THE CAUCHY PROBLEM FOR THIN FILM AND OTHER NONLINEAR PARABOLIC PDEs

Summer School on Nonlinear Parabolic Equations and Applications

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Lecture 3: PLAN

The Fourth-Order Porous Medium Equation (the PME-4)

$$u_t = -(|u|^n u)_{xxxx} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+,$$

with given bounded integrable initial data $u_0(x)$,

where $n > 0$ is a fixed constant.
(i) Existence-Uniqueness Theory in Sobolev Spaces (1960s);
(ii) Nonlinear Eigenfunction Theory, Behaviour as $t \to +\infty$;
(iii) Homotopy Approach,

$$\boxed{n \to 0^+} \Longrightarrow \text{convergence to } u_t = -u_{xxxx}. \tag{1}$$

(iv) Numerical Evidences by MatLab, as Unavoidable Tools of PDE Theory of the XXI Century... .

Lecture 3: The PME–4

The Cauchy problem (CP) for the PME–4

We first consider a **quasilinear equation**, with the crucial exponent

$$n > 0.$$

The setting of the CP is standard:

$$u_t = -(|u|^n u)_{xxxx} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+,$$

We are looking for *COMPACTLY SUPPORTED solutions*.
$|u|^n$ is essential: the solutions are *oscillatory* near finite interfaces (cf. the oscillatory eigenfunctions $\psi_k(y)$!)

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Models various higher-order NONLINEAR diffusion phenomena, many applications... .

Existence-Uniqueness Theory: Standard, Fortunately

Weak Solution

Since the PDE is in divergent form in both t and x, this naturally defines solutions in the weak sense, where all the derivatives are distributions: the equation is understood in the distribution sense:

$$-\iint u\,\chi_t = -\iint (|u|^n u)\,\chi_{xxxx} \;\; \forall\, \chi \in C_0^\infty,$$

where u and $|u|^n u$ are assumed to be in L^2, and initial data are satisfied in L^1- or L^2-sense,

$$\|u(\cdot,t) - u_0(\cdot)\|_{L^1(L^2)} \to 0, \quad t \to 0,$$

or in a weaker topology if necessary (in fact, L^2 is fine).

Existence by Galerkin Approximation

Existence: Bubnov–Galerkin Method

Fixing a large bounded interval $I_L = (-L, L)$ such that $u(x,t)$ is supposed to be supported in I_L for some $t \in (0,T)$, a solution is obtained by a finite-dimensional approximation:

$$u = \lim_{m\to\infty} u_m, \quad \text{where}$$

$$(|u_m|^n u_m)(x,t) = \sum_{k=1}^{m} c_k(t) V_k(x)$$

where $\{V_k\}$ are eigenfunctions of $-D^4 < 0$ in I_L with the Dirichlet conditions:

$$-V^{(4)} = \mu_k V, \quad V = V' = 0 \text{ at } x = \pm L.$$

Existence by Galerkin Approximation

Existence: Bubnov–Galerkin Method

A priori bounds for $\{u_m\}$ are obtained by multiplication by $|u|^n u$ in L^2:

$$\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^{n+1} = -\int \left[(|u|^n u)_{xx}\right]^2 \leq 0,$$

and also by $(|u|^n u)_t$:

$$\frac{4(n+1)}{(n+2)^2}\int \left[(|u|^{\frac{n}{2}} u)_t\right]^2 = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \left[(|u|u)_{xx}\right]^2 \leq 0.$$

This implies strong *a priori* bounds to pass to the limit $m \to \infty$ by compact embedding of Sobolev spaces involved.

