \in \mathbb{X}$. Consequently,

$$\|x_n - x\| \leq \|x_n - x_{N_k}\| + \|x_{N_k} - x\| \to 0,$$

because (x_n) is a Cauchy sequence. □

Example 4.12. The series

$$\sum_{k=1}^{\infty} \frac{t^k}{k!}$$

converges and absolute converges in Banach space $C([0,1])$, moreover

$$\sum_{k=1}^{\infty} \frac{t^k}{k!} = e^t.$$

Note that the same series converges absolutely in $\mathcal{P}([0,1])$, but does not converge in $\mathcal{P}([0,1])$ (see Example 4.1). This shows, in particular, that $\mathcal{P}([0,1])$ is not complete.

4.3 Completion of a normed space

It turns out that it is always possible to enlarge an incomplete normed space to a Banach space. Such a procedure is known as a *completion* of a normed space.

Definition 4.13. Let $(\mathbb{X}, \|\cdot\|)$ be a normed space. A normed space $(\tilde{\mathbb{X}}, \|\cdot\|_*)$ is called a *completion* of $(\mathbb{X}, \|\cdot\|)$ if:

(a) There exists a one-to-one mapping $\Phi : \mathbb{X} \to \tilde{\mathbb{X}}$ such that

$$\Phi(\lambda x + \mu y) = \lambda \Phi(x) + \mu \Phi(y), \qquad \text{for all } x, y \in \mathbb{X} \text{ and } \lambda, \mu \in \mathbb{F},$$

(b) $\|x\| = \|\Phi(x)\|_*$ for all $x \in \mathbb{X}$,

(c) $\Phi(\mathbb{X})$ is dense in $\tilde{\mathbb{X}}$;

(d) $\tilde{X}$ is complete, i.e. $\tilde{X}$ is a Banach space.

Example 4.14. The space $C([0,1])$ is a completion of the space $\mathcal{P}([0,1])$. As the map $\Phi : \mathcal{P}([0,1]) \to C([0,1])$ we simply take the *identity map*, defined as

$$\Phi(x) = x.$$

Remark 4.15. Given a normed space $(\mathbb{X}, \|\cdot\|)$, a completion of $\mathbb{X}$ can be always formally constructed as the spaces of equivalence classes of Cauchy sequences of elements of $\mathbb{X}$ with an induced norm. See [2] for further details.

4.4 Banach spaces $L_p([a,b])$

According to Example 4.8, the spaces $\mathscr{L}_p([a,b])$ are not complete. We are going to define spaces $L_p([a,b])$, the completion of $\mathscr{L}_p([a,b])$.

Definition 4.16. Let A be a subset of the real interval $[a,b]$. We say that A is a *set of measure zero*, and write mes$A = 0$, if for every $\varepsilon > 0$ there exists a finite or countable collection of intervals $\{[a_i, b_i] \subset [a,b] \,|\, i \in \mathcal{I}\}$, where $\mathcal{I} := \{1, 2, 3 \ldots N\}$ or $\mathcal{I} := \mathbb{N}$, such that:

i) $A \subset \cup_{i\in\mathcal{I}} [a_i, b_i]$;

ii) $\sum_{i\in\mathcal{I}} (b_i - a_i) < \varepsilon$.

Example 4.17. Every finite or countable subset of the real line has measure zero. In particular, the set of rational numbers $\mathbb{Q}$ has measure zero.

Exercise 4.18. Show that a finite or countable union of sets of measure zero has measure zero.

Definition 4.19. Let $x, y : [a,b] \to \mathbb{R}$ be *measurable*[6] functions. We say that that the function x is *equivalent* to function y, and write $x \sim y$, if

$$\text{mes}\{t \in [a,b] | x(t) \neq y(t)\} = 0.$$

[6]Definition of measurable functions was discussed in the *Measure and Probability* unit.

Example 4.20. The Dirichlet function $D : [a,b] \to \mathbb{R}$,

$$D(t) := \begin{cases} 1, & t \in [a,b] \cap \mathbb{Q}, \\ 0, & t \in [a,b] \cap (\mathbb{R} \setminus \mathbb{Q}) \end{cases}$$

is equivalent to the zero function $y(t) = 0$.

The equivalence of functions defines an equivalence relation on the set of measurable functions, i.e. for any measurable functions $x, y, z : [a,b] \to \mathbb{R}$:

(a) $x \sim x$;

(b) $x \sim y$ implies $y \sim x$;

(c) $x \sim y$ and $y \sim z$ implies $x \sim z$.

Given a measurable function $x : [a,b] \to \mathbb{R}$, we denote by $[x]$ the *equivalence class* of all measurable functions which are equivalent to x, that is

$$[x] := \{y : [a,b] \to \mathbb{R} \mid y \sim x\}.$$

Let $L([a,b])$ denotes the set of all equivalence classes of measurable functions on the interval $[a,b]$. Given $[x], [y] \in L([a,b])$ and $\lambda \in \mathbb{R}$, we define operations on $L([a,b])$ by:

$$[x] + [y] := [x + y], \qquad \lambda[x] := [\lambda x].$$

With these operations $L([a,b])$ becomes a *linear vector space*.

