Parabolic problems with dynamical boundary conditions

joint work with
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Abstract

An existence theory for local solutions of a parabolic problem under a dynamical boundary condition $\sigma u_t + u_n = 0$ is developed and a spectral representation formula is derived. It relies on the spectral theory of an associated elliptic problem with the eigenvalue parameter both in the equation and the boundary condition. The well-posedness of the parabolic problem holds in some natural space only if the number of negative eigenvalues is finite. This depends on the parameter σ in the boundary condition. If $\sigma \geq 0$ the parabolic problem is always well-posed. For $\sigma < 0$ it is well-posed only if the space dimension is 1 and ill-posed in space dimension ≥ 2. By means of the theory of compact operators the spectrum is analyzed and some qualitative properties of the eigenfunctions are derived. An interesting phenomenon is the "parameter-resonance", where for a specific parameter-value σ_0 two eigenvalues of the elliptic problem cross. Depending on the time some qualitative properties will be discussed.

References

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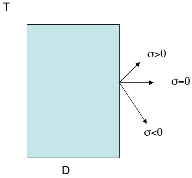
What is the problem?

$D \subset \mathbb{R}^N$ is a bounded Lipschitz domain with outer normal n, $q(x) \in L^\infty$ non-negative function, $\sigma(x) \in C^0(\partial D)$

$$u_t - \triangle u + qu = f(x,t) \quad \text{in} \quad D \times (0,T) \tag{1}$$
$$\sigma u_t + u_n = 0 \quad \text{on} \quad \partial D \times (0,T), \tag{2}$$
$$u(x,0) = u_0(x). \tag{3}$$



Separation of variable $f(x,t) = 0$

We seek for a solution of the form

$$u(x,t) = \phi(x)\alpha(t).$$

Then

$$\frac{\dot{\alpha}}{\alpha} - \frac{\Delta\phi}{\phi} + q = 0 \text{ in } D \times \mathbb{R}^+.$$

$\alpha(t) = e^{-\lambda t}$ and ϕ solves

$$\boxed{\Delta\phi - q\phi + \lambda\phi = 0 \text{ in } D, \quad \phi_n = \lambda\sigma\phi \text{ on } \partial D.}$$

Eigenvalue problem

$$\triangle\phi - q\phi + \lambda\phi = 0 \text{ in } D, \quad \phi_n = \lambda\sigma\phi \text{ on } \partial D.$$

An eigenfunction is a critical point of the Rayleigh quotient

$$R[v] := \frac{\int_D |\nabla v|^2\, dx + \int_D qv^2\, dx}{\int_D v^2\, dx + \oint_{\partial D} \sigma v^2\, ds} := \frac{< v, v >}{a(v, v)}.$$

Min-max principle

Assume $q \neq 0$ and $\sigma = \sigma^+ - \sigma^-$ whith $\sigma^- \neq 0$.

$$\lambda_1 = \inf_{W^{1,2}(D)} < v, v >, \quad a(v,v) = 1,$$

$$\lambda_{-1} = \sup_{W^{1,2}(D)} - < v, v >, \quad a(v,v) = -1.$$

$$\lambda_j = \inf_{W^{1,2}(D)} < v, v >, \quad a(v,v) = 1,\ a(\phi_i, v) = 0,\ i = 1, \ldots, j -$$

$$\lambda_{-j} = \sup_{W^{1,2}(D)} - < v, v >, \quad a(v,v) = -1,\ a(\phi_{-i}, v) = 0,\ i = -1, \ldots, -j +$$

Spectral theory

Theorem

(i) There exists a countable number of positive and negative eigenvalues.

$$\dots \lambda_{-n} \leq \dots \leq \lambda_{-2} < \lambda_{-1} < 0 < \lambda_1 < \lambda_2 \leq \dots \lambda_n \leq \dots$$

(ii) $\lambda_{\pm 1}$ *is simple,* $\phi_{\pm 1}$ *is of constant sign.*
(iii) $\lambda_n \to \infty$ *if* $n \to \infty$.
$(iv)_1$ *If* $N > 1$*, there exist infinitely many negative eigenvalues such that* $\lim_{n\to\infty} \lambda_{-n} = -\infty$.
$(iv)_2$ *If* $N = 1$ *and* $D = (0, L)$ *there exist exactly two negative eigenvalues provided* $\sigma(0)\sigma(L) > 0$*, otherwise there is exactly one negative eigenvalue.*

Proof

1. Apply the spectral theory for compact self-adjoint operators to $K : W^{1,2}(D) \to W^{1,2}$ where K is the solution operator of $\triangle v - qv + h = 0$ in D, $v_n = \sigma h$ on ∂D.
2. Show by means of the variational principle and a Harnack inequality that $\phi_{\pm 1} > 0$. Use the "Lagrange identity". Let $\phi > 0$ and ψ be two eigenfunctions corresponding to the same eigenvalue. Then

$$\int_D |\frac{\psi}{\phi}\nabla\phi - \nabla\psi|^2 \, dx = 0.$$

3. Construct suitable trial function.
4. Sturm's comparison theorem.

Representation formula

Theorem

$\{\phi_i\}_{i\in\mathbb{Z}}$ *is a complete orthonormal system such that* $<\phi_i,\phi_j>=\lambda_i a(\phi_i,\phi_j)=\delta_{ij}$, *in* $W^{1,2}(D)$.

