HARNACK INEQUALITIES FOR DEGENERATE AND SINGULAR PARABOLIC EQUATIONS

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INTRODUCTION

- Heat equation

$$u_t - \Delta u = 0$$

- u **non negative local** solution in $\Omega_T = \Omega \times (0, T]$

- $(x_0, t_0) \in \Omega_T$, $\quad u(x_0, t_0) > 0$

- **PROBLEM**: Does this information spread on a larger set at a **later** time?

- **ANSWER**: YES

THEOREM [Parabolic Harnack Inequality] There exist two positive constants that depend only on the dim N, c, ϑ such that, for all $B_{4R}(x_0) \times \{t_0 - \vartheta(4R)^2, t_0 + \vartheta(4R)^2\} \subseteq \Omega_T$

$$u(x_0, t_0) \leq c \inf_{B_R(x_0)} u(\cdot, t_0 + \vartheta R^2)$$

REMARKS

* It is a statement on the expansion of positivity
* It is also a statement on the controlled growth of u: we know that solutions are regular, and here we are saying that their behaviour cannot be arbitrary
* The proof is due to Pini and Hadamard (1954): it is based on the local representation of solutions in terms of heat potentials.
* It is a very linear proof
* Equivalent and alternative form

$$\sup_{B_R(x_0)} u(\cdot, t_0 - \theta R^2) \leq c\, u(x_0, t_0)$$

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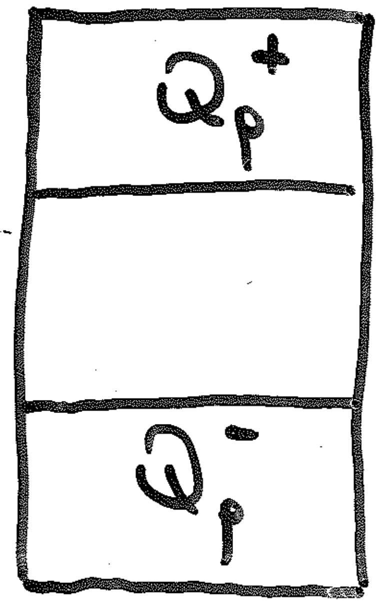
* General quasi-linear parabolic equation with growth of order 2

$$\begin{cases} u_t - \operatorname{div} \mathbb{A}(x,t,u,Du) = 0 \\ \mathbb{A}(x,t,u,Du)\cdot Du \geq c_0\, |Du|^2 \\ |\mathbb{A}(x,t,u,Du)| \leq c_1 |Du| \\ c_0,\ c_1 \ \text{positive constants} \end{cases}$$

* Main result : the Harnack inequality still holds for this general case
* Third equivalent form

$$\sup_{Q_\rho^-} u \leq c \inf_{Q_\rho^+} u$$



* The time gap cannot be eliminated (there are counterexamples)
* You need **sufficient** space between Q_ρ^+, Q_ρ^- and $\partial\Omega_T$ (just as before
* Highly **non linear** proof, purely measure - theoretic
* The Harnack Inequality implies the Hölder continuity of bounded local solutions

SUMMARY

* Three equivalent forms

$$u(x_0, t_0) \leq c \inf_{B_R(x_0)} u(\cdot, t_0 + \vartheta R^2)$$

$$\sup_{B_R(x_0)} u(\cdot, t_0 - \vartheta R^2) \leq c\, u(x_0, t_0)$$

$$\sup_{Q_R^-} u \leq c \inf_{Q_R^+} u$$

* The first proof is based on heat potentials, the general case is dealt with pure measure-theoretic tools.

QUESTION

Suppose we consider a **non linear** diffusion equation, like

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = 0, \qquad p > 1$$

[parabolic p-laplacian]

or

$$u_t - \operatorname{div}(Du^m) = 0 \qquad m > 0$$

[porous medium equation]

Does a Harnack Inequality still hold?

REMARKS

* In the following we concentrate on the first case
* Our question parallels going from

$$\Delta u = 0, \qquad \text{or} \qquad \begin{cases} \operatorname{div} \mathbb{A}(x,u,Du) = 0 \\ \mathbb{A} \cdot Du \geq C_0 |Du|^2 \\ |\mathbb{A}| \leq C_1 |Du| \end{cases}$$