Uniqueness by Monotonicity

Monotonicity in H^{-2}

$$\text{The Operator} \quad \mathbf{A}(u) = -(|u|^n u)_{xxxx}$$

is monotone in the metric of H^{-2} (negative Sobolev space): for any $u,\ v \in C_0^\infty$,

$$\begin{aligned}\langle \mathbf{A}(u) - \mathbf{A}(v), u - v\rangle_{H^{-2}} &\equiv \int (\mathbf{A}(u) - \mathbf{A}(v))(D^4)^{-1}(u - v) \\ &= -\int (|u|^n u - |v|^n v)(u - v) \le 0\end{aligned}$$

(extension to weak solutions by closure...).
This implies uniqueness by classic theory of monotone operators: let there exist two solutions $u(x,t)$ and $v(x,t)$ for the same data u_0, then by the above monotonicity:

$$\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\|u(t) - v(t)\|_{H^{-2}} = -\int (|u|^n u - |v|^n v)(u - v) \le 0$$

Uniqueness by Monotonicity

Monotonicity in H^{-2}

Therefore:

$$\Longrightarrow \|u(t) - v(t)\|_{H^{-2}} \equiv 0 \Longrightarrow u(t) \equiv v(t).$$

Thus, as usual, we arrive at the next PROBLEM: describing the actual evolution properties of solutions.

Similarity Solutions: Nonlinear Eigenfunctions

Similarity Solutions

By scaling invariance, the PME–4 formally possesses the following similarity solutions:

$$u_S(x,t) = t^{-\alpha} f(y), \quad y = \frac{x}{t^\beta}, \quad \beta = \frac{1-\alpha n}{4}, \tag{2}$$

where $\alpha > 0$ is a parameter (a nonlinear eigenvalue).

Similarity Solutions: Nonlinear Eigenfunctions

Kernel Equation

The similarity kernel $f(y) \in C_0$, $f \neq 0$ (**compactly supported!**) satisfies the *nonlinear eigenvalue problem*

$$\mathbf{B}_n(f) \equiv -(|f|^n f)^{(4)} + \tfrac{1-\alpha n}{4}\, y f' + \alpha f = 0 \quad \text{in} \quad \mathbb{R}. \tag{3}$$

Collecting all terms with the eigenvalue α on the right-hand side yields

$$\begin{aligned} \mathbf{B}_n^1(f) &\equiv -(|f|^n f)^{(4)} + \tfrac{1}{4}\, y f' \\ &= \alpha\left(\tfrac{n}{4}\, y f' - f\right) \equiv \alpha \mathcal{L}_n f. \end{aligned} \tag{4}$$

An eigenvalue problem for a *linear pencil* of two, nonlinear $\mathbf{B}_n^1$ and linear $\mathcal{L}_n$, ordinary differential operators.

Nonlinear Eigenfunction Setting

Even and Odd Eigenfunctions

The ODE for f is invariant under the group of scaling transformations

$$f \mapsto \epsilon^{\frac{4}{n}} f, \quad y \mapsto \epsilon y \quad (\epsilon > 0), \tag{5}$$

so that, for a unique representation of necessary solutions, one needs an additional normalization.
For solutions $f_l(y)$ with even $l = 0, 2, ...$, the following normalization and the symmetry conditions at the origin $y = 0$:

$$f(0) = 1, \quad \text{and} \quad f'(0) = 0, \;\; f'''(0) = 0, \;\; \text{and} \tag{6}$$

$$f'(0) = 1, \quad \text{and} \quad f(0) = 0, \;\; f''(0) = 0, \;\; l = 1, 3, 5, \ldots. \tag{7}$$

Nonlinear Eigenfunction Setting

A General Approach Needed

For the above degenerate nonlinear fourth-order operator, any simple geometric approach is not possible (no phase-plane analysis!), and the shooting problem is always at least 3D.

