THE CAUCHY PROBLEM FOR THIN FILM AND OTHER NONLINEAR PARABOLIC PDEs

Summer School on Nonlinear Parabolic Equations and Applications

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Lectures 4 and 5: PLAN

The Fourth-Order Thin Film Equation (the TFE-4)

$$u_t = -(|u|^n u_{xxx})_x \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+;$$

$n > 0$; **compactly supported** solutions.
(i) Self-Similar Solutions, Oscillatory Sign-Changing Behaviour, Nonlinear Eigenfunction Theory,
(ii) Finite Interfaces, Homoclinic Bifurcation Parameter

$$\boxed{n_h = 1.758665...\,.}$$

(iii) Existence-Uniqueness Concepts (no Proof still!) by a Homotopy Approach,

$$\boxed{n \to 0^+} \Longrightarrow \text{convergence to the bi-harm. eq.} \tag{1}$$

(iv) ALL Supported by Numerical Evidences by MatLab... .

The Cauchy problem (CP) for the TFE–4

The setting of the CP is standard:

$$u_t = -(|u|^n u_{xxx})_x \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+,$$

We are looking for **compactly supported** solutions.

Lectures 4-5: The TFE–4

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A Classification

The distribution of the derivatives in $(3, 1)$: 3 inside and 1 outside; the TFE–(3,1) = TFE–4, the canonical one.

Other TFE-like Equations

Back to PME–4

Then the *fully divergent* PME–4 is

$$u_1 = -(|u|^n u)_{xxxx} \Longrightarrow \text{the TFE–(0,4).}$$

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TFE–(2,2)

Another non-fully divergent PDE:

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The Cauchy problem (CP) for the TFE–4

Again: $|u|^n$ is essential: the solutions are *oscillatory* near finite interfaces (cf. the oscillatory eigenfunctions $\psi_k(y)$!)

The TFE–4: Derivation

Application and Derivation

Models various NONLINEAR thin-film phenomena... .

Example: A Hele–Shaw flow between two parallel plates,

$$\begin{cases} \text{conservation of mass:} \quad u_t + (uv)_x = 0, \quad \text{and} \\ \text{Darcy's law:} \quad v = -\frac{h_0^2}{12\mu} p_x, \end{cases} \tag{2}$$

v is the average velocity of the fluid in the film, μ is the fluid viscosity, and p is the pressure. Here: $\boxed{p = -\gamma\kappa \equiv -\gamma u_{xx},}$ where γ is the surface tension and κ is the curvature of the surface. Substituting p_x into the second equation in (2) and the resulting v into the first equation yields the TFE–4 with $n = 1$:

$$u_t + \frac{\gamma h_0^2}{12\mu}(uu_{xxx})_x = 0.$$

The TFE–4: Application

Main Applications

- spreading of thin Newtonian liquid drops,
- $n = 1$: flows in a PM, Hele–Shaw cell,
- $n = 2$: Reynolds' equation for Stokes flow,

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The case $n = 2$ represents Navier-slip-dominated Stokes flows of thin film. Then the analog is

$$(h^2)_y = -2\left(h^2 h_{xxx}\right)_x.$$

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Physical Experiments

Formation of travelling waves in thin films:

P. Kapitza (Noble Prize, 1979) and **S. Kapitza**, 1949.

Source-type Solution: First Nonlinear Eigenfunction

First Similarity Pattern: Existence and Uniqueness

The source-type solution for the TFE–4

$$u_*(x,t) = t^{-\frac{1}{n+4}} f(y), \quad y = x/t^{\frac{1}{n+4}}, \tag{3}$$

$$-(|f|^n f''')' + \tfrac{1}{n+4}(yf)' = 0 \Longrightarrow |f|^n f''' = \tfrac{1}{n+4} fy, \quad \boxed{f''' = \tfrac{1}{n+4} |f|^{-n} fy.}$$

Source-type Solution: First Nonlinear Eigenfunction

Comparison with the PME-4

$$u_t = -(|u|^{\hat{n}} u)_{xxxx}, \quad u_*(x,t) = t^{-\frac{1}{\hat{n}+4}} f(y), \quad y = x/t^{\frac{1}{\hat{n}+4}}, \qquad (4)$$

$$-(|f|^{\hat{n}} f)^{(4)} + \frac{1}{\hat{n}+4} (yf)' = 0 \Longrightarrow (|f|^{\hat{n}} f)''' = \frac{1}{\hat{n}+4} fy.$$

