THE CAUCHY PROBLEM FOR THIN FILM AND OTHER NONLINEAR PARABOLIC PDEs

Summer School on Nonlinear Parabolic Equations and Applications

Math. Dept., Swansea Univ., 7-11 July 2008

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July, 2008

Lecture 2: PLAN

The Bi-Harmonic Equation, the fourth-order parabolic equation:

$$u_t = -u_{xxxx} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+,$$

again, the Cauchy Problem.

Twenty-First Century Theory (2004).

SHARP Asymptotic Theory:

(i) as $t \to +\infty$, large-time behaviour, and

(ii) blow-up behaviour, as $t \to T^- < \infty$.

Hermitian Spectral Theory of **Non** Self-Adjoint Operators (2004).

The Cauchy problem for the bi-harmonic equation

In order to move ahead to higher-order diffusion-like equation, using the lines of our previous analysis, we consider the Cauchy Problem for the *bi-harmonic equation*. Then we will underline the main principal differences between second- and fourth-order linear parabolic PDEs:

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with given bounded integrable initial data $u_0(x)$.

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Models various higher-order diffusion phenomena, a well-known canonical PDE.

An Application in Hydrodynamics: Burnett Equations are Fourth-Order (a Non-Standard Fact)

Two Main Models of Hydrodynamics

As customary, higher-order viscosity terms occur via Grad's method in Chapman–Enskog expansions for hydrodynamics, where the viscosity part occurs as follows via "singular" expansion of the kernels of collision-like operators by using kernels with pointwise supports:

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} \equiv \mathbf{u}_t + (\mathbf{u} \cdot \nabla \mathbf{u})\mathbf{u} = -\nabla p + \sum_{n=0}^{\infty} \epsilon^{2n+1} \Delta^n (\mu_n \Delta \mathbf{u})$$
$$= \epsilon\left(\mu_0 \Delta \mathbf{u} + \epsilon^2 \mu_1 \Delta^2 \mathbf{u} + \dots\right),$$

where $\epsilon > 0$ is essentially the Knudsen number; $\mathbf{u}$ is the solenoidal (div-free) velocity field and p the pressure.

An Application in Hydrodynamics: Burnett Equations are Fourth-Order

Navier–Stokes Equations: $n = 0$

In a full model, truncating such series at $n = 0$ leads to the Navier-Stokes equations:

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla \mathbf{u})\mathbf{u} = -\nabla p + \epsilon \mu_0 \Delta \mathbf{u}, \quad \operatorname{div} \mathbf{u} = 0.$$

Global existence and Uniqueness of classical bounded solutions are unknown:
The **Millennium Problem** of the Clay Institute!
One of the most important for hydrodynamics and PDE theory of the XXI century... .

An Application in Hydrodynamics: Burnett Equations are Fourth-Order

Burnett Equations: $n = 1$

In a full model, truncating such series at $n = 1$ leads to the *Burnett equations*:

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla \mathbf{u})\mathbf{u} = -\nabla p - \hat{\mu}_2 \Delta^2 \mathbf{u}, \quad \operatorname{div} \mathbf{u} = 0.$$

Global Existence and Uniqueness of classical bounded solutions are also unknown....
(Not any Millennium Problem but seems to be much more difficult mathematically; a problem for the XXII century!? Or next Millennium?)

The Fundamental (Similarity) Solution

The Bi-Harmonic Equation

$$u_t = -u_{xxxx} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+,$$

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The Bi-Harmonic Equation

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The Fundamental Solution

$$b(x,t) = t^{-\frac{1}{4}} F(y), \quad y = \frac{x}{t^{1/4}}.$$

$$\mathbf{B}F \equiv -F^{(4)} + \tfrac{1}{4}(yF)' = 0, \quad \int_{\mathbb{R}} F = 1$$

$$\Longrightarrow \quad -F''' + \tfrac{1}{4} yF = 0.$$

Applying the Fourier transform yields

$$\mathcal{F}(b(\cdot,t))(\xi) = \mathrm{e}^{-\xi^4 t}, \quad \text{and} \quad \hat{F}(\omega) = \mathcal{F}(F(\cdot))(\omega) = \mathrm{e}^{-\omega^4}. \qquad (1)$$