Remark 4.21. In what follows, when it does not cause a confusion, we write x instead of $[x]$, assuming that x is a representative of its own equivalence class $[x]$.

The *Lebesgue spaces* $L_p([a,b])$ are defined as linear subspaces of $L([a,b])$ which consist of the equivalences classes of measurable functions on $[a,b]$ of the finite $\|\cdot\|_p$–norm. The precise definition is as follows.

Definition 4.22. For $p \in [1, \infty)$, the space $L_p([a,b])$ is the set of equivalence classes of p–integrable measurable functions, that is

$$L_p([a,b]) := \left\{ x \in L([a,b]) \mid \|x\|_p := \left(\int_a^b |x(t)|^p dt \right)^{1/p} < \infty \right\},$$

where the integral is understood in the *Lebesgue* sense. [7]

The space $L_\infty([a,b])$ is the set of equivalence classes of *essentially bounded* measurable functions, that is

$$L_\infty([a,b]) := \left\{ x \in L([a,b]) \mid \|x\|_\infty := \operatorname{ess}\sup_{t \in [a,b]} |x(t)| < \infty \right\},$$

[7] Definition of the Lebesgue integral was discussed in the *Measure and Probability* unit. An importnat property of the Lebesgue integral is that $x \sim y$ implies $\int_a^b x(t)dt = \int_a^b y(t)dt$, so the value of the Lebesgue integral does not depend on the choice of a particular representative of an equivalence class.

where the *essential supremum*, denoted ess sup, (of the equivalence class) of a measurable function x is defined by

$$\operatorname*{ess\,sup}_{t\in[a,b]} |x(t)| := \inf_{y\in[x]} \sup_{t\in[a,b]} |y(t)|.$$

Example 4.23. For example, the function $x : [0,1] \to \mathbb{R}$ defined by

$$x(t) := \begin{cases} n, & t = 1/n, \\ 0, & t \neq 1/n, \end{cases}$$

is essentially bounded, and thus $x \in L_\infty([0,1])$. In fact $x \sim 0$ and $\|x\|_\infty = 0$.

Exercise 4.24. Show that $L_p([0,1]) \neq L_q([0,1])$ if $p \neq q$.

Hint. Consider the function $x(t) = t^\alpha$. For which values of $\alpha \leq 0$ the function x belongs to $L_p([0,1])$?

Exercise* 4.25. Let $p, q \in [1,\infty]$ and $p < q$. Prove that

$$L_q([a,b]) \subset L_p([a,b]).$$

Theorem 4.26. *$L_p([a,b])$ is a Banach space, for every $p \in [1,\infty]$.*

Proof. It is easy to see that $L_p([a,b])$ is a linear subspace of $L([a,b])$ and that $\|\cdot\|_p$ defines a norm on $L_p([a,b])$, so $L_p([a,b])$ is a normed space. The proof of the completeness of $L_p([a,b])$ requires some advanced facts from Measure theory and Lebesgue integration. We omit the details. □

Remark 4.27. For $p \in [1,\infty)$, the space $L_p([a,b])$ is a completion of the space $\mathscr{L}_p([a,b])$. As the one-to-one map $\Phi : \mathscr{L}_p([a,b]) \to L_p([a,b])$ we take

$$\Phi(x) := [x].$$

Note also that $C([a,b])$ can be considered as a closed subspace of $L_\infty([a,b])$ if we identify continuous functions with their equivalence classes.

One can prove that Lebesgue integrable functions satisfy the Hölder inequality.

Lemma 4.28. (Hölder inequality in $L_p([a,b])$) *Let $x \in L_p([a,b])$ and $y \in L_{p'}([a,b])$, where p' is the conjugate*[8] *to p. Then*

$$\boxed{\int_a^b |x(t)y(t)|dt \leq \|x\|_p \|y\|_{p'}.} \tag{4.5}$$

[8] See (2.6).

5 Hilbert spaces

5.1 Inner product spaces

Definition 5.1. Let $\mathbb{H}$ be a real linear vector space. An *inner product* on $\mathbb{H}$ is a function $\langle \cdot, \cdot \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ such that the following axioms are satisfied for all $x, y, z \in \mathbb{H}$ and $\lambda \in \mathbb{R}$:

$(H1)$ $\langle x, y \rangle = \langle y, x \rangle$;

$(H2)$ $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$;

$(H3)$ $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$;

$(H4)$ $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.

A *real inner product space* is a pair $(\mathbb{H}, \langle \cdot, \cdot \rangle)$, where $\mathbb{H}$ is a linear vector space and $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{H}$.

Remark 5.2. An inner product on *complex* linear vector spaces is defined differently. In what follows we consider only *real* inner product spaces and omit the word real if there is no ambiguity.

Remark 5.3. Inner product spaces are sometimes also called *prehilbert spaces*.

Exercise 5.4. Prove from the axioms of an inner product that for any $x, y, z \in \mathbb{H}$ and $\lambda, \mu \in \mathbb{R}$ one has:

i) $\langle x, \lambda y + \mu z \rangle = \lambda \langle x, y \rangle + \mu \langle x, z \rangle$

ii) $\langle x + y, x + y \rangle = \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle$.

