Ordinary Differential Equations
MAA121 / MAG131

Lecture Notes 2009/2010

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Notations

$\mathbb{N}$ the set of positive integers

$\mathbb{Z}$ the set of all integers

$\mathbb{R}$ the field of real numbers

Recommended Texts

This lecture notes follow closely first chapters of the book:

J. C. Robinson, *An Introduction to Ordinary Differential Equations.* Cambridge University Press, 2004.

Robinson's book is recommended as the principal reading for this course.

The following book (amongst many others) could be recommended as a complementary reading for the Mathematica based part of the course:

M. L. Abell, J. P. Braselton, *Differential Equations with Mathematica.* Academic Press, 2004.

See also numerous resources available at `http://www.wolfram.com/` and especially at Mathworld website `http://mathworld.wolfram.com`

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1 Radioactive decay and carbon dating

1.1 Radioactive decay model

Radioactive decay is the process in which an unstable atomic nucleus spontaneously loses energy by emitting ionizing particles and radiation. This decay, or loss of energy, results in an atom of one type, called the parent nuclide transforming to an atom of a different type, named the daughter nuclide. For example: a Carbon-14 atom (the "parent") emits radiation and transforms to a Nitrogen-14 atom (the "daughter"). It is impossible to predict when a given atom will decay, but given a large number of similar atoms the decay rate, on average, is predictable.[1]

Let $N(t)$ denote the number of radioactive atoms in some sample of material at time t. Then the differential equation

$$\boxed{\frac{dN}{dt} = -kN,} \tag{1.1}$$

is a model for the *radioactive decay*, where $k > 0$ is the *rate of decay*. Assume that at time T_0 there are N_0 atoms. A direct computation verifies that the solution of (1.1) is

$$\boxed{N(t) = N_0 e^{-k(t-T_0)}.} \tag{1.2}$$

Note that $N(t)$ depends only on the *time interval* $t - T_0$.

The *half–life* $T_{1/2}$ of a particular radioactive isotope is the time it takes for half of the radioactive atoms to decay, that is

$$N(T_{1/2}) = \frac{1}{2}N(0),$$

and hence, using (1.2) with $T_0 = 0$ we obtain

$$N_0 e^{-kT_{1/2}} = \frac{1}{2}N_0.$$

Taking the (natural) logarithm of both sides we conclude that

$$\boxed{T_{1/2} = \frac{\log(2)}{k}.} \tag{1.3}$$

Note that $T_{1/2}$ does not depend on N_0.

[1] http://en.wikipedia.org/wiki/Radioactive_decay

1.2 Radiocarbon dating

The solution (1.2) forms the basis of the technology of *radiocarbon dating*. The essence of the method is as follows. Living matter is constantly taking up carbon from the air. The result is that within such material the ratio of the number of isotopes of radioactive Carbon-14 to the number of isotopes of stable Carbon-12 is essentially constant. Once the specimen is dead (for example, a tree is cut down for its wood) the radioactive Carbon-14 atoms begin to decay according to the model (1.1). The half–life $T_{1/2}$ of Carbon-14 is approximately 5700 years, so the decay rate constant k in (1.1) to be

$$\boxed{k = \frac{\log(2)}{5700} \approx 1.216 \times 10^{-4}.} \tag{1.4}$$

By examining the ratio of the number of isotopes of Carbon-12 to Carbon-14 in the sample of material that we want to date, it is possible to work out the proportion remaining of the Carbon-14 atoms that were initially present. Suppose that the sample stopped taking up carbon from the air at time T_0, and that the number of Carbon-14 atoms present then was $N(T_0)$. If we know that the sample now (at time t) contains only a fraction p of the initial level of Carbon-14, then $N(t) = pN(T_0)$. Using the explicit solution (1.2) we should have

$$pN(T_0) = N(t) = N(T_0)e^{-k(t-T_0)}.$$

Taking the (natural) logarithm of both sides we conclude that $\log(p) = -k(t - T_0)$ and so the year T_0 from which the sample dates is given by

$$\boxed{T_0 = t + \frac{\log p}{k}.} \tag{1.5}$$

Example. In 1988, the Shroud of Turin[2] was dated by several independent groups of scientists. Fibres from the shroud were found to contain about 92% of the initial level of Carbon-14 in the living matter. Using (1.5), we therefore obtain

$$T_0 = 1988 + \frac{\log 0.92}{0.0001216} \simeq 1302,$$

which puts the origin of the Shroud in the Middle Ages.