Local Oscillatory Behaviour at Interfaces

Asymptotic Semilinear ODE

Finite propagation for the PME–4 can be proved by energy methods (developed in the lines of Saint–Venant's Principle from solid mechanics, mid of XIX century).
Let $y_0 > 0$ be the right-hand interface of a solution $f(y)$. Then from the ODE

$$-(|f|^n f)^{(4)} + \beta y f' + \alpha f = 0$$

on integration once and neglecting some terms, making for convenience the **reflection** $y \mapsto y_0 - y$, with $y > 0$ small enough, for small $y > 0$, we have

$$(|f|^n f)''' = -\beta y_0 f + ... \quad (\beta > 0).$$

Local Oscillatory Behaviour at Interfaces

Asymptotic Semilinear ODE

We scale out the positive constant βy_0 to get the ODE

$$(|f|^n f)''' = -f \quad \text{for} \quad y > 0, \quad f(0) = 0. \tag{8}$$

Next, it is convenient to use the natural change

$$F = |f|^n f \quad \Longrightarrow \quad \boxed{F''' = -\left|F\right|^{-\frac{n}{n+1}} F.} \tag{9}$$

Local Oscillatory Behaviour at Interfaces

ODE for the Oscillatory Component

We need to describe oscillatory solution of changing sign of the ODE (9), with zeros concentrating at the given interface point $y = 0^+$.

We look for the solutions of the form

$$F(y) = y^{\mu}\varphi(s), \quad s = \ln y, \quad \mu = \tfrac{3(n+1)}{n} > 3, \tag{10}$$

where $\varphi(s)$ is called the *oscillatory component*.

Local Oscillatory Behaviour at Interfaces

ODE for the Oscillatory Component

Substituting yields

$$P_3(\varphi) = -|\varphi|^{-\frac{n}{n+1}}\varphi, \tag{11}$$

where P_k denote linear differential polynomials

$$\begin{aligned} P_1(\varphi) &= \varphi' + \mu\varphi, \\ P_2(\varphi) &= \varphi'' + (2\mu - 1)\varphi' + \mu(\mu - 1)\varphi, \\ P_3(\varphi) &= \varphi''' + 3(\mu - 1)\varphi'' \\ &+ (3\mu^2 - 6\mu + 2)\varphi' + \mu(\mu - 1)(\mu - 2)\varphi. \end{aligned}$$

Local Oscillatory Behaviour at Interfaces

Periodic Oscillatory Component

We are interested in uniformly bounded global solutions $\varphi(s)$ that are well defined as $s = \ln y \to -\infty$, i.e., as $y \to 0^+$. The best candidates for such global orbits are periodic solutions $\varphi_*(s)$:

Lemma

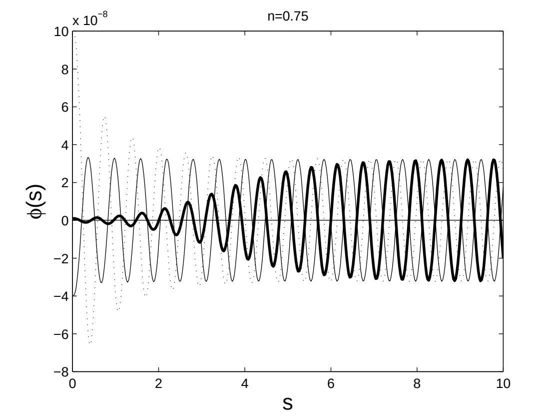
For any $n > 0$, (11) has a periodic solution $\varphi_*(s)$ of changing sign.

Proof. By 2D shooting... . □

Galaktionov, Adv. Differ. Equat. (2008).

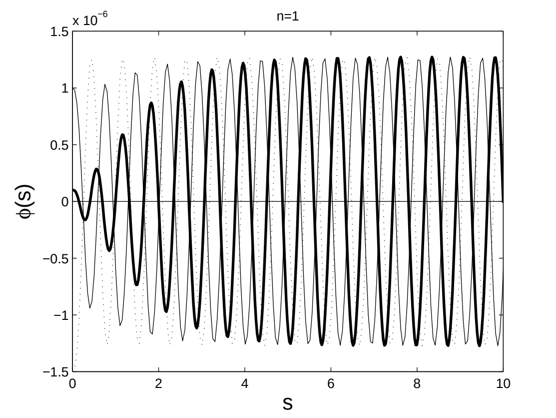
Local Oscillatory Behaviour at Interfaces: $n = 0.75$