Theorem 5.5. (Cauchy–Schwartz inequality) *Let* $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ *be an inner product space. Then for all* $x, y \in \mathbb{H}$ *one has*

$$|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \sqrt{\langle y, y \rangle}.$$

Proof. In the case $y = 0$ the inequality is obvious, so we assume that $y \neq 0$. Observe that for $\alpha \in \mathbb{R}$ we have

$$0 \leq \langle x + \alpha y, x + \alpha y \rangle = \langle x, x \rangle + 2\alpha \langle x, y \rangle + \alpha^2 \langle y, y \rangle. \tag{5.1}$$

Set

$$\alpha := -\frac{\langle x, y \rangle}{\langle y, y \rangle}$$

and multiply (5.1) by $\langle y, y \rangle$. Then we obtain

$$0 \leq \langle x, x \rangle \langle y, y \rangle - 2\langle x, y \rangle^2 + \langle x, y \rangle^2 = \langle x, x \rangle \langle y, y \rangle - \langle x, y \rangle^2,$$

and hence

$$\langle x, y \rangle^2 \leq \langle x, x \rangle \langle y, y \rangle.$$

Thus the assertion follows. □

The next theorem shows, in particular, that every inner product space is a normed space.

Theorem 5.6. (Norm on inner product spaces) *Let* $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ *be an inner product space. Then the function*

$$\|x\| := \sqrt{\langle x, x \rangle}$$

defines a norm on $\mathbb{H}$*. In particular, every inner product space is a normed space.*

Proof. Axioms $(N1)$ and $(N2)$ of the norm follow immediately from $(H4)$ and $(H3)$. To get the triangle inequality $(N3)$, observe that for every $x, y \in \mathbb{H}$, using Exercise 5.4 (ii) and Cauchy–Schwartz inequality we obtain

$$\|x+y\|^2 = \langle x+y, x+y \rangle = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2 \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2,$$

so the assertion follows. □

Remark 5.7. In what follows, we always assume that an inner product space $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ is equipped with the standard norm $\|x\| := \sqrt{\langle x, x \rangle}$.

Remark 5.8. In view of Theorem 5.6, the *Cauchy–Schwartz inequality* can be written as

$$\boxed{|\langle x, y \rangle| \leq \|x\|\|y\|.}$$

Exercise* 5.9. Let $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ be an inner product space and $x, y \in \mathbb{H}$. Prove that

$$|\langle x, y \rangle| = \|x\|\|y\| \qquad \Longleftrightarrow \qquad x = \alpha y \text{ for some } \alpha \in \mathbb{R}.$$

Theorem 5.10. (Parallelogram identity) *Let* $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ *be an inner product space and* $\|\cdot\|$ *the standard norm generated by the inner product. Then for every* $x, y \in \mathbb{H}$ *one has*

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Proof. For every $x, y \in \mathbb{H}$ one has

$$\|x+y\|^2 = \langle x+y, x+y \rangle = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2,$$
$$\|x-y\|^2 = \langle x-y, x-y \rangle = \|x\|^2 - 2\langle x, y \rangle + \|y\|^2,$$

Adding the above two identities, we obtain the required parallelogram identity. □

Remarkably, a norm satisfies the parallelogram identity if and and only if it is generated by an inner product, as the following exercise shows.

Exercise* 5.11. (Polarization identity) Let $(\mathbb{X}, \|\cdot\|)$ be a normed space. Assume that the norm $\|\cdot\|$ satisfies the parallelogram identity, that is for every $x, y \in \mathbb{X}$ one has

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Prove that the following *polarization identity*

$$\langle x, y \rangle := \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$$

defines a scalar product on $\mathbb{X}$, and therefore $\mathbb{X}$ is an inner product space.

Hint. Verify the axioms of the scalar product.

Examples of inner product spaces. We show that the normed spaces $\mathbb{R}_2^n$, l_2, $\mathscr{L}_2([a,b])$ and $L_2([a,b])$ are in fact inner product spaces.

Example 5.12. (Inner product on $\mathbb{R}_2^n$) For $x, y \in \mathbb{R}_2^n$, the function

$$\langle x, y\rangle := \sum_{i=1}^{n} x_i y_i$$

defines an inner product on $\mathbb{R}_2^n$. The corresponding norm is given by

$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{\sum_{i=1}^{n} |x_i|^2} = \|\cdot\|_2.$$

Example 5.13. (Inner product on l_2) For $x, y \in l_2$, the function

$$\langle x, y\rangle := \sum_{i=1}^{\infty} x_i y_i$$

defines an inner product on l_2. The corresponding norm is given by

$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{\sum_{i=1}^{\infty} |x_i|^2} = \|\cdot\|_2.$$

Note that in view of the Cauchy–Schwartz inequality,

$$|\langle x, y\rangle| \leq \|x\|\|y\| < \infty,$$

so the series in the definition of the inner product indeed converges.