Finally,

$$\hat{f} = |f|^{\hat{n}} f \Longrightarrow \boxed{\hat{f}''' = \frac{1}{\hat{n}+4} |\hat{f}|^{-\frac{\hat{n}}{\hat{n}+1}} fy.}$$

Source-type Solution: First Nonlinear Eigenfunction

First Similarity Pattern: TFE=PME for $n \in (0, 1)$

Comparing the boxed equation shows that these coincide (up to easy scaling) if

$$n = \frac{\hat{n}}{\hat{n}+1}, \quad \text{i.e., for any } \; n \in (0, 1).$$

Hence they have the same unique (up to scaling) similarity profiles that are *oscillatory near interfaces*!

This means a certain **universality** of formation of evolution patterns for rather different PDE models: the TFE–4 and the PME–4.

Source-type Solution: First Nonlinear Eigenfunction

Warning 1: $n \geq 1$?

For $n \geq 1$, the ODEs for the TFE–4 and the PME–4 are **completely different**!

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Warning 2: other nonlinear eigenfunctions?

Even for $n \in (0, 1)$, other similarity patterns $\{F_l\}$ (nonlinear eigenfunctions) for the TFE–4 and the PME–4 are also **completely different**!

Very difficult to study...

First Nonlinear Eigenfunction

$F_0(y)$

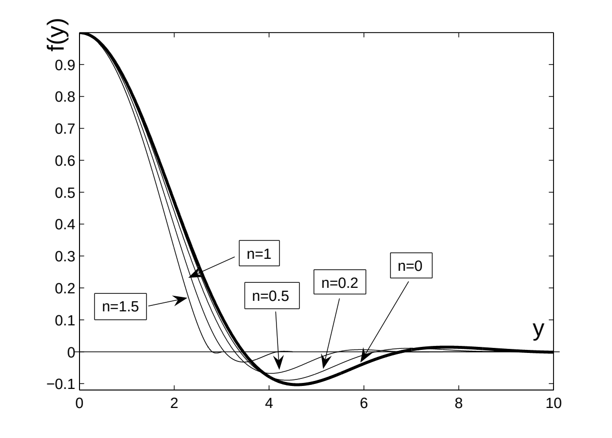

Figure: The first pattern $f_0(y)$ for various n.

Local Oscillatory Behaviour at Interfaces

ODE for the Oscillatory Component

For $n \in (0,1)$ is similar to the PME–4, but for $n \geq 1$ is **different**!

Let $y_0 > 0$ be the right-hand interface of a solution $f(y)$. Then, for $y \approx y_0^-$ ($\frac{y_0}{n+4}$ scaled out),

$$f''' = |f|^{-n} f + ... \implies$$

$$f(y) = (y_0 - y)^{\gamma} \varphi(s), \quad s = \ln(y_0 - y), \gamma = \frac{3}{n} \implies$$

$$P_3[\varphi] = |\varphi|^{-n} \varphi, \quad \text{where} \tag{5}$$

$$P_3[\varphi] = \varphi''' + 3(\gamma - 1)\varphi'' + (3\gamma^2 - 6\gamma + 2)\varphi' + \gamma(\gamma - 1)(\gamma - 2)\varphi,$$

Local Oscillatory Behaviour at Interfaces

Periodic Oscillatory Component

Existence of a periodic oscillatory component for $n \in [1, \frac{3}{2} + \epsilon)$, $\forall\, \epsilon > 0$ is rather easy, but uniqueness is still open.

For which $n \geq \frac{3}{2}$ a periodic connection exists ?

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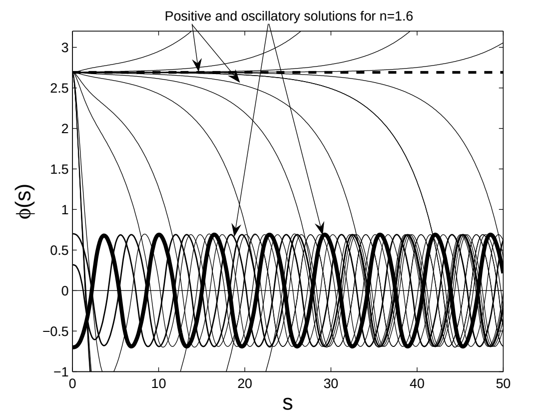
Two Types of Solutions: Positive and Oscillatory

Positive are known from the 1970s: Greenspan (1978); Smyth and Smyth (1988);... Bernis, Peletier, and Williams (1992),... .
Oscillatory: in the XXI century... .