The Fundamental Rescaled Kernel

Hence, F is given by:

$$F(y) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-s^4} (s|y|)^{\frac{1}{2}} J_{-\frac{1}{2}}(s|y|)\, ds.$$

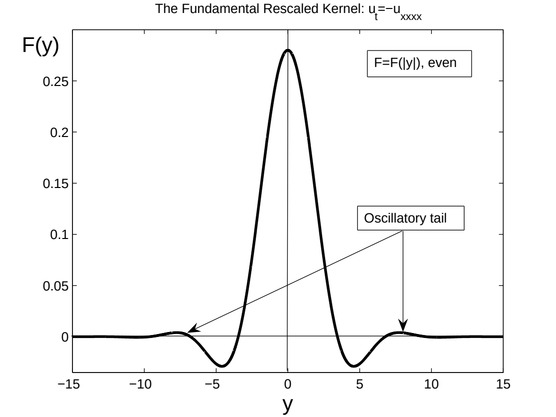
where $J_{-\frac{1}{2}}$ is Bessel's function:

$$J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cos z.$$

Oscillatory Behaviour of Changing Sign!

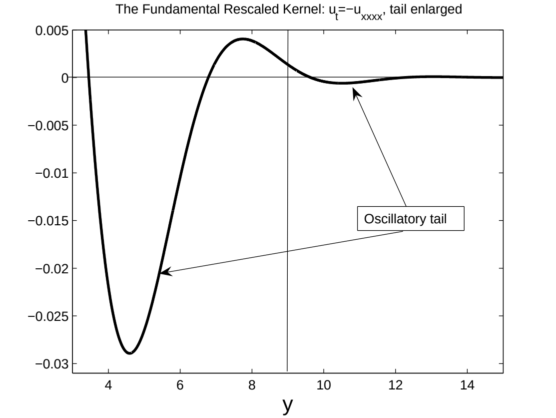
The Oscillatory Kernel for the Bi-Harmonic Equation

Rescaled Kernel of the Fundamental Solution to $u_t = -u_{xxxx}$



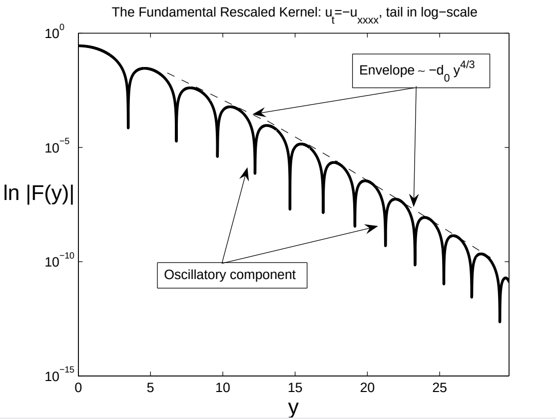
The Oscillatory Kernel for the Bi-Harmonic Equation

Rescaled Kernel for $u_t = -u_{xxxx}$: tail enlarged



The Oscillatory Kernel for the Bi-Harmonic Equation

Rescaled Kernel for $u_t = -u_{xxxx}$: tail in log-scale



Oscillatory Rescaled Kernel of Changing Sign

Consequences:

(i) No order-preserving properties of the bi-harmonic flow,
(ii) No comparison,
(iii) No Maximum Principle,
(iv) No Sturm zero set properties (No Sturm Theorems),...
...

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No Symmetry at All

(k) $\mathbf{B}$ IS NOT SELF-ADJOINT, no symmetry of the operator !

Sharp Asymptotics of the Oscillatory Rescaled Kernel

WKBJ Expansion (1920s)

The ODE is

$$\mathbf{B}F \equiv -F^{(4)} + \tfrac{1}{4}\,(yF)' = 0 \Longrightarrow F''' + \tfrac{1}{4}\,yF = 0.$$

Using standard classic WKBJ-type asymptotics (1920s!), substitute the function

$$F(y) \sim y^{-\delta_0}\,\mathrm{e}^{ay^{4/3}}, \quad y \to +\infty.$$

This gives the algebraic equation for a,

$$\left(\tfrac{4}{3}a\right)^3 = \tfrac{1}{4}, \quad \text{and} \quad \boxed{\delta_0 = \tfrac{1}{3} > 0\,.}$$