Example 5.14. (Inner product on $\mathscr{L}_2([a,b])$) For $x, y \in \mathscr{L}_2([a,b])$, the function

$$\langle x, y\rangle := \int_a^b x(t)y(t)dt$$

defines an inner product on $\mathscr{L}_2([a,b])$. The corresponding norm is given by

$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{\int_a^b |x(t)|^2 dt} = \|\cdot\|_2.$$

Example 5.15. (Inner product on $L_2([a,b])$) For $x, y \in L_2([a,b])$, the function

$$\langle x, y\rangle := \int_a^b x(t)y(t)dt$$

defines an inner product on $L_2([a,b])$. The corresponding norm is given by

$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{\int_a^b |x(t)|^2 dt} = \|\cdot\|_2.$$

Note that in view of the Cauchy–Schwartz inequality,

$$|\langle x, y\rangle| \leq \|x\|\|y\| < \infty,$$

so the integral in the definition of the inner product indeed converges.

Exercise 5.16. Verify the axioms of the inner product for defined above inner products on $\mathbb{R}_2^n$, l_2, $L_2([a,b])$.

Exercise* 5.17. Show that if $p \neq 2$ then the spaces $\mathbb{R}_p^n$, l_p, $L_p([a,b])$ do not satisfy the parallelogram identity, and therefore, are not inner product spaces.

Geometrical meaning of the inner product. Consider the space $\mathbb{R}_2^2$. Let

$$x := (x_1, x_2), \qquad e := (1, 0).$$

Then

$$\frac{\langle x, e\rangle}{\|x\|\|e\|} = \frac{x_1}{\sqrt{x_1^2 + x_2^2}} = \cos(x, e),$$

the *cosine* of the angle between vectors x and e. Notice, in particular, that if $\langle x, e\rangle = 0$ then $\cos(x, e) = 0$, so the vectors x and e are orthogonal in that case.

Generally, given two vectors x and y in an abstract inner product space $\mathbb{H}$, the quantity

$$\frac{\langle x, y\rangle}{\|x\|\|y\|}$$

can be interpreted as the *cosine* of the angle between x and y. In this sense, the inner product encodes a geometrical information about the angle between two vectors. We will explore this geometrical meaning of the inner product in the next sections.

5.2 Hilbert spaces. Orthogonal and orthonormal systems

Definition 5.18. Let $(\mathbb{H}, \langle\cdot,\cdot\rangle)$ be an inner product space and $\|\cdot\|$ the standard norm generated by the inner product. We say that $\mathbb{H}$ is a *Hilbert space* if $\mathbb{H}$ is complete with respect to the norm $\|\cdot\|$.

Example 5.19. Spaces $\mathbb{R}_2^n$, l_2, $L_2([a,b])$ are Hilbert spaces. The space $\mathscr{L}_2([a,b])$ is an inner product space which is not a Hilbert spaces (because it is not complete).

Definition 5.20. Let $(\mathbb{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $x, y \in \mathbb{H}$. We say that x is *orthogonal* to y if

$$\langle x, y\rangle = 0.$$

Example 5.21. Vectors $e_1 = (1, 0, 0, 0, \dots)$ and $e_2 = (0, 1, 0, 0, \dots)$ are orthogonal in l_2. Vectors $x(t) = 1$, $y(t) = \sin(t)$ are orthogonal in $L_2([-\pi, \pi])$, but are not orthogonal in $L_2([0, 1])$.

Definition 5.22. Let $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $S \subset \mathbb{H}$ a subset of $\mathbb{H}$. We say that S is an *orthogonal system* in $\mathbb{H}$ if $0 \notin S$ and

$$x, y \in S \quad \text{and} \quad x \neq y \qquad \Longrightarrow \qquad \langle x, y \rangle = 0.$$

We say that S is an *orthonormal system* in $\mathbb{H}$ if S is an orthogonal system and, in addition,

$$x \in S \qquad \Longrightarrow \qquad \|x\| = 1.$$

Remark 5.23. Let $S \subset \mathbb{H}$ be an orthogonal system. Note that if $x \in S$ then

$$\left\| \frac{x}{\|x\|} \right\| = 1.$$

Hence

$$\tilde{S} := \left\{ \frac{x}{\|x\|} \, : \, x \in S \right\}$$

is an orthonormal system, generated by S. In other words, every orthogonal system can be *normalized* into an orthonormal system.

Remark 5.24. If $S = (e_k)_{k \in \mathbb{N}}$ is a sequence of vectors in $\mathbb{H}$ then instead of the term *system* we use the word *sequence*.

Example 5.25. (Standard orthonormal sequence in l_2) The sequence $(e_k)_{k \in \mathbb{N}} \subset l_2$, where

$$\begin{aligned} e_1 &:= (1, 0, 0, 0, 0, \dots), \\ e_2 &:= (0, 1, 0, 0, 0, \dots), \\ e_3 &:= (0, 0, 1, 0, 0, \dots), \\ &\dots \end{aligned}$$

is an orthonormal sequence in l_2, which is often called *the standard orthonormal sequence*.