Which ones do correspond to the CP?

Local Oscillatory Behaviour at Interfaces

Numerics: Periodic Oscillatory Component for $n = 1.6$



Heteroclinic Bifurcation: Destruction of Periodic Oscillations

Main Conjecture

Conjecture 1. A stable periodic solution $\varphi_*(s)$ exists for all $n \in (0, n_\mathrm{h})$, where $n_\mathrm{h} \in (\frac{3}{2}, 2)$ is a subcritical heteroclinic bifurcation point of equilibria.

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$$\boxed{n_h = 1.758665\ldots.} \tag{6}$$

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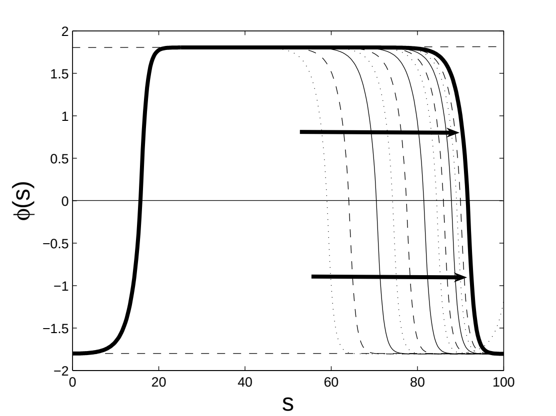
Best Analytical Estimate:

$$n_{\rm h} < n_* = \frac{9}{3+\sqrt{3}} = 1.901923\dots.$$

For extra details: Evans, Galaktionov, and King, Euro J. Appl. Math., **18** (2007), 273–321.

Formation of a heteroclinic connection in as $n \longrightarrow n_{\mathrm{h}}^{-}$

Standard Heteroclinic Bifurcation (Non-Local!)



On "Homotopy" ODE Approach: $n \longrightarrow 0$

Continuous connection with $\mathbf{B}$ and ψ_0

For small $n > 0$, the ODE from THE theory is

$$|f|^n f''' = \boxed{\tfrac{1}{4}}\, yf, \quad \text{where } n \to 0^+. \tag{7}$$

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WKBJ Asymptotics

Then the WKBJ concepts suggest a double-scale expansion

$$y = n^{-\frac{3}{4}} Y, \quad f = n^{\frac{1}{2}\left(N - \frac{1}{p-1}\right)} \mathrm{e}^{-\frac{1}{n}\phi_0(Y)} + \ldots,$$

where we use a complex representation:

$$\phi_0(Y) = u(Y) + \mathrm{i}v(Y) \Longrightarrow |f|^n \sim \mathrm{e}^{-u(Y)}. \tag{8}$$

On "Homotopy" ODE Approach: $n \to 0$

Asymptotic Calculus

By differentiating and keeping the leading term:

$$f' = \mathrm{e}^{-\frac{1}{n}\,\phi_0}\Big(-\frac{1}{n}\,\frac{\mathrm{d}\phi_0}{\mathrm{d}Y}\Big) + \ldots,\ \ \ldots$$
$$\Longrightarrow f''' = \mathrm{e}^{-\frac{1}{n}\,\phi_0}\Big(-\frac{1}{n}\,\frac{\mathrm{d}\phi_0}{\mathrm{d}Y}\Big)^3 + \ldots.$$

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$$\implies f''' = \mathrm{e}^{-\frac{1}{n}\phi_0}\Big(-\frac{1}{n}\frac{\mathrm{d}\phi_0}{\mathrm{d}Y}\Big)^3 + \dots.$$

Complex ODE for $\{u(Y), v(Y)\}$

Substituting yields the following equation:

$$\mathrm{e}^{-u}\Big(\frac{\mathrm{d}\phi_0}{\mathrm{d}Y}\Big)^3 = -\frac{1}{4}Y, \qquad \boxed{\mathrm{e}^{-\frac{u}{3}}(u_Y + \mathrm{i}v_Y) = \Big(\frac{1}{2} \pm \mathrm{i}\frac{\sqrt{3}}{2}\Big)\Big(\frac{Y}{4}\Big)^{\frac{1}{3}}.}$$