[2]The Shroud of Turin (or Turin Shroud) is a linen cloth bearing the image of a man who appears to have suffered physical trauma in a manner consistent with crucifixion. It is kept in the royal chapel of the Cathedral of Saint John the Baptist in Turin, Italy. The origins of the shroud and its image are the subject of intense debate among scientists, theologians, historians and researchers. See `http://en.wikipedia.org/wiki/Radiocarbon_14_dating_of_the_Turin_Shroud`

2 Classification of differential equations

Unknown function and independent variable(s). Differential equations date back to the mid–seventeenth century, when calculus was discovered by Newton and Leibniz. Most generally, one can say that differential equations is an equation which contains an unknown function of one or several independent variables and derivatives that function. The analysis of a given differential equation starts from identifying correctly an unknown function and independent variable(s). For example, in the Equation of Radioactive Decay

$$\frac{dN}{dt} = -kN \tag{2.1}$$

the unknown function is $N(t)$ (number of elements) and the independent variable is t (time). However, we should not be confused by possible change of notations. For instance, in the equation

$$\frac{dy}{dx} = -ky, \tag{2.2}$$

the unknown function is denoted by $y(x)$ and the independent variable is denoted by x, but otherwise it is exactly the same equation as (2.1). The only difference is that the notation $y(x)$ is used instead of $N(t)$. Similarly,

$$z' = -kz \tag{2.3}$$

is exactly the same equation as (2.1) and (2.2). The difference now is that the notation z is used to denote the unknown function (because z' denotes the differentiation, so z must be a function!). We are free to decide ourself how to denote the independent variable. For instance, we can denote the independent variable by x, as before, so that the unknown function is $z(x)$.

Classification of differential equations. The most significant distinction in Differential Equations is between Ordinary and Partial Differential Equations.

In a *Partial Differential Equations* the unknown function depends on more then one independent variable, and the derivatives are therefore *partial* derivatives. For example, the heat h in a rod at position x and time t obeys the *heat equation*

$$\frac{\partial h}{\partial t} = k\frac{\partial^2 h}{\partial x^2}.$$

Here the unknown function $h(t,x)$ depends on two independent variables, x and t. Another example of a Partial Differential Equations is the *Laplace equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Here the unknown function $u(x, y)$ depends on two independent variables, x and y. We do not study Partial Differential Equations in this course.

In *Ordinary Differential Equations* there is only one independent variable, for example, as in the equation of Radioactive Decay (2.1). Another example is the equation of *Energy Conservation*, which is written as

$$\frac{1}{2}m\dot{x}^2 + V(x) = E.$$

Here the unknown function $x(t)$ depends on one independent variable, t, which is interpreted as time, and $\dot{x}$ denotes the derivative of x with respect to the independent variable t. The term $\frac{1}{2}m\dot{x}^2$ has the physical meaning of the kinetic energy, while $V(x)$ is the potential energy at point x.

Another example of an Ordinary Differential Equations of 2nd order is *Newton's 2nd Law of Motion*

$$m\ddot{x} = F,$$

where $\ddot{x}$ denotes the 2nd derivative of an unknown function x with respect to the independent variable t.

The equation

$$\psi''' + \frac{1}{2}\psi\psi' = 0$$

occurs in the theory of Fluid Boundary Layers. It is a 3rd order ordinary differential equation, where the unknown function is $\psi(x)$ and independent variable is x.

Generally, *the order* of a differential equation is the order of the highest order derivative which appears in the equation. To be more precise, we introduce the following.

Definition 2.1. An *n-th Ordinary Differential Equations* for an unknown function $y(t)$ is an equation of the form

$$\mathcal{F}(y^{(n)}, y^{(n-1)}, \ldots, y', y, t) = 0. \tag{2.4}$$

A *solution* of the ODE (2.4) on the interval (a, b) is a function $y(t)$ which is n–times differentiable on the interval (a, b) and satisfies the equation $(*)$ for all $t \in (a, b)$.