Periodic Oscillatory Component for $n = 0.75$



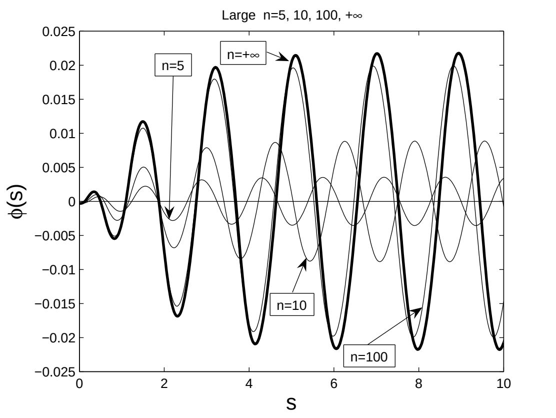
Local Oscillatory Behaviour at Interfaces: $n = 1$

Periodic Oscillatory Component for $n = 1$



Local Oscillatory Behaviour at Interfaces: $n \gg 1$

Periodic Oscillatory Component for $n \gg 1$



First four n-branches: explicit eigenvalues

Moments Conservation

We use the following conservation laws reflecting highly divergent structure of the operator: for $u_0 \in C_0(\mathbb{R})$ and $l = 0, 1, 2, 3$,

$$\begin{aligned} &\tfrac{\mathrm{d}}{\mathrm{d}t} \int x^l u(x,t)\,\mathrm{d}x = 0 \\ \Longrightarrow &\int x^l u(x,t)\,\mathrm{d}x = \int x^l u_0(x)\,\mathrm{d}x \quad \text{for} \quad t \geq 0. \end{aligned} \tag{12}$$

For the similarity solutions u_S, this yields

$$\int x^l u_S(x,t)\,\mathrm{d}x = t^{-\alpha+(l+1)\beta} \int y^l f(y)\,\mathrm{d}y, \quad \text{so:} \tag{13}$$

$$\begin{aligned} &-\alpha + (l+1)\,\tfrac{1-\alpha n}{4} = 0 \quad \Longrightarrow \\ &\boxed{\alpha_l(n) = \tfrac{l+1}{4+(l+1)n}} \quad \text{for} \;\; l = 0, 1, 2, 3. \end{aligned} \tag{14}$$

First four n-branches: explicit eigenvalues

Bernis–McLeod: $l = 0, 1, 2, 3$ (1991)

The corresponding nonlinear eigenfunctions $f_l(y)$ of (3) were constructed in

Bernis–McLeod, Nonl. Anal., TMA, **17** (1991), 1039–1068.

The proof of existence and uniqueness is not easy at all! There is still no any proof for f_4 and others!

Very difficult and advanced mathematics!

For $l \geq 4$, the ODE is true **FOURTH***-order and the known techniques fail*.

Numerical Construction of Nonlinear Eigenfunctions

MatLab: Reliable Evidence with Tols up to 10^{-13}

The *nonlinear eigenvalue problem*:

$$F = |f|^n f \implies$$
$$-F^{(4)} + \beta(1-\mu)|F|^{-\mu}F'y + \alpha|F|^{-\mu}F = 0, \ \ \mu = \tfrac{n}{n+1}.$$

We next present numerical results concerning existence and multiplicity of solutions and stress some principal properties and difficulties.

In the next Figure constructed by `MatLab`, we show the first basic symmetric pattern that is again called the $F_0(y)$ for $n = 0$, 0.5, 1, and 2. The negative $n = -\frac{1}{2}$ is included (for further thinking: **FAST** diffusion).