On “Homotopy” ODE Approach: $n \to 0$

Real System for $\{u(Y), v(Y)\}$

This is a system of the two first-order ODEs,

$$\begin{cases} e^{-\frac{u}{3}} u_Y = \frac{1}{2}\left(\frac{Y}{4}\right)^{\frac{1}{3}}, \\ e^{-\frac{u}{3}} v_Y = \pm\frac{\sqrt{3}}{2}\left(\frac{Y}{4}\right)^{\frac{1}{3}}. \end{cases} \tag{9}$$

Solving the independent first equation gives, up to omitted constant,

$$u(Y) = -3\ln\left(1 - \frac{1}{2}\left(\frac{Y}{4}\right)^{\frac{4}{3}}\right). \tag{10}$$

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Blow-up Behaviour of Interface as $n \to 0$

Therefore, the leading order interface position is

$$y_0(n) \sim Y_0\, n^{-\frac{3}{4}} \quad \text{as} \quad n \to 0^+; \quad Y_0 = 2^{\frac{11}{4}}.$$

On “Homotopy” ODE Approach: $n \to 0$

WKBJ Expansion as $n \to 0$: Convergence to the Linear ODE

Finally, this yields the following expansion as $n \to 0^+$ of the similarity profile:s

$$f(y) \sim k\left(1 - \tfrac{1}{2}\left(\tfrac{Y}{4}\right)^{\frac{4}{3}}\right)^{\frac{3}{n}} \cos\left[\tfrac{3\sqrt{3}}{n}\, \ln\left(1 - \tfrac{1}{2}\left(\tfrac{Y}{4}\right)^{\frac{4}{3}}\right) + k_1\right],$$

$$\text{where} \quad Y = n^{\frac{3}{4}} y \quad \text{and} \quad Y_0 = n^{\frac{3}{4}} y_0(n),$$

$k > 0$ and k_1 being parameters. Not easy multi-scale... .

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$n = 0$: Linear ODE for the Fundamental Kernel $F(y) = \psi_0(y)$

$$\mathbf{B}F \equiv -F^{(4)} + \tfrac{1}{4}\,(yF)' = 0 \Longrightarrow F(y) = \psi_0(y).$$

Existence-Uniqueness Theory: Non-Standard, Very Difficult, Open

Standard Weak Solutions of the CP are not Available

The TFE–4,

$$u_t = -(|u|^n u_{xxx})_x$$

is not fully divergent, so do not admit a standard definition of weak solutions via integration by parts.

This is a **principal difficulty**!

On the other hand:

Existence-Uniqueness Theory: Non-Standard, Very Difficult, Open

SECOND Fundamental Result of TFE Theory

Bernis and **Friedman**, J. Differ. Equat., **83** (1990), 179–206.

CIT.: 83

Construction of **nonnegative** solutions for any $n \in (0,3)$

$$u(x,t) = \lim_{\epsilon \longrightarrow 0} u_\epsilon(x,t) \geq 0,$$

where $\{u_\epsilon\}$ solve the regularized "singular" parabolic equation with

$$u_\epsilon : \quad |u|^n \mapsto \frac{|u|^{n+4}}{\epsilon |u|^n + u^4} \longrightarrow |u|^n, \quad \epsilon \longrightarrow 0.$$

Existence-Uniqueness Theory: Non-Standard, Very Difficult, Open

FIRST Fundamental Result of TFE Theory

RECALL: **Bernis–McLeod**, Nonl. Anal., TMA, **17** (1991), 1039–1068. CIT.: 7!

Oscillatory similarity Solutions of the PME–4.

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Oscillatory similarity Solutions of the PME–4.

Free-Boundary Problem (FBP)

At least for $n < n_{\rm h} = 1.7587...$, positive are not solutions of the CP (which are oscillatory and changing sign).

These are solutions of a special FBP.

Uniqueness for the TFEs is not settled (an open problem), since there is no mechanism to distinguish the CP and VARIOUS FBPs. A typical difficulty for higher-order parabolic PDE theory: it is not clear which solutions you are dealing with... .