Another important concept in the classification of ODEs is linearity. Generally, linear ODEs are relatively "easy" and nonlinear problems are "hard".

Definition 2.2. An n-th order ODE for an unknown function $y(t)$ is said to be linear if it can be written in the form

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = f(t). \tag{2.5}$$

Such a linear equation is called *homogeneous* if $f(t) = 0$ and *inhomogeneous* if $f(t) \neq 0$.

For example, the equation of Radioactive Decay is a linear homogeneous first order ODE, the Energy Conservation equation is a 1st order nonlinear ODE, while Newton's 2nd Law of Motion is a 2nd order linear inhomogeneous (if $F \neq 0$) equation. The equation of Fluid Boundary Layers is a 3rd order nonlinear ODE.

3 Trivial ODEs

In this section we consider the simplest possible type of ODEs, equations of the form

$$y' = f(x),$$

where $y(x)$ is an unknown and $f(x)$ is a given function of the independent variable x. The equation asks us to find a function y whose graph has the slope $f(x)$ at the point x. Such ODEs can be solved by direct integration.

3.1 The Fundamental Theorem of Calculus

Any function $F : (a, b) \to \mathbb{R}$ that satisfies

$$\boxed{F'(x) = f(x) \quad \text{for all} \quad x \in (a, b)}$$

is called an *antiderivative* or a *primitive* of the function f on the interval (a, b). Antiderivatives of a function f are also frequently referred to as *indefinite integrals* of the function f and denoted

$$\int f(x)dx,$$

so that, by the definition,

$$\frac{d}{dx}\left(\int f(x)dx\right) = f(x).$$

Clearly, if F is an antiderivative of f on (a, b), then so is $F(x)+C$, for any constant $C \in \mathbb{R}$.

Example 3.1. $\sin(x) + C$ is an antiderivative of $\cos(x)$ on $\mathbb{R}$, for any constant $C \in \mathbb{R}$.

Example 3.2. $\log(x) + C$ is an antiderivative of x^{-1} on $(0, +\infty)$, for any constant $C \in \mathbb{R}$.

Given a function $F : [a, b] \to \mathbb{R}$, the *definite integral*

$$\int_a^b f(x)\, dx,$$

is defined informally to be the signed area of the region in the xy-plane bounded, by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$.

Thus the definite integral is always a real number ("the area") and antiderivative (indefinite integral) is always a function of real variable. Definite integrals are related to antiderivatives through the Fundamental Theorem of Calculus: the definite integral of a function over an interval is equal to the difference between the values of an antiderivative evaluated at the endpoints of the interval. More precisely, we formulate the following.

Theorem 3.3. (THE FUNDAMENTAL THEOREM OF CALCULUS) *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. For $x \in [a, b]$ define*

$$G(x) = \int_a^x f(s)ds. \tag{3.1}$$

Then $G(a) = 0$ and G is an antiderivative of f on (a, b), that is

$$G'(x) = f(x) \quad \text{for all} \quad x \in (a, b).$$

Furthermore, for any antiderivative F of f on (a, b) one has

$$G(x) = F(x) - F(a). \tag{3.2}$$

In particular,

$$\int_a^b f(s)ds = F(b) - F(a). \tag{3.3}$$

Essentially, this theorem says that differentiation reverses integration, since if we know an antiderivative F of f then we can calculate integrals of f using formula (3.3) and if we know how to calculate integrals (e.g. numerically using a software package) then we can calculate an antiderivative using formula (3.1).

Note that identity (3.3) is frequently written in the form

$$\int_a^b f(s)ds = \big[F(s)\big]_{y=a}^{b}.$$

Remark 3.4. In *Mathematica*,

$$\texttt{Integrate}[\texttt{f}, \texttt{x}]$$

gives the indefinite integral $\int f(x)dx$, and

$$\texttt{NIntegrate}[\texttt{f}, \{\texttt{x}, \texttt{a}, \texttt{b}\}]$$

gives a (numerical approximation) to the definite integral $\int_a^b f(x)dx$. Definite integral also could be computed using

$$\texttt{Integrate}[\texttt{f}, \{\texttt{x}, \texttt{a}, \texttt{b}\}]$$

which essentially performs (3.3) to evaluate definite integral from an antiderivative.