Sharp Asymptotics of the Oscillatory Rescaled Kernel

WKBJ Oscillatory Asymptotics

Thus:

$$a = \frac{3}{4^{4/3}} \left[\cos\left(\frac{2\pi}{3}\right) + \mathrm{i} \sin\left(\frac{2\pi}{3}\right) \right] \equiv -d_0 + \mathrm{i}\, b_0.$$

This gives the following double-scale asymptotics as $y \to +\infty$:

$$F(y) = y^{-\delta_0}\, \mathrm{e}^{-d_0 y^{4/3}} \left[C_1 \sin(b_0 y^{4/3}) + C_2 \cos(b_0 y^{4/3}) \right] + \dots ,$$

where $C_{1,2}$ are real constants, $|C_1| + |C_2| \neq 0$. Here

$$d_0 = 3 \cdot 2^{-\frac{11}{3}}, \quad b_0 = 3^{\frac{3}{2}} \cdot 2^{-\frac{11}{3}},\ \delta_0 = \frac{1}{3}.$$

By Convolution Theorem for Fourier Transforms

For bounded L^1 data, $\exists\,!$ solution

$$u(x,t) = b(t) * u_0 \equiv t^{-\frac{1}{4}} \int_{\mathbb{R}} F\left(\tfrac{x-z}{t^{1/4}}\right) u_0(z)\, \mathrm{d}z,$$

in the corresponding Tikhonov-like class of not more than exponentially growing initial data:

$$|u(x,t)| \le C \mathrm{e}^{c|x|^{4/3}}.$$

Precise Asymptotic Behaviour as $t \to +\infty$

Rescaled Variables

Equation:

$$u_t = -u_{xxxx}, \quad y \in \mathbb{R}, \quad t > 0.$$

$$u(x,t) = t^{-\frac{1}{4}}\, v(y,\tau), \quad y = \frac{x}{t^{1/4}}, \quad \tau = \ln t \gg 1.$$

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The Rescaled Equation

$$v_\tau = \mathbf{B}v \equiv -v_{yyyy} + \frac{1}{4}\, y v_y + \frac{1}{4}\, v, \quad y \in \mathbb{R}, \ \tau > 0.$$

$\mathbf{B}$ IS NOT Self-Adjoint Operator

From 1836 to the XXI century

One can see that $\mathbf{B}$ does not admit any symmetric form in any L^2_ρ-space for any $\rho > 0$ (easy negative calculus: too many conditions imposed to be symmetric for 4th-order operator)! We did not find any trace of such a $\mathbf{B}$-spectral theory in existing literature.

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Non-Self Adjoint Theory Developed in 2004

Egorov, Galaktionov, Kondratiev, and Pohozaev, Adv. Differ. Equat., **9** (2004), 1009–1038.

Expansion of the Semigroup

Using Convolution

$$u(x,t) = t^{-\frac{1}{4}} \int_{\mathbb{R}} F\left(\frac{x-z}{t^{1/4}}\right) u_0(\mathrm{z})\,\mathrm{dz}.$$

Hence, for the rescaled solution $v(y,\tau) = t^{1/4} u(x,t)$,

$$v(y,\tau) = \int_{\mathbb{R}} F(y - \mathrm{z}\mathrm{e}^{-\tau/4})\, u_0(\mathrm{z})\,\mathrm{dz}.$$

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Analytic Kernel Expansion

By Taylor's expansion

$$F(y - \mathrm{z}e^{-\tau/4}) = \sum_{(k)} \frac{1}{k!} F^{(k)}(y)(-1)^k \mathrm{z}^k \mathrm{e}^{-k\tau/4},$$

which converges uniformly on compact subsets (rather easy). We next substitute this into the semigroup expression:

Eigenfunction Expansion in L^2_ρ

Expansion

We have for the semigroup $\{\mathrm{e}^{\mathbf{B}\tau}\}_{\tau\geq 0}$

$$v(y,\tau) = \sum_{(k)} \mathrm{e}^{-k\tau/4} \frac{(-1)^k}{\sqrt{k!}} F^{(k)}(y) \, \frac{1}{\sqrt{k!}} \int_{\mathbb{R}} z^k u_0(z) \, \mathrm{d}z.$$

Here we see: REAL spectrum and both sets of eigenfunctions!