Homotopic PDE Approach to the Cauchy Problem

Extension of Analytic Semigroups

Unlike the FBPs, we need **oscillatory** solutions.
As usual, we will use *homotopy* of the TFE to the bi-harmonic PDE

$$u_t = -u_{xxxx}.$$

We construct a *homotopic path* via equations: if $\exists$ a family of uniformly parabolic PDEs (a *homotopy deformation*) with coefficient $\varphi_\epsilon(u)$ analytic in both variables $u \in \mathbb{R}$ and $\epsilon \in (0,1]$,

$$u_\epsilon : \quad u_t = -\big(\varphi_\epsilon(u) u_{xxx}\big)_x \tag{11}$$

such that $\varphi_1(u) \equiv 1$ and as $\epsilon \to 0$, uniformly

$$\varphi_\epsilon(u) \to |u|^n. \tag{12}$$

Homotopic PDE Approach to the Cauchy Problem

Example

For instance:

$$\varphi_\epsilon(u) = \epsilon^n + (1-\epsilon)(\epsilon^2 + u^2)^{\frac{n}{2}}, \quad \epsilon \in (0,1].$$

Homotopic PDE Approach to the Cauchy Problem

Extension of Analytic Semigroups

For an $\epsilon \in (0,1]$, let $u_\epsilon(x,t)$ be the unique solution of the CP for the regularized equation with same data u_0. By classic parabolic theory, u_ϵ is continuous (and analytic) in $\epsilon \in (0,1]$ in any natural functional topology.

Main problem: the limit $\epsilon \to 0$ (regularized PDE loses its uniform parabolicity).

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Proper (Extended) Solution

$u(x,t)$ is called a *proper solution* of the CP for the TFE if

$$u(x,t) = \lim_{\epsilon \to 0} u_\epsilon(x,t). \tag{13}$$

A typical definition in extended semigroup theory (e.g., in blow-up theory admitting $u(x,t) \equiv \infty$).

Homotopic PDE Approach to the Cauchy Problem

Extension of Analytic Semigroups

Proof of existence and uniqueness: difficult, simpler for *Riemann's problems* with particular clear step-like geometry of $u_0(x)$.

Uniqueness: no $O(1)$-oscillations in $\epsilon \to 0$.

In general: **open problem**: a typical difficulty with higher-order parabolic PDEs with not monotone and potential operators... .

Homotopic PDE Approach to the Cauchy Problem

Example of Riemann Problem: Regularized TFE in Inner Region

Close to singular points near interface, $\exists$ scaling

$$u_\epsilon(x,t) = \epsilon\, v_\epsilon(y,\tau), \quad y = \tfrac{x}{\epsilon^{\hat\alpha}},\ \tau = \tfrac{t}{\epsilon^{\hat\beta}},$$

where $\hat\beta = 4\hat\alpha - n$. This gives at $\epsilon = 0$ the corresponding uniformly parabolic *matching* TFE (mTFE)

$$v_\tau = -\left(\left[1 + (1+v^2)^{\frac{n}{2}}\right] v_{yyy}\right)_y \tag{14}$$

or, simply,

$$v_\tau = -\left[(1+v^2)^{\frac{n}{2}} v_{yyy}\right]_y.$$

Homotopic PDE Approach to the Cauchy Problem

Extension of Analytic Semigroups

CRUCIAL: well-posedness of the mTFE in a given class of data. Can be proved for some Riemann's problems, very difficult... .

The analytic parabolic flow describes a smooth ϵ-transition through a singular layer occurring as $\epsilon \to 0$.

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Proper Extended Solution via a Formal Asymptotic Series

If the mTFE is uniquely solved (for a given Riemann Problem), this gives an opportunity, by standard asymptotic theory, to define

$$u_\epsilon(x,t) = \sum_{(k \geq 0)} V_k(x,t,\epsilon), \quad V_0 = v,$$

where $\{V_k\}_{k\geq 1}$ are defined from LINEAR (linearized) parabolic PDEs; also a difficult problem... .

Some Nonlinear PDEs of the XXI Century

New Ideas Needed!

Local oscillatory structure near finite interfaces for solutions of maximal regularity and homotopy approaches $\exists$ for :

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The TFE–6

$$u_t = (|u|^n u_{xxxxx})_x \pm (|u|^{p-1} u)_{xx};$$

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DLSS: from Hierarchy of Dispersion Models

$$u_t = \left[u\left(-\frac{u_{xx}}{u} + \frac{2u_x u_{xx}}{u^2} - \frac{(u_x)^3}{u^3}\right)\right]_x;$$

Further Nonlinear PDEs of the XXI Century

New Ideas Needed!