Example 5.26. (Trigonometric orthonormal sequence in $L_2[(-\pi, \pi])$) The sequence

$$\frac{1}{\sqrt{2\pi}}, \ \frac{\cos(t)}{\sqrt{\pi}}, \ \frac{\sin(t)}{\sqrt{\pi}}, \ \frac{\cos(2t)}{\sqrt{\pi}}, \ \frac{\sin(2t)}{\sqrt{\pi}}, \ \dots \ \frac{\cos(kt)}{\sqrt{\pi}}, \ \frac{\sin(kt)}{\sqrt{\pi}}, \ \dots \ .$$

is an important example of an orthonormal sequence in $L_2[(-\pi, \pi])$, which is called *trigonometric orthonormal sequence*.

Theorem 5.27. *Let $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $S \subset \mathbb{H}$ an orthogonal system in $\mathbb{H}$. Then S is a linearly independent system, that is for any $x_1, \dots, x_m \in S$ and any $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ one has*

$$\lambda_1 x_1 + \dots + \lambda_m x_m = 0 \qquad \Longrightarrow \qquad \lambda_1 = \dots = \lambda_m = 0.$$

Proof. Assume that $\lambda_1 x_1 + \cdots + \lambda_m x_m = 0$. Then a direct computation shows that

$$0 = \sum_{i=1}^{m} \langle 0, \lambda_i x_i \rangle = \sum_{i=1}^{m} \left\langle \underbrace{\sum_{i=1}^{m} \lambda_i x_i}_{=0} \,,\, \lambda_i x_i \right\rangle = \sum_{i=1}^{m} |\lambda_i|^2 \underbrace{\|x_i\|^2}_{>0}.$$

We conclude that $\lambda_1 = \cdots = \lambda_m = 0$. □

Remark 5.28. In particular, this implies that if a Hilbert space $\mathbb{H}$ admits an orthogonal sequence then $\mathbb{H}$ is infinite dimensional.

Apparently, the converse to the above theorem is not true, that is a linearly independent system in a Hilbert space may not be orthogonal, as it is shown by the following example.

Example 5.29. In $L^2([-1,1])$, consider the sequence of vectors

$$1,\ t,\ t^2,\ \ldots,\ t^k,\ \ldots\ .$$

It is easy to verify that vectors in the sequence are linearly independent (see Exercise 1.30). However, for every $k \in \mathbb{N}$ we obtain

$$\int_{-1}^{1} t^k t^{k+2} dt = \int_{-1}^{1} t^{2k+2} dt \neq 0,$$

so the sequence is not orthogonal, moreover, it contains infinitely many different nonorthogonal pairs of vectors. Note however that for every $k \in \mathbb{N}$ one has

$$\int_{-1}^{1} t^k t^{k+1} dt = \int_{-1}^{1} t^{2k+1} dt = 0.$$

Gram–Schmidt orthogonalisation process. Given a linearly independent sequence of vectors in a Hilbert space one can always construct an associated orthogonal sequence of vector using the so-called *Gram–Schmidt orthogonalisation process*.

Theorem 5.30. (Gram–Schmidt orthogonalisation) *Let $(\mathbb{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $(x_k)_{k\in\mathbb{N}} \subset \mathbb{H}$ a linearly independent sequence of vectors in $\mathbb{H}$. Then the sequence of vectors*

$$\begin{aligned} e_1 &= x_1, \\ e_2 &= x_2 - \lambda_{21} e_1, \quad \textit{where} \quad \lambda_{21} = \frac{\langle x_2, e_1\rangle}{\|e_1\|^2}, \\ &\ldots \\ e_k &= x_k - \sum_{i=1}^{k} \lambda_{ki} e_i, \quad \textit{where} \quad \lambda_{ki} = \frac{\langle x_k, e_i\rangle}{\|e_i\|^2}, \\ &\ldots \end{aligned}$$

is an orthogonal sequence in $\mathbb{H}$.

Proof. A direct computation shows that $(e_k)_{k\in\mathbb{N}}$ is indeed an orthogonal sequence in $\mathbb{H}$, that is

(i) for every $k \in \mathbb{N}$ one has $\|e_k\| \neq 0$, and

(i) for every $k, l \in \mathbb{N}$, $k \neq l$ one has $\langle e_k, e_l\rangle = 0$.

Verify this as an exercise! □

Exercise 5.31. In $L^2([-1,1])$, consider a linearly independent sequence of vectors

$$1,\ t,\ t^2,\ \ldots,\ t^k,\ \ldots\ .$$

Then, using Gram–Schmidt orthogonalisation process, we obtain the first three terms of the corresponding orthogonal sequence as follows:

$$\begin{aligned}
e_1 &= 1;\\
e_2 &= t - \lambda_{21}.1 = t, \quad \text{as } \lambda_{21} = \frac{\int_{-1}^1 t.1\,dt}{\int_{-1}^1 1^2\,dt} = 0;\\
e_3 &= t^2 - \lambda_{31}.1 - \lambda_{32}.t = t^2 - \frac{1}{3}, \quad \text{as } \lambda_{31} = \frac{\int_{-1}^1 t^2.1\,dt}{\int_{-1}^1 1^2\,dt} = \frac{1}{3}, \quad \lambda_{32} = \frac{\int_{-1}^1 t^2.t\,dt}{\int_{-1}^1 |t|^2\,dt} = 0.
\end{aligned}$$