3.2 General solutions and initial conditions

Let us return to the trivial ODE

$$y' = f(x). \tag{3.4}$$

Any antidrivative of f is a solution of this equation, hence we could simply write

$$y(x) = \int f(s)ds.$$

If we choose one particular antiderivative F of f, then we know that

$$y(x) = F(x) + C$$

is an antiderivative of f for any $C \in \mathbb{R}$. Thus (3.4) has many solutions. We say that

$$y(x) = F(x) + C$$

is the *general solution* of the (3.4) since every possible solution of (3.4) could be obtained by choosing C appropriately. This is not a surprise: we can move a graph 'up and down' and not to change its slope.

A possible way to pick out a particular solution of ODE (3.4) is to prescribe a point that must lie on the graph of y, in other words to specify the value $y(x_0)$ at some particular x. We refer to such a restriction

$$y(x_0) = y_0$$

as an *initial condition*. The idea is that we construct a solution of an ODE starting at point (x_0, y_0) on the (x, y)–plane.

Example 3.5. Find the solution of the initial value problem

$$y' = x + 10\sin(x), \quad y(\pi) = 0. \tag{3.5}$$

Solution. Using standard integrals we see that

$$F(x) = \frac{1}{2}x^2 - 10\cos(x)$$

is an antiderivative of $f(x) = x + 10\sin(x)$. Thus the general solution of the equation (3.5) is

$$y(x) = \frac{1}{2}x^2 - 10\cos(x) + C,$$

for any $C \in \mathbb{R}$. To find the solution that satisfies $y(\pi) = 0$ we must have

$$0 = y(\pi) = \frac{1}{2}\pi^2 - 10\cos(\pi) + C.$$

Hence

$$C = -\frac{1}{2}\pi^2 - 10$$

and so

$$y(x) = \frac{1}{2}x^2 - 10\cos(x) - \left(\frac{\pi^2}{2} + 10\right)$$

is the solution of the initial value problem (3.5).

Example 3.6. Find the equation of a curve passing through the point $(1,0)$ on the (x,y)–plane and has slope $\log(x)$.

Solution. We have to solve the initial value problem

$$y' = \log(x), \quad y(1) = 0. \tag{3.6}$$

Using formula (3.1) for an antiderivative, we obtain

$$y(x) - \underbrace{y(1)}_{=0} = \int_1^x \log(s)ds = \big[s\log(s) - s\big]_{s=1}^{x} = (x\log(x) - x) - (0-1).$$

Therefore, the solution of the initial value problem (3.6) is given by

$$y(x) = x\log(x) - x + 1.$$

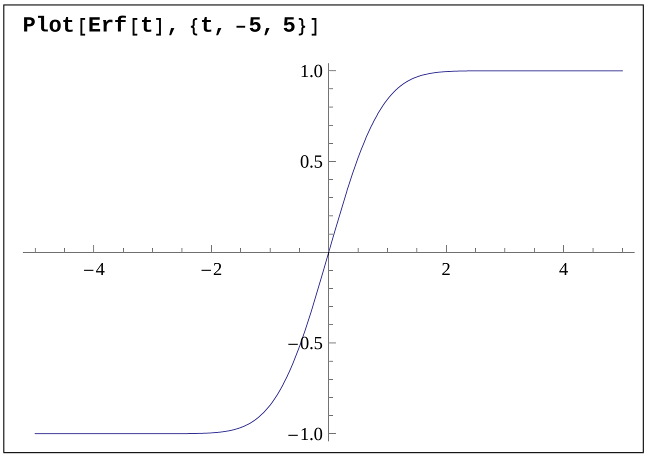

Figure 1: *Mathematica* code which plots the graph of the error function $\mathrm{Erf}(t)$.