First Nonlinear Eigenfunction

$F_0(y)$

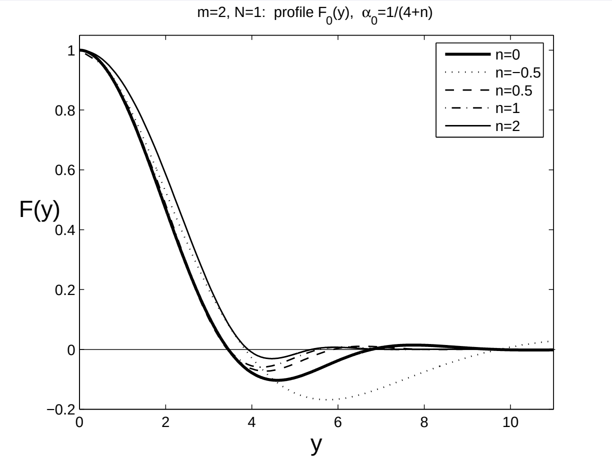


Figure: The first solution $F_0(y)$ for various n.

1. Further Nonlinear Eigenfunctions

$F_l(y)$

In the next Figure, we show next four nonlinear eigenfunctions from the family

$$\Phi = \{F_l,\ l = 0, 1, 2, ...\}$$

for the same values of n.

2. Further Nonlinear Eigenfunctions

$F_1(y)$, the odd dipole-like profile

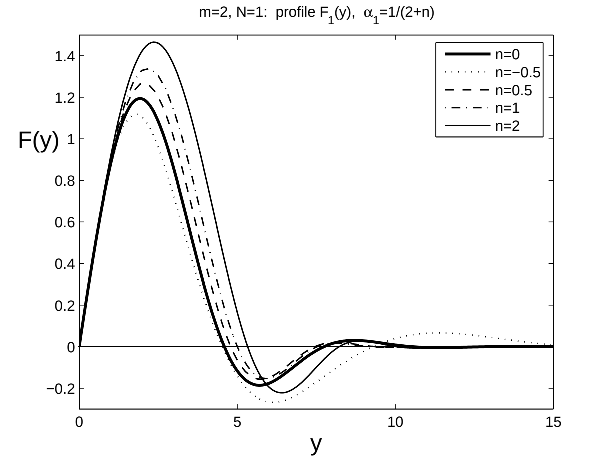


Figure: The dipole $F_1(y)$ for various n.

3. Further Nonlinear Eigenfunctions

$F_2(y)$, even

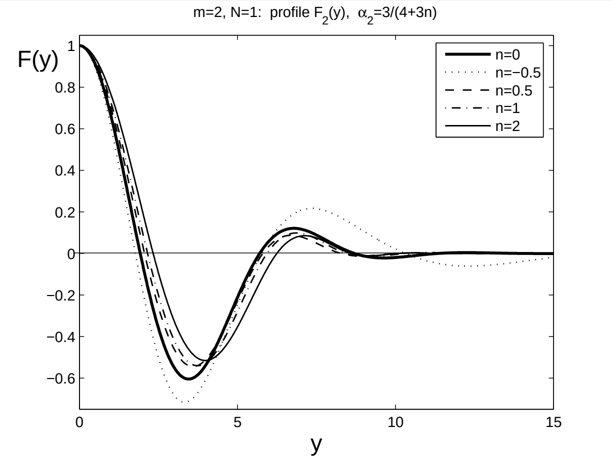


Figure: The third solution $F_2(y)$ for various n.

4. Further Nonlinear Eigenfunctions

$F_3(y)$, odd (second dipole)

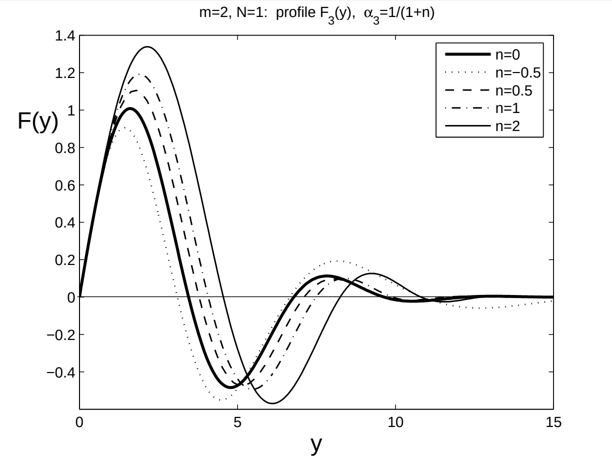


Figure: The odd profile $F_3(y)$ for various n.