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Bi-Orthonormal Sets of Eigenfunctions

For $u_0 \in L^2_\rho$, this defines the eigenfunction expansion

$$v(y,\tau) = \sum_{(k)} e^{-k\tau/4} \psi_k(y) \langle \psi_k^*, u_0 \rangle, \quad \psi_k(y) = \frac{(-1)^k}{\sqrt{k!}} F^{(k)}(y),$$

and $\psi_k(y)$ MUST be polynomials, called the *generalized Hermite polynomials* (have nothing to do with any self-adjoint theory).

Domain of $\mathbf{B}$

Exponential Weight and Domain

$\mathbf{B}$ if defined in the weighted space $L^2_\rho(\mathbf{R})$ with the exponential weight

$$\rho(y) = \mathrm{e}^{a|y|^{4/3}} > 0, \quad a \in (0, 2d_0). \tag{2}$$

The domain is a Hilbert space of functions H^4_ρ with the inner product and the norm

$$\langle v, w \rangle_\rho = \int_{\mathbb{R}} \rho(y) \sum_{k=0}^{4} D^k v(y) \overline{D^k w(y)} \, \mathrm{d}y,$$

$$\|v\|^2_\rho = \int_{\mathbb{R}} \rho(y) \sum_{k=0}^{4} |D^k v(y)|^2 \, \mathrm{d}y.$$

Then $H^4_\rho \subset L^2_\rho \subset L^2$, and $\mathbf{B}$ is a bounded linear operator from H^4_ρ to L^2_ρ.

Discrete Real Spectrum of $\mathbf{B}$

Spectral Properties of $\mathbf{B}$ (Non-Self Adjoint)

Lemma

(i) *The spectrum of* $\mathbf{B}$ *comprises real simple eigenvalues only,*

$$\sigma(\mathbf{B}) = \left\{\lambda_k = -\tfrac{k}{4},\ k = 0, 1, 2, \ldots\right\}. \tag{3}$$

(ii) *The eigenfunctions* $\psi_k(y)$ *are given by*

$$\psi_k(y) = \frac{(-1)^k}{\sqrt{k!}} D^k F(y) \tag{4}$$

and form a complete subset in L^2 *and in* L^2_ρ*.*
(iii) *The resolvent* $(\mathbf{B} - \lambda I)^{-1} : L^2_{\rho^*} \to L^2_\rho$ *for* $\lambda \notin \sigma(\mathbf{B})$ *is a compact integral operator (*$\rho^* = 1/\rho$*, see below).*

Domain of the Adjoint Operator $\mathbf{B}^*$

Definition of $\mathbf{B}^*$ by Blow-up Scaling

$$u_t = -u_{xxxx}, \quad y \in \mathbb{R}, \quad -1 < t < 0; \quad \boxed{u(0,0) = 1}.$$

The adjoint operator $\mathbf{B}^*$ occurs after the blow-up (multiple-zero-like) scaling

$$u(x,t) = v(y,\tau), \quad y = \frac{x}{(-t)^{1/4}}, \quad \tau = -\ln(-t),$$

so that $v(y,\tau)$ solves the rescaled equation

$$v_\tau = \mathbf{B}^* v \equiv -v_{yyyy} - \tfrac{1}{4}\, y v_y.$$

Here $\mathbf{B}^*$ is formally adjoint to $\mathbf{B}$ in the standard (dual) L^2-metric.

Domain of the Adjoint Operator $\mathbf{B}^*$

Domain of $\mathbf{B}^*$

The weight is:

$$\rho^*(y) \equiv \frac{1}{\rho(y)} = \mathrm{e}^{-a|y|^{4/3}} > 0, \tag{5}$$

and we ascribe to $\mathbf{B}^*$ the domain $H^4_{\rho^*}$, which is dense in $L^2_{\rho^*}$.