Example 5.32. (Legendre Polynomials) Continuing the Gram–Schmidt orthogonalisation process, one can show that the orthogonal sequence in $L^2([-1,1])$, generated by the linearly independent sequence

$$1,\ t,\ t^2,\ \ldots,\ t^k,\ \ldots\ .$$

can be expressed (up to normalization) in the form of the *Legendre Polynomials*:

$$\begin{aligned}
P_0(t) &= 1;\\
P_k(t) &= \frac{1}{2^k k!}\frac{d^k}{dx^k}(x^2-1)^k \qquad (k \in \mathbb{N}).
\end{aligned}$$

Legendgre polynomials is an example of a *polynomial orthonormal sequence* in $L_2([-1,1])$.

5.3 Fourier series

Definition 5.33. Let $(\mathbb{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $(e_k)_{k\in\mathbb{N}} \subset \mathbb{H}$ an orthonormal sequence in $\mathbb{H}$. Given a vector $x \in \mathbb{H}$, the number

$$\alpha_k := \langle x, e_k\rangle \qquad (k \in \mathbb{N})$$

is called *k–th Fourier coefficient of* x (with respect to the orthonormal sequence $(e_k)_{k\in\mathbb{N}}$), and the series

$$\sum_{k=1}^{\infty} \alpha_k e_k$$

is called *Fourier series of* x (with respect to the orthonormal sequence $(e_k)_{k\in\mathbb{N}}$).

Example 5.34. Consider the orthonormal *trigonometric sequence* in $L_2([-\pi,\pi])$

$$\begin{aligned} e_0 &= \frac{1}{\sqrt{2\pi}}, \\ e_{2k-1} &= \frac{\sin(kt)}{\sqrt{\pi}} \qquad (k \in \mathbb{N}), \\ e_{2k} &= \frac{\cos(kt)}{\sqrt{\pi}} \qquad (k \in \mathbb{N}), \end{aligned}$$

Let $x = \cos^2(t)$. By a direct computation (taking into account that $\sin(kt)$ are odd and $\cos^2(t)$ and $\cos(kt)$ are even functions) we obtain the *Fourier coefficients* of x as:

$$\begin{aligned} \alpha_0 &= \int_{-\pi}^{\pi} \cos^2(t)\,\frac{1}{\sqrt{2\pi}}\,dt = \frac{\sqrt{2\pi}}{2}, \\ \alpha_{2k-1} &= \int_{-\pi}^{\pi} \cos^2(t)\,\frac{\sin(kt)}{\sqrt{\pi}}\,dt = 0 \quad \text{for all } k \in \mathbb{N}, \\ \alpha_2 &= \int_{-\pi}^{\pi} \cos^2(t)\,\frac{\cos(t)}{\sqrt{\pi}}\,dt = 0, \\ \alpha_4 &= \int_{-\pi}^{\pi} \cos^2(t)\,\frac{\cos(2t)}{\sqrt{\pi}}\,dt = \frac{\sqrt{\pi}}{2}, \\ \alpha_{2k} &= \int_{-\pi}^{\pi} \cos^2(t)\,\frac{\cos(kt)}{\sqrt{\pi}}\,dt = 0 \quad \text{for all } k = 3,4,5,\dots. \end{aligned}$$

Thus the *Fourier series* of x given by the (finite) series:

$$\frac{\sqrt{2\pi}}{2}e_0 + \frac{\sqrt{\pi}}{2}e_2.$$

The classical trigonometric formula

$$2\cos^2(t) = \cos(2t) + 1$$

confirms that in this example the Fourier series of x actually coincides with x, that is

$$x = \frac{\sqrt{2\pi}}{2}e_0 + \frac{\sqrt{\pi}}{2}e_2.$$

Convergence of Fourier series. Given an orthonormal sequence $(e_k)_{k\in\mathbb{N}} \subset \mathbb{H}$, we are going to answer *two fundamental questions* concerning the Fourier series of a vector $x \in \mathbb{H}$ with respect to $(e_k)_{k\in\mathbb{N}}$:

($Q1$) When Fourier series of $x \in \mathbb{H}$ converges in $\mathbb{H}$, that is there exists a vector $S_\infty \in \mathbb{H}$ such that

$$S_\infty = \sum_{k=1}^{\infty} \alpha_k e_k \ ?$$

$(Q2)$ If Fourier series of $x \in \mathbb{H}$ converges in $\mathbb{H}$, when the sum S_∞ of the series coincides with x, that is

$$x = S_\infty \, ?$$

In order to answer $(Q1)$ and $(Q2)$ we need to prove several preliminary results.

Theorem 5.35. (Pythagoras Theorem) *Let $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space and*

$$\{x_1, x_2, \ldots, x_m\} \subset \mathbb{H}$$

a finite orthogonal system in $\mathbb{H}$. Then

$$\left\| \sum_{k=1}^{m} x_k \right\|^2 = \sum_{k=1}^{m} \|x_k\|^2. \tag{5.2}$$

Proof. We proceed by induction.