5. Further Nonlinear Eigenfunctions

$F_4(y)$, even

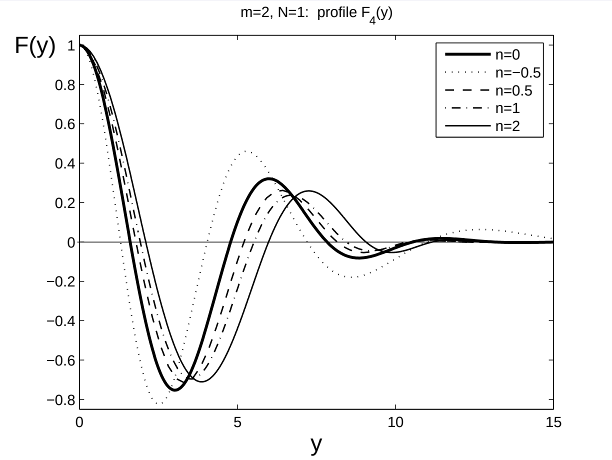


Figure: The even pattern $F_4(y)$ for various n.

$F_6(y)$, even

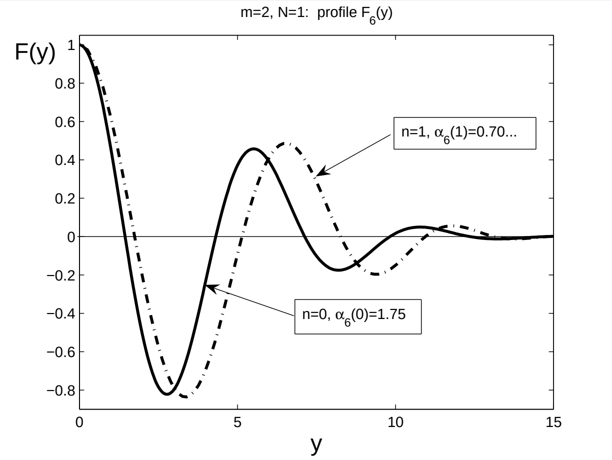


Figure: The even pattern $F_6(y)$ for various n.

7. Further Nonlinear Eigenfunctions

$F_{10}(y)$, even

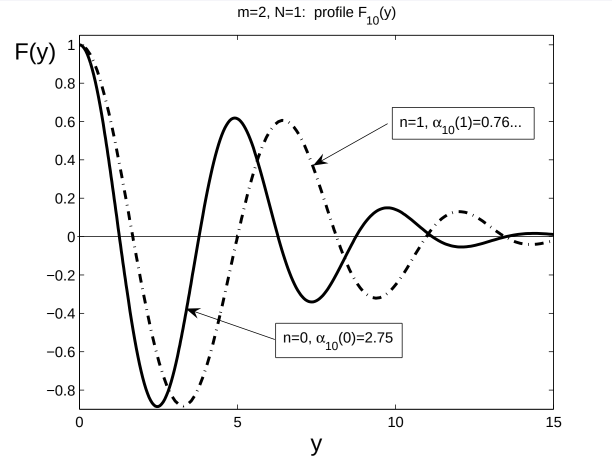


Figure: The even pattern $F_{10}(y)$ for various n.

