THEOREM [DiBenedetto - G. - Vespri, Acta Math. (2008), Duke Math. J. (2008)]
Let u be a local weak non-negative solution of

$$u_t - \operatorname{div} \mathbb{A}(x,t,u,Du) = 0, \quad p>2$$

with

$$\mathbb{A}(x,t,u,Du) \geq C_0 |Du|^p$$

$$|\mathbb{A}(x,t,u,Du)| \leq C_1 |Du|^{p-1}$$

$$C_0,\ C_1 > 0.$$

Then, as before,

$$u(x_0,t_0) \leq \gamma \inf_{B_R} u(\cdot, t_0+\theta R^p)$$

$$\theta = \left(\frac{c}{u(x_0,t_0)}\right)^{p-2}$$

with the same assumptions on γ, c, and the reference domain

REMARKS

* The second alternative form holds true, namely

$$\sup_{B_R(x_0)} u(\cdot, t_0 - \vartheta R^p) \leq \gamma\, u(x_0, t_0)$$

$$\vartheta = \left(\frac{c}{u(x_0, t_0)} \right)^{p-2}$$

[unpublished]

* The third alternative form holds true,

$$\sup_{Q_R^-} u \leq c \inf_{Q_R^+} u$$

provided the two cylinders are intrinsically stretched
[Kuusi, Annali SNS, to appear]

FURTHER REMARKS

* The method is purely measure theoretic (back to Moser)
* Harnack implies Hölder
* The two papers give two different proofs
* Quite unexpectedly, nonlinear potentials are back! Indeed

THEOREM Let u be a non-negative solution of

$$u_t - \operatorname{div} \mathbb{A}(x,t,u,Du) = 0$$

satisfying

$$u(\cdot, t_o) \geq k > 0 \qquad \text{in } B_R(x_o)$$

for some $(x_o, t_o) \in \Omega_T$. Then for all $(x,t) \in E_T$ with $x \neq x_o$, $t_o < t < \frac{3}{2} t_o$,

$$u(x,t) \geq \frac{1}{2} \frac{k R^{\nu}}{S^{\nu/\lambda}(t)} \left[1 - \gamma_o \lambda^{\frac{1}{p-1}} \left(\frac{|x - x_o|}{S^{1/\lambda}(t)} \right)^{\frac{p}{p-1}} \right]_+^{\frac{p-1}{p-2}}$$

provided $\lambda k^{p-2} \rho^{\nu(p-2)} \geq 2\rho^{\lambda}$. Here

$$\lambda = \nu(p-2) + p > 0, \quad \nu > 0$$

λ (and hence ν) depends on the data,
$S(t) = \lambda k^{p-2} \rho^{\nu(p-2)} (t - t_o) + \rho^{\lambda}$.

Barenblatt similarity solution

$$\mathcal{B}_p = \frac{1}{t^{N/\lambda}} \left(1 - \gamma_p(N,p) \left(\frac{|x|}{t^{1/\lambda}} \right)^{\frac{p}{p-1}} \right)_+^{\frac{p-1}{p-2}}$$

$$\lambda = N(p-2)+p$$

Lower bound

$$\Gamma_p = \frac{1}{2} \frac{kR^\nu}{S^{\nu/\lambda}(t)} \left[1 - \gamma \lambda^{\frac{1}{p-1}} \left(\frac{|x-x_0|}{S^{1/\lambda}(t)} \right)^{\frac{p}{p-1}} \right]_+^{\frac{p-1}{p-2}}$$

$$\lambda = \nu(p-2)+p \qquad S(t) = \lambda k^{p-2} \rho^{\nu(p-2)}(t-t_0)+\rho^\lambda$$

REMARKS

* A family of parametrized subpotentials
* We close the circle

  Before: $\mathcal{B}_p \Longrightarrow$ Harnack

  Now : Harnack $\Longrightarrow \Gamma_p$
* Physical interpretation: Degeneracy in physical models surfaces only if one insists in a flat Euclidean description

* A dizect link between subpotentials and the Harnack inequality

QUESTION: What about the $1<p<2$ case?

REMARKS

* When $|Du|=0$, the diffusion coefficient $|Du|^{p-2}$ blows up
* When $1<p\leq \frac{2N}{N+1}$ solutions with initial datum $u_0\in L^1(\mathbb{R}^N)$ become extipt abzuptly in finite time
* When $\frac{2N}{N+1}<p<2$, we have a unique space decay, namely $u(x,t)\simeq |x|^{-\frac{p}{2-p}}$

but

* When $1<p\leq \frac{2N}{N+1}$, we have both a

fast decay $\quad u(x,t)\simeq |x|^{-\frac{N-p}{p-1}}$

and a

slow decay $\quad u(x,t)\simeq |x|^{-\frac{p}{2-p}}$

* In the whole range, $1<p<2$ solutions are Hölder continuous [Chen-DiBenedetto, (1992)], but this is not a contradiction.