Example 3.7. Find the solution of the initial value problem

$$\dot{x} = e^{-t^2}, \quad x(0) = x_0. \tag{3.7}$$

Solution. Using (3.1), we obtain that

$$x(t) = x_0 + \int_0^t e^{-s^2} ds$$

is a solution of the initial value problem (3.7). However, there is no explicit form for the antiderivative of e^{-s^2} in terms of elementary functions! In fact,

$$\mathrm{erf}(t) := \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2} ds \tag{3.8}$$

is known as *error function*. It is particularly important in statistics. However the integral in the right hand side of (3.8) is merely the definition of erf. The graph of the error function produced using *Mathematica* `Plot` instruction is displayed on Figure 1.

3.3 Newton's 2nd Law of Motion

Newton's 2nd Law of Motion states that the velocity v of an object satisfies the differential equation

$$m\dot{v} = F(t), \tag{3.9}$$

where $F(t)$ is the force applied to the object at time t and $m > 0$ is the mass of the object.

Clearly equation (3.9) is a particular case of trivial differential equation (3.4), since (3.9) can be rewritten in the form

$$\dot{v} = \frac{F(t)}{m}. \tag{3.10}$$

Example 3.8. A car of mass m is traveling at a speed v_0, when it suddenly starts to brake. The brakes apply a constant force k. Using Newton's second law of motion

$$m\dot{v} = -k,$$

where $v(t)$ is the speed of the car at time t, find out how long does it take the car to stop, and how far does the car travel before it comes to rest.

Solution. First of all, we need to solve the initial value problem

$$m\dot{v} = -k, \qquad v(0) = v_0.$$

The general solution of the differential equation is

$$v(t) = -\frac{k}{m}t + C.$$

To satisfy the initial condition $v(0) = v_0$ we need to set $C = v_0$, so the particular solution of the initial value problem is

$$v(t) = -\frac{k}{m}t + v_0.$$

The *stopping time* T_{stop} of the car is the solution of the equation

$$v(T_{stop}) = 0,$$

or, using the explicit form of the solution $v(t)$, we rewrite

$$-\frac{k}{m}T_{stop} + v_0 = 0.$$

Thus we obtain

$$\boxed{T_{stop} = \frac{mv_0}{k}.}$$

Since the velocity of the car is the time derivative of the position of the car x,

$$\dot{x} = v,$$

or, using again the explicit form of the solution $v(t)$ we have

$$\dot{x} = -\frac{k}{m}t + v_0.$$

Integrating again, we obtain

$$x(t) = v_0 t - \frac{k}{2m}t^2 + C.$$

Assume that at time $t = 0$ the car is at the zero position, that is

$$x(0) = 0.$$

Then $C = 0$ and therefore the *stopping distance* is computed as

$$\boxed{x(T_{stop}) = v_0 T_{stop} - \frac{k}{2m}T_{stop}^2 = \frac{m}{2k}v_0^2.}$$

Notice that the stopping distance depends on the *square* of the initial speed v_0. For instance, the stopping distance of the car, traveling at 40 mph is 1.78 times longer then the stopping distance of the car traveling at 30 mph!

4 Separable ODEs

Separable ODE is a differential equation in the form

$$\frac{dx}{dt} = f(x)g(t).$$

here $x(t)$ is the unknown function and t is an independent variable. Note that in contrast to trivial ODEs, the right hand side of the equation now depends not only on t, but also on the unknown function x!

4.1 The solution 'recipe'

The practical solution 'recipe' is to divide by $f(x)$ and multiply up by dt, to obtain

$$\frac{dx}{f(x)} = g(t)\,dt.$$

This is 'separation of variables', since we now have all the xs on one side and all the ts on the other, and that is why the equation is called separable. For the general solution we integrate both sides to get

$$\boxed{\int \frac{dx}{f(x)} = \int g(t)\,dt.}$$

Note that the integral on the left is taken with respect to dx, while on the right it is with respect to dt!

Example 4.1. Find the solution of the initial value problem

$$\frac{dx}{dt} = x^2, \qquad x(0) = x_0. \tag{4.1}$$

Solution. This is a 1st order nonlinear *separable* equation. The unknown function is $x(t)$ and the independent variable is t.