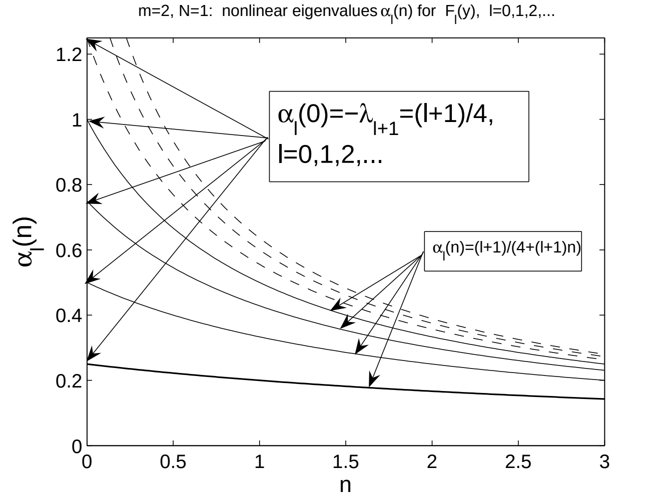
1. n-Branching of Nonlinear Eigenfunctions

n-Bifurcation Diagram

Here, we show first explicit n-branches of eigenfunctions; other branches are not explicit.
We next show how to estimate their behaviour via branching at the *branching point* $n = 0$ from eigenfunctions of the corresponding linear eigenvalue problem.

1. n-Branching of Nonlinear Eigenfunctions

n-Bifurcation Diagram



2. n-Branching of Nonlinear Eigenfunctions

Countable branching of eigenfunctions at $n = 0$

We study the behaviour of nonlinear eigenfunction curves appeared at the branching point $n = 0$ from *linear* eigenfunctions: looking forward to seeing the operator $\mathbf{B}$! The analysis is based on classic bifurcation-branching theory going back to Lyapunov and Schmidt (turn of the XXth century).

Classic Monographs by **Vainberg–Trenogin** (1974), **Krasnosel'skii–Zabreiko** (1984), **Deimling** (1985), etc.

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Equation for f

The similarity solutions kernels $f \in C_0$ solve:

$$\mathbf{B}_n(f) \equiv -(|f|^n f)^{(4)} + \tfrac{1-\alpha n}{4}\, y f' + \alpha f = 0. \qquad (15)$$

3. n-Branching of Nonlinear Eigenfunctions

Countable branching of eigenfunctions at $n = 0$

For $n > 0$ small enough, the nonlinear eigenvalue problem is a "perturbation" of the linear one for the linear operator $\mathbf{B} \equiv \mathbf{B}_0$! Indeed, since $\beta = \frac{1}{4} - \frac{\alpha}{4}n$, we write the equation as

$$\mathbf{B}f = g(f,n) \equiv [(|f|^n - 1)f]^{(4)} + n\tfrac{\alpha}{4}yf' - \big(\alpha - \tfrac{1}{4}\big)f. \tag{16}$$

Next, since $\mathbf{B}$ has compact resolvent in L^2_ρ, we form the strictly negative operator $\mathbf{B} - I$ and, instead of (16), consider the equivalent integral equation

$$f = \mathbf{A}(f,n) \equiv (\mathbf{B}_0 - I)^{-1}(g(f,n) - f); \tag{17}$$

the nonlinear operator being treated as compact in suitable metrics.

4. n-Branching of Nonlinear Eigenfunctions

FORMAL n-branching analysis in $\mathbb{R}^2$

There exist two parameters, n and α, so we deal with bifurcation (branching) problem for

$$\mu = (n, \alpha)^T \in \mathbb{R}^2. \tag{18}$$

the first n-branch is supposed to appear from the rescaled kernel F at the branching point

$$\mu_0 = \left(0, \tfrac{1}{4}\right)^T.$$

The branching equations are famous scalar Lyapunov–Schmidt ones:

4. n-Branching of Nonlinear Eigenfunctions

Asymptotic expansion of branches for small $n > 0$

Branching is possible under the following non-trivial kernel assumption: for $n = 0$,

$$\begin{gathered} \alpha - \tfrac{1}{4} = -\lambda_l = \tfrac{l}{4} \quad \Longrightarrow \\ \alpha_l(0) = \tfrac{1+l}{4}, \quad l \geq 0. \end{gathered} \tag{19}$$

This gives an approximation of the countable sequence of critical exponents $\{\alpha_l(n), \beta_l(n)\}$ (to be determined).