Let $m = 2$. Then

$$\|x_1 + x_2\|^2 = \|x_1\|^2 + 2 \underbrace{\langle x_1, x_2 \rangle}_{=0} + \|x_2\|^2 = \|x_1\|^2 + \|x_2\|^2.$$

Next, assume that (5.2) holds for $m = k$. Then for $m = k + 1$ we obtain

$$\begin{aligned} \|(x_1 + \cdots + x_k) + x_{k+1}\|^2 &= \left(\|x_1\|^2 + \cdots + \|x_k\|^2\right) + 2 \underbrace{\langle x_1 + \cdots + x_k, x_{k+1} \rangle}_{=0} + \|x_{k+1}\|^2 \\ &= \|x_1\|^2 + \cdots + \|x_k\|^2 + \|x_{k+1}\|^2. \end{aligned}$$

Thus the assertion follows. $\square$

Theorem 5.36. (Bessel Identity and Inequality) *Let $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space and*

$$\{e_1, e_2, \ldots, e_m\} \subset \mathbb{H}$$

be a finite orthonormal system in $\mathbb{H}$. Given $x \in \mathbb{H}$, let

$$\alpha_k = \langle x, e_k \rangle \quad \textit{and} \quad S_m = \sum_{k=1}^{m} \alpha_k e_k.$$

Then

$$\|x - S_m\|^2 = \|x\|^2 - \sum_{k=1}^{m} \alpha_k^2, \tag{5.3}$$

and

$$\sum_{k=1}^{m} \alpha_k^2 \leq \|x\|^2. \tag{5.4}$$

Proof. By Pythagoras Theorem we obtain

$$\|S_m\|^2 = \left\|\sum_{k=1}^{m} \alpha_k e_k\right\|^2 = \sum_{k=1}^{m} \|\alpha_k e_k\|^2 = \sum_{k=1}^{m} \alpha_k^2.$$

Then

$$\begin{aligned}
\|x - S_m\|^2 &= \|x\|^2 - 2\langle x, S_m\rangle + \|S_m\|^2 = \|x\|^2 - 2\sum_{k=1}^{m}\left\langle x, \sum_{k=1}^{m}\alpha_k e_k\right\rangle + \sum_{k=1}^{m}\alpha_k^2 \\
&= \|x\|^2 - 2\sum_{k=1}^{m}\alpha_k \underbrace{\langle x, e_k\rangle}_{=\alpha_k} + \sum_{k=1}^{m}\alpha_k^2 = \|x\|^2 - 2\sum_{k=1}^{m}\alpha_k^2 + \sum_{k=1}^{m}\alpha_k^2 \\
&= \|x\|^2 - \sum_{k=1}^{m}\alpha_k^2,
\end{aligned}$$

which proves the identity (5.3). Next, we simply rewrite (5.3) as

$$\sum_{k=1}^{m}\alpha_k^2 = \|x\|^2 - \underbrace{\|S_m\|^2}_{\geq 0} \leq \|x\|^2,$$

which establishes the inequality (5.4). □

We are now in a position to answer question ($Q1$).

Theorem 5.37. (Convergence of Fourier Series) *Let $(\mathbb{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $(e_k)_{k\in\mathbb{N}} \subset \mathbb{H}$ an orthonormal sequence. Then for any $x \in \mathbb{H}$ the Fourier series of x*

$$\sum_{k=1}^{\infty}\alpha_k e_k$$

converges in $\mathbb{H}$, that is there exists $S_\infty \in \mathbb{H}$ such that

$$S_\infty := \sum_{k=1}^{\infty}\alpha_k e_k.$$

Moreover,

$$\|S_\infty\|^2 = \sum_{k=1}^{\infty}\alpha_k^2 \leq \|x\|^2. \tag{5.5}$$

Proof. To prove the convergence of the Fourier series of x it is sufficient to show that the sequence of partial sums

$$S_m := \sum_{k=1}^{m}\alpha_k e_k$$

is a Cauchy sequence in $\mathbb{H}$.

Observe that the real sequence of partial sums

$$\sigma_m := \sum_{k=1}^{m} \alpha_k^2$$

is monotone increasing and, by Bessel Inequality, is bounded above by $\|x\|^2$, that is

$$\sigma_m \leq \|x\|^2.$$

Therefore, (σ_m) converges and, in particular (σ_m) is a Cauchy sequence of real numbers. Hence for all $m, n \in \mathbb{N}$ such that $m > n$, using Pythagoras Theorem, we obtain

$$\|S_m - S_n\|^2 = \left\|\sum_{k=n}^{m} \alpha_k e_k\right\|^2 = \sum_{k=n}^{m} \alpha_k^2 = \sigma_m - \sigma_n \to 0.$$

Thus (S_m) is a Cauchy sequence in $\mathbb{H}$ and therefore (S_m) converges in $\mathbb{H}$ to a vector $S_\infty \in \mathbb{H}$:

$$S_\infty := \lim_{m\to\infty} S_m := \sum_{k=1}^{\infty} \alpha_k e_k.$$

Further, passing to the limit as $m \to \infty$ in the Bessel inequality (5.4), we conclude that

$$\|S_\infty\|^2 = \sum_{k=1}^{\infty} \alpha_k^2 \leq \|x\|^2.$$

The proof is completed. □

To answer the question $(Q2)$, we introduce the following definition.