Separating the variables in the equation we obtain

$$\frac{dx}{x^2} = dt.$$

Integrating both sides

$$\int \frac{dx}{x^2} = \int dt,$$

we obtain

$$-\frac{1}{x} + C_1 = t + C_2,$$

or, absorbing arbitrary constants C_1 and C_2 into one constant $C := C_1 - C_2$, we get

$$-\frac{1}{x} = t + C.$$

Therefore,

$$\boxed{x(t) = -\frac{1}{t+C}}$$

is the *general solution*.

Substituting the initial condition $x(0) = x_0$ we obtain

$$x(0) = -\frac{1}{0+C} = x_0,$$

so that $C = -x_0^{-1}$. Therefore

$$\boxed{x(t) = \frac{1}{x_0^{-1} - t}}$$

is the *particular solution* of the initial value problem, provided that $x_0 \neq 0$ (otherwise the formula is not well defined).

Note that if $x_0 > 0$ then this solution 'blows-up' (has a vertical asymptote) when $t = x_0^{-1}$, while for $x_0 < 0$ the solution is well defined for all $t > 0$. Finally, note that if $x_0 = 0$ then $x(t) = 0$ is the 'obvious' solution of the initial value problem. This completes our analysis.

Exercise 4.2. Find the solution of the initial value problem

$$\frac{dx}{dt} = x^\alpha, \qquad x(0) = x_0,$$

where $\alpha > 1$. Show that solutions with $x_0 > 0$ blow up in a finite time.

Example 4.3. Find the solution of the initial value problem

$$\frac{dy}{dx} = -2xy, \qquad y(x_0) = y_0.$$

Solution. This is a 1st order linear *separable* equation. The unknown function is $y(x)$ and the independent variable is x.

Separating the variables in the equation we obtain

$$\frac{dy}{y} = -2x\,dx.$$

Integration both sides

$$\int \frac{dy}{y} = \int -2x dx,$$

```
ode = DSolve[{x'[t] == x[t]^2, x[0] == x0}, x[t], t]
```

$$\left\{\left\{x[t] \to -\frac{x0}{-1+t\,x0}\right\}\right\}$$

```
x[t_] = First[x[t] /. ode]
```

$$-\frac{x0}{-1+t\,x0}$$

```
Plot[{x[t] /. {x0 → 1/5}, x[t] /. {x0 → 1/6}, x[t] /. {x0 → -1}, x[t] /. {x0 → -5}},
 {t, 0, 5}, PlotRange → {{0, 5}, {-5, 10}}]
```

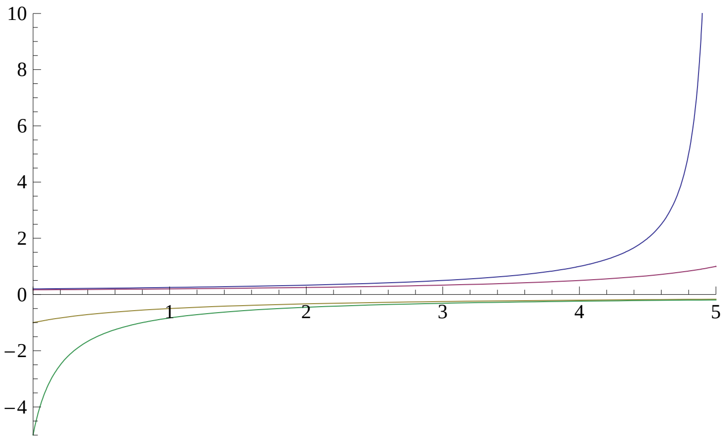

Figure 2: *Mathematica* code which solves initial value problem (4.1) and plots its solution $x(t)$ for $x_0 = -1$ and $x_0 = 0.2$ in the interval $t \in [0, 10]$. Note that the vertical asymptote and the lower part of the hyperbola (when $t > 5$) are not parts of the solution with initial data $x_0 = 0.2$, however they are plotted by Mathematica.

and absorbing arbitrary constants, we obtain

$$\log|y| = -x^2 + C.$$

Exponentiating both sides we get

$$|y| = e^{-x^2+C},$$

or

$$|y(x)| = Ae^{-x^2},$$

where $A := e^C$ is positive. Taking $|y(x)| = y(x)$ gives a positive solution, while taking $|y(x)| = -y(x)$ gives a negative solution. Notice that $y(x) = 0$ is also a solution of the equation. Thus the *general solution* is

$$\boxed{y(x) = A\,e^{-x^2}}$$

allowing any $A \in \mathbb{R}$.