5. n-Branching of Nonlinear Eigenfunctions

FORMAL n-branching analysis in $\mathbb{R}^2$

To this end, we use in (16) the expansion

$$|f|^n f = f + nf \ln|f| + o(n) \quad \text{as} \quad n \to 0^+; \tag{20}$$

true uniformly on bounded intervals in f, and in the weak sense... .
Substituting all expansions into the equation yields, still formally:

$$\mathbf{B}_n(f) \equiv \mathbf{B}f + \left(\alpha - \tfrac{1}{4}\right)f + n\mathcal{L}(f) + o(n) = 0, \tag{21}$$

with the perturbation operator

$$\mathcal{L}(f) = -\left[(f \ln|f|)^{(4)} + \tfrac{\alpha}{4}\, yf'\right]. \tag{22}$$

6. n-Branching of Nonlinear Eigenfunctions

FORMAL n-branching analysis in $\mathbb{R}^2$

We apply the classical Lyapunov-Schmidt method to the above equation. In this linearized setting, we naturally arrive at the functional framework that is suitable for the linear operator $\mathbf{B}$, i.e., it is L^2_ρ, with the domain H^4_ρ, etc., and a similar setting for the adjoint operator $\mathbf{B}^*$.

7. n-Branching of Nonlinear Eigenfunctions

Asymptotics as $n \to 0$

We perform linearization about f being a certain linear eigenfunction ψ_l of $\mathbf{B}$, with the eigenvalue $\lambda_l = -\frac{l}{4}$. $\psi_l(y) \sim D^l F(y)$, so the nodal (zero) set of $f(y)$ is well understood: consists of isolated points (not easy to prove!) concentrated as $y \to \infty$, where

$$\psi_l(y) \to 0 \quad \text{as} \quad y \to \infty \text{ uniformly and exponentially fast.} \tag{23}$$

All zeros are transversal (in the usual sense) a.e., which is necessary for checking the key hypothesis on the nonlinearity:

$$\mathcal{L}(\psi_l) \in L^2_\rho. \tag{24}$$

8. n-Branching of Nonlinear Eigenfunctions

Branching Formalities

By Spectral Theory from Lecture 2, the kernel of the linearized operator

$$E_0 = \ker\left(\mathbf{B} - \lambda_l I\right)$$

is always 1D! (Simple eigenvalues simplify).
Hence denoting by E_1 the complementary (orthogonal to E_0) invariant subspace, we set

$$f = \phi_l + V_1, \quad \phi_l \in E_0, \quad V_1 = \textstyle\sum_{k>l} c_k \psi_k \in E_1. \tag{25}$$

According to classic theory, we set

$$V_1 = nY + o(n) \ \ (Y \perp \psi_l),$$
$$\alpha_l(n) = \tfrac{l+1}{4} + c_l n + o(n).$$

9. n-Branching of Nonlinear Eigenfunctions

Branching Equationd for any $l \geq 0$

Then in the $O(n)$-approximation:

$$(\mathbf{B} + \tfrac{l}{4})Y + c_l\psi_l + \mathcal{L}(\psi_l) = 0. \qquad (26)$$

Multiplying by ψ_l^* yields the scalar equation:

$$c_l = -\langle \mathcal{L}(\psi_l), \psi_l^* \rangle,$$

and then Y (not from the kernel) is uniquely determined from the inhomogeneous equation (26).

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Final Branching Conclusion

Thus, under the fixed hypothesis, for small $n > 0$, there exists a *countable set of nonlinear eigenfunctions*.

10. n-Branching of Nonlinear Eigenfunctions: Open Problems

Open Problem 1

Global continuation of branches for larger n is unknown (numerics confirm their global existence).

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Open Problem 1

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Open Problem 2

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Open Problem 3

Nonlinear eigenfunctions of the operator $\mathbf{B}_n^*$ for zero set blow-up analysis, and convergence to Hermite polynomials as $n \to 0^+$.