$\Longrightarrow$ formal solution of the inhomogeneous parabolic problem

$$u(x,t)=\underbrace{\sum_{i\in\mathbb{Z}}<u_0,\phi_i>\phi_i(x)e^{-\lambda_i t}}_{\text{solution of homogeneous equation}}$$

$$+\quad\underbrace{\sum_{i\in\mathbb{Z}}\lambda_i\left\{\int_0^t e^{\lambda_i(\tau-t)}a(\phi_i,f)\,d\tau\right\}\phi_i(x)}_{\text{solution of the inhomogeneous problem with zero initial condition}}\quad.$$

Consequences

Corollary

If $\lambda_{-n} \to -\infty$ *then the parabolic problem has no weak solution* $u \in C([0,T]; W^{1,2}(D))$ *for arbitrary* $u_0 \in W^{1,2}(D)$ *and* $f \in W^{1,2}((0,T); L^2(D))$*.*

REMARK If $N > 1$ the parabolic problem is well-posed only if $\sigma \geq 0$ everywhere.

Corollary

If $N = 1$ *the problem is well-posed. It has a unique solution for all* t*.*

Blow up

Assume $\sigma \geq 0$, $u_0 \geq 0$, $u_0 \not\equiv 0$, $f(s), f'(s) > 0$ for $s > 0$ and

$$\int^{\infty} \frac{ds}{f(s)} < \infty.$$

Theorem

All solutions of

$$\begin{aligned} u_t - \triangle u &= f(u) \text{ in } D \times (0,T), \\ \sigma u_t + u_n &= 0 \text{ on } \partial D \times (0,T), \\ u(x,0) &= u_0(x) \end{aligned}$$

blow up in finite time.

Spectrum in the case $q \equiv 0$.

$$\triangle\phi + \lambda\phi = 0 \text{ in } D, \quad \phi_n = \lambda\sigma\phi \text{ on } \partial D.$$

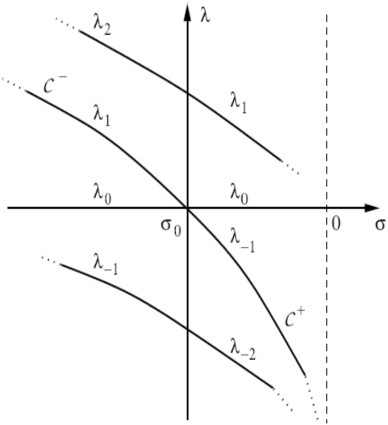
$\lambda_0 = 0$ is a *simple* eigenvalue with $\phi_0 =$const.

Does it correspond to $\lim_{q\to 0} \lambda_1(q)$ or to $\lim_{q\to 0} \lambda_{-1}(q)$?

This depends on the mean value $\boxed{\overline{\sigma} := \frac{1}{|\partial D|} \oint_{\partial D} \sigma \, ds.}$

The critical threshold of the mean is $\boxed{\sigma_0 = -\frac{|D|}{|\partial D|}.}$

Bifurcation



The asymptotic behavior of λ and ϕ on the branch C near σ_0 can be computed.

Theorem

Let $q \equiv 0$.

- *If $\overline{\sigma} < \sigma_0$ then λ_1 is simple, ϕ_1 is of constant sign and ϕ_{-1} changes sign.*
- *If $\overline{\sigma} > \sigma_0$ then λ_{-1} is simple, ϕ_{-1} is of constant sign and ϕ_1 changes sign.*
- *If $\overline{\sigma} = \sigma_0$ then ϕ_1 and ϕ_{-1} both changes sign.*

REMARK If $\overline{\sigma} = \sigma_0$ the eigenfunctions are not complete.

Heuristic explanation

$$\lambda_1 = \inf_{W^{1,2}(D)} < v, v >, \quad a(v,v) = 1, a(v,1) = 0$$

$$\lambda_{-1} = \sup_{W^{1,2}(D)} - < v, v >, \quad a(v,v) = -1, a(v,1) = 0.$$

Observe that if $a(1,1) < 0 \Longleftrightarrow \overline{\sigma} < \sigma_0$, then

$$a(v+c, v+c) = a(v,v) + 2ca(v,1) + c^2 a(1,1) \leq a(v,v) - \frac{a(v,1)^2}{a(1,1)}.$$

Equality if and only if $a(v+c,1) = 0$. Hence

$$\lambda_1 = \inf_{W^{1,2}(D)} < v, v >, \quad a(v,v) = 1.$$