to

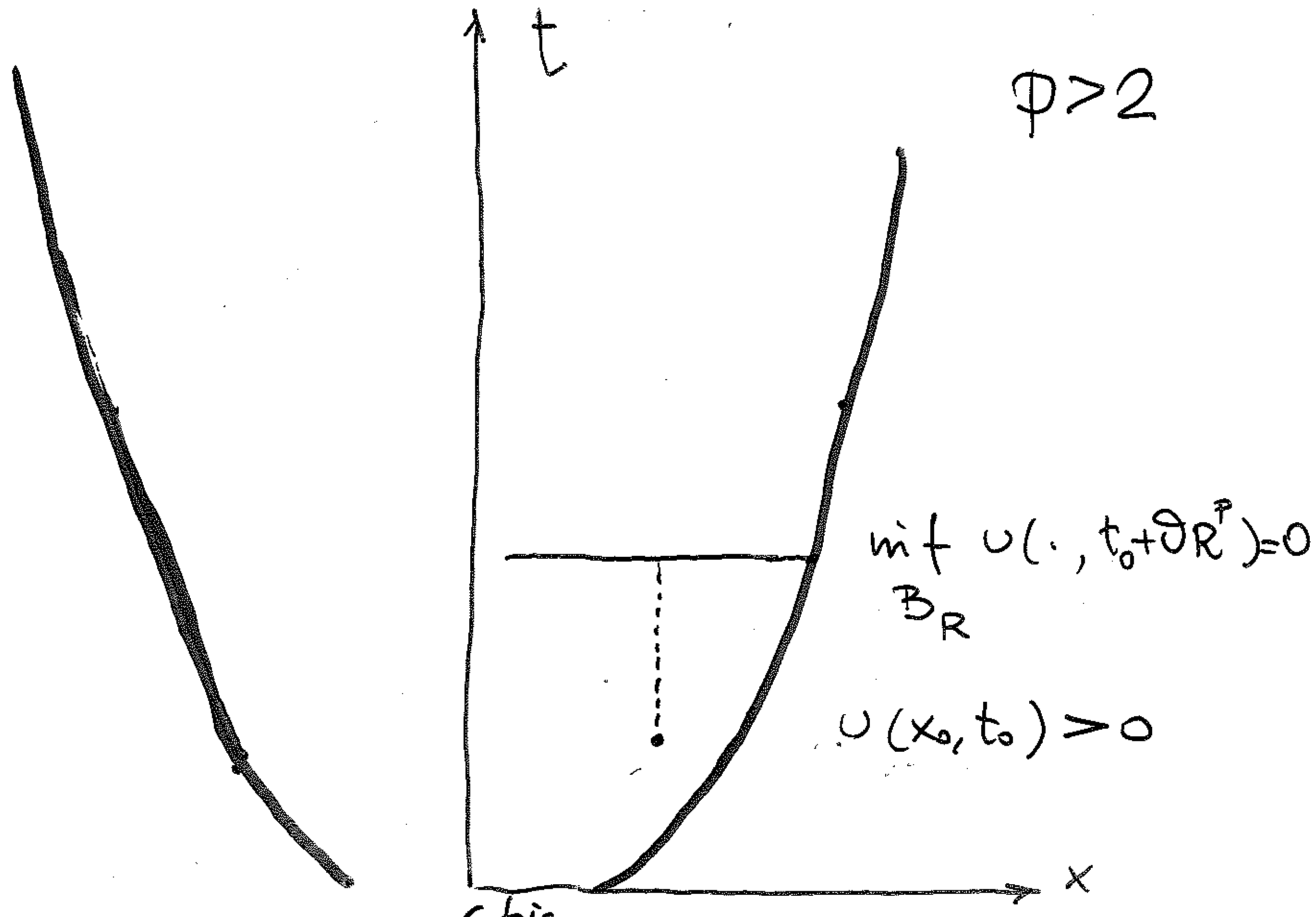
$$\operatorname{div}(|Du|^{p-2}Du) = 0$$

* In the elliptic case the answer is yes, just track down the topology of L^p versus the topology of L^2.
* A simple change in the space-time scaling seems enough: instead of (ρ, ρ^2) take (ρ, ρ^p). Unfortunately this is not correct

COUNTEREXAMPLE: the Barenblatt similarity solution

$$\mathcal{B}_p(x,t) = \frac{1}{t^{N/\lambda}}\left(1 - \gamma_p\left(\frac{|x|}{t^{1/\lambda}}\right)^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}$$

$$t>0, \qquad \gamma_p = \gamma_p(N,p), \qquad \lambda = N(p-2)+p$$



The reason for this lies in the lack of homogeneity

$$u_t - \Delta u = 0 \quad \text{vs} \quad u_t - \operatorname{div}(|Du|^{p-2} Du) = 0$$

HINT: Use an **intrinsic** scaling, namely scale time as $u^{2-p} \rho^p$

The first suggestion is somehow given in Trudinger (1968)

THEOREM [DiBenedetto 1988 ($p > 2$), DiBenedetto-Kwong ($\frac{2N}{N+1} < p < 2$)]

Let u be a non-negative local weak solution to $u_t - \operatorname{div}(|Du|^{p-2} Du) = 0$

There exist two constants c and γ depending only on N and p, such that for all cylinders

$$B_{4R}(x_0) \times \{ t_0 - \theta (4R)^p, t_0 + \theta (4R)^p \}$$

contained in Ω_T, we have

$$u(x_0, t_0) \leq \gamma \inf_{B_R(x_0)} u(\cdot, t_0 + \theta R^p)$$

where

$$\theta = \left(\frac{c}{u(x_0, t_0)} \right)^{p-2}$$

REMARKS

* Range of p: $p > 2$; $\frac{2N}{N+1} < p < 2$

* It holds just for the prototype equation, but not for the equation with the full quasi-linear structure

* The proof uses the comparison principle and the Barenblatt fundamental solution. Somehow we are back to Pini and Hadamard.

* We have only the first form

* No Harnack Inequality when $1 < p \leq \frac{2N}{N+1}$: there are explicit counterexamples

* We cannot say that Harnack implies Hölder

QUESTION 1: What can we say for the general case? Can we extend the previous result?

QUESTION 2: If the answer is yes, can we recover all three equivalent forms?

Let us first concentrate on the $p>2$ case. It turns out there is a considerable difference between $p>2$ and $1<p<2$

For $p>2$, when $|Du|=0$, the diffusion coefficient $|Du|^{p-2}$ vanishes: no diffusion at all!

Two competing phenomena:

- Diffusion

vs

- Time evolution

The Barenblatt fundamental solution reflects this.