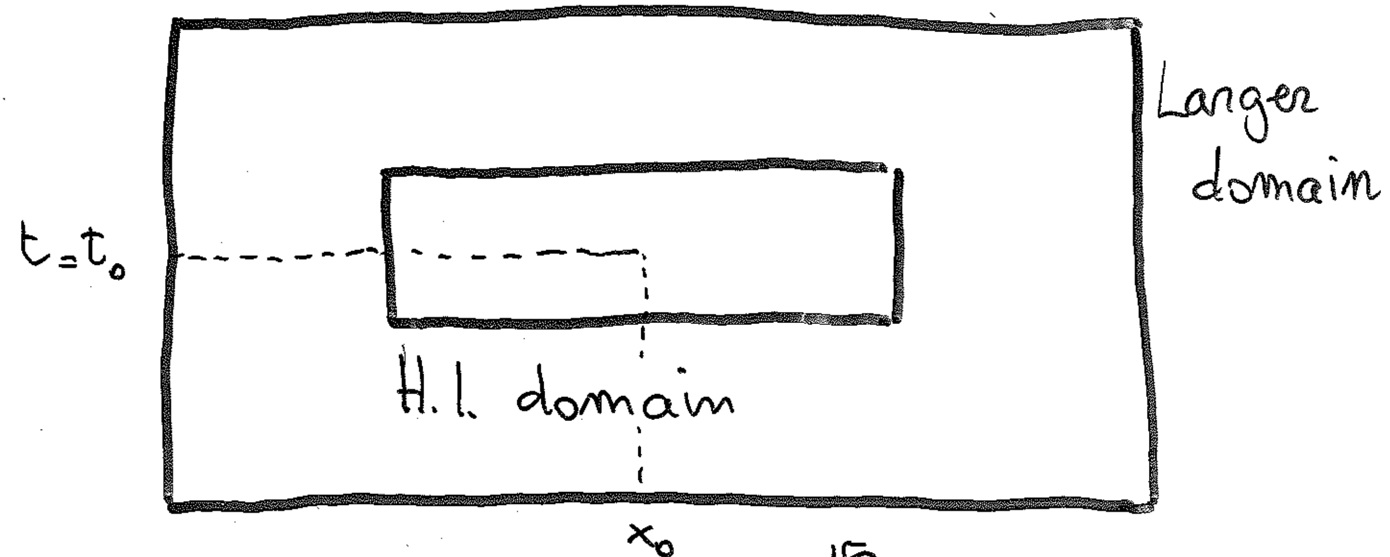
THEOREM Let u be a non-negative weak solution for $\frac{2N}{N+1}<p<2$. There exist positive constant δ and c, depending only upon the data, such that for all $(x_0,t_0)\in\Omega_T$ and for all cylinders

$$B_{8R}(x_0)\times\left\{t_0-\left[\frac{u(x_0,t_0)}{c^4}\right]^{2-p}(8R)^p,\right.$$
$$\left.t_0+\left[\frac{u(x_0,t_0)}{c^4}\right]^{2-p}(8R)^p\right\}\subseteq\Omega_T$$

we have

$$c\,u(x_0,t_0)\le\inf_{B_R(x_0)} u(\cdot,t)$$

$$\forall t\in\left[t_0-\delta[u(x_0,t_0)]^{2-p}R^p,\ t_0+\delta[u(x_0,t_0)]^{2-p}R^p\right]$$



The previous result is simultaneously a forward in time, elliptic, and backward in time Harnack inequality

FORWARD IN TIME

With the same assumptions as before on the reference domain

$$c\, u(x_o,t_o) \leq \inf_{B_R(x_o)} u(\cdot,\, t_o + \delta [u(x_o,t_o)]^{2-p} R^p)$$

* Exactly as in the prototype case
* c and δ can be stabilized as $p \to 2$, but they tend to zero as $p \to \frac{2N}{N+1}$.

ELLIPTIC

Same assumptions as before

$$c\, u(x_0, t_0) \leq \inf_{B_R(x_0)} u(\cdot, t_0)$$

* In this case c tends to zero as either $p \to 2$ or $p \to \frac{2N}{N+1}$.
* Diffusion dominates over time evolution
* Parabolic character is not lost

BACKWARD IN TIME

Same assumptions as before

$$c\, u(x_0, t_0) \leq \inf_{B_R(x_0)} u(\cdot, t_0 - \delta [u(x_0, t_0)]^{2-p} R^p)$$

* c and δ tend to zero as either $p \to 2$ or $p \to \frac{2N}{N+1}$
* Time is not reversed
* New and unexpected; previous example by Fabes-Garofalo-Salsa, but in a very different context

REMARKS

* Harnack implies Hölder
* Subpotentials lower bounds, as for $p>2$

$$\frac{u(x,t)}{u(x_0,t_0)} \geq \left[1 + \gamma(\text{data})\left(\frac{[u(x_0,t_0)]^{2-p}|x-x_0|^p}{t-t_0}\right)^{\frac{1}{p-1}}\right]^{\frac{p-1}{p-2}}$$

* We have upper bounds too
* Now the time decay is not optimal
* Once more, a connection between the Harnack inequality and potentials

FINAL QUESTION: What about $1 < p \leq \frac{2N}{N+1}$?

* Explicit examples rule out any of the previous forms
* Result by Bonforte and Vazquez
* A **local** statement?

Example 1

$$p = \frac{2N}{N+2} < \frac{2N}{N+1}$$

$$u(x,t) = (T-t)_+^{\frac{N+2}{4}} \left[a + b|x|^{\frac{2N}{N-2}} \right]^{-\frac{N}{2}}, \quad N > 2$$

$a > 0$, T arbitrary

$$b = b(N,a) = \frac{N-2}{N^2} \left(\frac{N+2}{4Na} \right)^{\frac{N+2}{N-2}}$$

Example 2

$$p = \frac{2N}{N+1}$$

$$u(x,t) = \left[|x|^{\frac{2N}{N-1}} + e^{bt} \right]^{-\frac{N-1}{2}}, \qquad N \geq 2$$

$$b = b(N) = \frac{2N}{N-1}^{\frac{2N}{N+1}}$$