Definition 5.38. Let $(\mathbb{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space. We say that an orthonormal sequence $(e_k)_{k\in\mathbb{N}} \subset \mathbb{H}$ is *complete* if for every $x \in \mathbb{H}$ the Fourier series of x converges to x, that is

$$x = \sum_{k=1}^{\infty} \alpha_k e_k.$$

Remark 5.39. It follows from (5.5) that if

$$x = \sum_{k=1}^{\infty} \alpha_k e_k$$

then

$$\|x\|^2 = \sum_{k=1}^{\infty} \alpha_k^2, \tag{5.6}$$

so Bessel inequality becomes an identity if the orthonormal sequence is complete.

Remark 5.40. A complete orthonormal sequence is also called an *orthonormal basis* in $\mathbb{H}$.

Exercise** **5.41.** Show that the *standard orthonormal sequence* in l_2, which was defined in Exercise 5.25, is complete.

Not every orthonormal sequence is complete, as the following example shows.

Example 5.42. In $L_2([-\pi, \pi])$, consider an orthonormal sequence

$$e_k(t) = \frac{\sin(kt)}{\sqrt{\pi}} \qquad (k \in \mathbb{N}).$$

Let $x(t) = 1$. Then

$$\alpha_k = \int_{-\pi}^{\pi} 1. \sin(kt)\, dt = 0,$$

and hence Fourier series of x do not converge to x. In fact, we have

$$S_\infty = \sum_{k=1}^{\infty} \alpha_k e_k = 0 \neq x.$$

Verification of the completeness of an orthonormal sequences is often complicated. We state without proofs that:

- *Trigonometric orthonormal sequence in $L_2([-\pi, \pi])$, as defined in Exercise 5.26, is complete*
- *Legendre orthonormal sequence in $L_2([-1, 1])$, as defined in Exercise 5.32, is complete*

The proofs of the completeness as well as other examples of complete orthonormal sequences are beyond the scope of this course.

5.4 Separable Hilbert spaces

Definition 5.43. Let $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space. We say $\mathbb{H}$ is *separable* if $\dim \mathbb{H} = \infty$ and $\mathbb{H}$ admits a complete orthonormal sequence.

Example 5.44. Spaces l_2 and $L_2([a, b])$ are separable.

Theorem 5.45. *The unit ball in a separable Hilbert space in not compact.*

Proof. Consider an orthonormal sequence $(e_n) \subset \mathbb{H}$. Let $n, m \in \mathbb{N}$ and $n \neq m$. Then

$$\|e_n - e_m\|^2 = \|e_n\|^2 - 2\langle e_n, e_m \rangle + \|e_m\|^2 = 2.$$

Hence (e_n) does not contain any Cauchy subsequence and therefore (e_n) does not contain any convergent subsequence. Since $(e_n) \subset \bar{B}_1(0)$, this means that $\bar{B}_1(0)$ is not compact. □

Isomorphism of separable Hilbert spaces. Let $(\mathbb{H}, \langle\cdot,\cdot\rangle)$ be a separable Hilbert space with a complete orthonormal sequence $(e_k) \subset \mathbb{H}$. Then every $x \in \mathbb{H}$ is represented as the sum of its Fourier series:

$$x = \sum_{k=1}^{\infty} \alpha_k e_k.$$

Denote by α the sequence of the Fourier coefficients of x:

$$\alpha := (\alpha_1, \alpha_2, \dots, \alpha_k, \dots).$$

From (5.6) we know that

$$\|x\|^2 = \sum_{k=1}^{\infty} \alpha_k^2.$$

Therefore, the sequence of Fourier coefficients α can be considered as an element of the space l_2, moreover

$$\|\alpha\|_2 = \left(\sum_{k=1}^{\infty} \alpha_k^2\right)^{1/2} = \|x\|.$$

In other words, to every $x \in \mathbb{H}$ corresponds an element $\alpha \in l_2$, the sequence of Fourier coefficients of x.

On the other hand, given an arbitrary sequence $\alpha \in l_2$, we can consider the Fourier series of the form

$$x := \sum_{k=1}^{\infty} \alpha_k e_k \in \mathbb{H},$$

and moreover, $\|x\| = \|\alpha\|_2$ by Bessel's inequality. So, to every $\alpha \in l_2$ corresponds an element $x \in \mathbb{H}$, the Fourier series with the sequence of coefficients α.

In such a way, we established a one–to–one correspondence between $\mathbb{H}$ and l_2:

$$\mathbb{H} \supset x \leftrightarrow \alpha \in l_2.$$

Moreover,

$$\|x\| = \|\alpha\|_2.$$

Such a correspondence is called an *isomorphism* between the Hilbert spaces $\mathbb{H}$ and l_2, and the above argument shows that *every separable Hilbert space is isomorphic to the Hilbert space l_2.*