Generalized Hermite Polynomials for $\mathbf{B}^*$

Discrete Spectrum

First, there holds

$$\langle \mathbf{B}v, w\rangle = \langle v, \mathbf{B}^* w\rangle \text{ for any } v \in H^4_\rho,\ \ w \in H^4_{\rho^*}. \tag{6}$$

Lemma

(i) $\sigma(\mathbf{B}) = \sigma(\mathbf{B}^*)$.
(ii) *The eigenfunctions* $\psi^*_k(y)$ *of* $\mathbf{B}^*$ *are polynomials:*

$$\psi^*_k(y) = \tfrac{1}{\sqrt{k!}} \sum_{j=0}^{[k/4]} \tfrac{1}{j!} D^{4j} y^k, \quad k = 0, 1, 2, \ldots, \tag{7}$$

and form a complete subset in $L^2_{\rho^*}$.
(iii) $\mathbf{B}^*$ *has compact resolvent* $(\mathbf{B}^* - \lambda I)^{-1}$ *in* $L^2_{\rho^*}$ *for* $\lambda \notin \sigma(\mathbf{B}^*)$.

Bi-Orthonormality of Φ and Φ^*

Corollary

With the given definitions of eigenfunctions, the orthonormality condition holds

$$\langle \psi_k, \psi_l^* \rangle = \delta_{kl} \quad \forall \; k, l \geq 0, \tag{8}$$

where δ_{kl} is the Kronecker delta.

Proof. Integrating by parts... . $\square$

Some Applications of Hermite Polynomials for $\mathbf{B}^*$

Unique Continuation Theorem

The fundamental uniqueness concepts of general PDE theory:

Holmgren (1901)–Carleman–Calderon–Pliss–Nirenberg–....

Since, by blow-up scaling the generalized Hermite polynomials describe ALL types of multiple-zero formation, we state the following unique continuation results:

Consider a solution $u(x,t)$ of the bi-harmonic equations defined in $\mathbb{R} \times (-1,1)$ such that, say,

$$u(0,0) = 0.$$

Some Applications of Hermite Polynomials for $\mathbf{B}^*$

Theorem 1: no infinite-order zeros

A traditional theorem (Carleman-type):

Theorem

If $(0,0)$ *is infinite-order zero of* $u(x,t)$ *(in any integral sense), then* $u(x,t) \equiv 0$*.*

Proof. A Hermite polynomial for $k = \infty$ does not exist... . $\square$

Some Applications of Hermite Polynomials for $\mathbf{B}^*$

Theorem 2: Hermitian Structure Involved

Theorem

If formation of the multiple zero at $x = 0$ as $t \to 0^-$ **DOES NOT** *asymptotically follow zero curves any of generalized Hermite polynomials $\psi_k^*(y)$, then $u(x,t) \equiv 0$.*

Proof. Ψ^* is complete.... $\square$

A new theorem: uses deep new results of Hermitian Spectral Theory developed: a complete knowledge of "micro-structure" of the PDE is available....

Hermitian Spectral Theory Were Absent for 170 Years!

Classic Theory: C. Sturm, 1836...

(i) The Heat Equation:

$$u_t = u_{xx};$$

(ii) the rescaled (blow-up) operator:

$$\mathbf{B}^* v = v'' - \tfrac{1}{2} y v',$$

(iii) $\sigma(\mathbf{B}^*) = \{-\frac{k}{2}\}$, Φ^* consists of Hermite classic polynomials... .

Hermitian Spectral Theory Were Absent for 170 Years!

Third-order Linear Dispersion Equation: Absent

(i) The LDE

$$u_t = u_{xxx};$$

(ii) the rescaled (blow-up) operator:

$$\mathbf{B}^* v = v''' - \tfrac{1}{3} y v',$$

(iii) $\sigma(\mathbf{B}^*) = \{-\frac{k}{3}\}$, Φ^* consists of generalized Hermite polynomials... .

Fernandes, Galaktionov (2009 ?).

Hermitian Spectral Theory for the 1D LSE!

1D Linear Schrödinger Equation: Absent

(i) The LSE, scattering theory, Quantum Mechanics...

$$\mathrm{i}\, u_t = u_{xx}; \quad \text{Hamiltonian:} \quad \int |u(x,t)|^2 \mathrm{d}x = \text{const.};$$

E. Schrödinger (1926), the most citable PDE EVER!

(ii) the rescaled (blow-up) operator:

$$\mathbf{B}^* v = v'' - \tfrac{\mathrm{i}}{2}\, y v';$$

(iii) $\sigma(\mathbf{B}^*) = \{-\frac{k}{2}\}$, Φ^* consists of generalized Hermite polynomials... .

Galaktionov, Kamotski (2008?).

Etc.