CAHN–HILLIARD (C-H) EQUATIONS WITH DEGENERATE MOBILITY

$$u_t = -(|1-u^2|^n u_{xxx})_x \pm (|1-u^2|^m u_x)_x;$$

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DNTFE–6

DOUBLY NONLINEAR TFE (a model by King)

$$u_t = (|u|^m |u_{xxxxx}|^n u_{xxxxx})_x;$$

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DOUBLY NONLINEAR TFE (a model by King)

$$u_t = (|u|^m |u_{xxxxx}|^n u_{xxxxx})_x;$$

TFE–8

EIGHTH-ORDER TFE (another King's model)

$$u_t = -(u^n u_{xxxxxxx})_x;$$

More Nonlinear PDEs of the XXI Century

New Ideas Needed!

Third (**odd**) order ROSENAU–HYMAN (RH) EQUATION

$$u_t = (u^2)_{xxx} + (u^2)_x;$$

Entropy theory still not fully developed...

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NDEs

NONLINEAR DISPERSION EQUATIONS (NDE):

$$u_t = \alpha(u^2)_{xxxxx} + \beta(u^2)_{xxx} + \gamma(u^2)_x;$$

Entropy theory ?

More Nonlinear PDEs of the XXI Century

The NDE–$2m+1$

$(2m+1)$TH-ORDER NONLINEAR DISPERSION EQUATION (NDE–$2m$)

$$u_t = D_x(|u|^n D_x^{2m} u);$$

More Nonlinear PDEs of the XXI Century

The NDE–$2m+1$

$(2m+1)$TH-ORDER NONLINEAR DISPERSION EQUATION (NDE–$2m$)

$$u_t = D_x(|u|^n D_x^{2m} u);$$

The Rosenau equation

The ROSENAU EQUATION

$$u_t + u_{xxt} = 3uu_x + \left[uu_{xx} + \tfrac{1}{2}(u_x)^2\right]_x,$$

and higher-order extensions;

More and More Nonlinear PDEs of the XXI Century

New Ideas Needed!

THE FFCH EQUATION

$$u_t - u_{xxt} = -3uu_x + 2u_x u_{xx} + uu_{xxx},$$

and other extensions, not integrable... .

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and other extensions, not integrable... .

Higher-order in Time Models

HIGHER-ORDER DISPERSION EQUATIONS such as

$$u_{ttt} = -(|u|^n u_{xxxx})_{xxx}, \ \dots ;$$

More and More Nonlinear PDEs of the XXI Century

Quasilinear wave equations, QWE–4

$$u_{tt} = -(u^n u_{xx})_{xx} \pm u^p \quad \text{(shocks!)};$$

More and More Nonlinear PDEs of the XXI Century

Quasilinear wave equations, QWE–4

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Higher-Order Monge–Ampère PDEs

HIGHER-ORDER HESSIAN EQUATIONS SUCH AS

$$u_t = -|D^{2m} u| \pm u^p, \quad m = 1, 2, ... ;$$

where $|D^{2m} u|$ is the determinant of catalecticant determinant of the Hessian matrix of $2m$th-order derivatives; non fully convex flows, ALL open... , ETC., see examples in:
Galaktionov and S.R. Svirshchevskii, Exact Solutions and Invariant Subspaces of Nonlinear Partial Differential Equations in Mechanics and Physics, Chapman & Hall/CRC, Boca Raton, Florida, 2007.

Final Slides:

A Tendency of PDE Theory in the XXI Century:

- Incredibly difficult NEW Models,
- Classic Fundamental Techniques of the XX century hardly apply,
- A FULL Theory CANNOT be developed (too many hypotheses...),

CONCLUSION

Those who want to develop XXI Century PDE Theory MUST know all fundamental tools developed earlier, even these do not apply directly, and MUST work in ALL the key directions, concerning various PDEs, in order to use this exceptional experience for

understanding and feeling crucial nonlinear properties of the models under the pressure of the absence of fully rigorous tools of the analysis...

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THANKS

THANK YOU FOR YOUR PATIENCE AND ATTENTION