Substituting the initial condition $y(x_0) = y_0$, we obtain

$$y(x_0) = Ae^{-x_0^2} = y_0,$$

so that $A = y_0 e^{x_0^2}$. Therefore

$$\boxed{y(x) = y_0 e^{x_0^2 - x^2}}$$

is the *particular solution* of the initial value problem, for any $x_0 \in \mathbb{R}$.

Remark 4.4. Note that

$$\boxed{\int \frac{dy}{y} = \log|y| + C.}$$

Indeed, for $y > 0$ we have

$$\frac{d}{dy}\Big(\log|y| + C\Big) = \frac{d}{dy}\Big(\log(y) + C\Big) = \frac{1}{y},$$

while for $y < 0$ we obtain

$$\frac{d}{dy}\Big(\log|y| + C\Big) = \frac{d}{dy}\Big(\log(-y) + C\Big) = -\frac{1}{(-y)} = \frac{1}{y}.$$

Exercise 4.5. Show, using separation of variables, that the solution of the initial value problem for the *radioactive decay model* (1.1):

$$\frac{dN}{dt} = -kN, \qquad N(T_0) = N_0,$$

is indeed given by the formula (1.2):

$$\boxed{N(t) = N_0 e^{-k(t-T_0)}.}$$

4.2 Malthus' population model

The simple linear equation

$$\frac{dp}{dt} = kp \tag{4.2}$$

was proposed in 1798 by the English economist Thomas Malthus as a basic model for population growth. Here $p(t)$ is the population level at time t. The increase in the population is taken to be proportional to the total number of people and $k > 0$ is the constant representing the rate of growth (the difference between the birthrate and the deathrate).

As it is easy to see by the method of separation of variables the model predicts exponential growth of the population

$$p(t) = p(t_0)e^{k(t-t_0)}, \tag{4.3}$$

where $p(t_0)$ is the population a given time t_0. Note that the size of the population grows without bound and doubles every $\log(2)/k$ years.

Analysis of Census Data. We shall compare the predictions of the model with census data. The population of Great Britain and Ireland in 1801, 1851, 1901 can be found in the Census:

(4.4)

Year	*Population*
1801	16 345 646
1851	27 533 755
1901	41 609 091

We can use the data from 801 and 1851 to estimate k. Our solution formula (4.3) predicts

$$p(1851) = p(1801)e^{50k},$$

so

$$k = \frac{\log(p(1851)) - \log(p(1801))}{50} \simeq 0.010.$$

Using this value of k, solution (4.3) gives a reasonable prediction for population in 1901:

$$p(1901) = p(1801)e^{100k} \simeq 46\,\text{million}.$$

However, it vastly overestimates the population in 2001: [3]

$$p(2001) = p(1801)e^{200k} \simeq 131\,\text{million}.$$

We shall see in the next section that another model gives a better result.

4.3 Logistic population model

The so called 'logistic equation' is

$$\frac{dp}{dt} = kp\Big(1 - \frac{p}{M}\Big). \tag{4.5}$$

Here $p(t)$ is the population level at time t, coefficient $k > 0$ is the constant representing the rate of growth, and the parameter $M > 0$ is the maximum sustainable population.

Rewrite equation (4.5) in the form

$$\frac{dp}{dt} = K(p)p,$$

where

$$K(p) = k\Big(1 - \frac{p}{M}\Big)$$

[3] The 2001 census found about 59 million. This does not include the figures for the Republic of Ireland. The census held there in 2002 found around 4 million. So the total figure for 2001 should be approximately 63 million.

can be considered as a nonconstant rate of growth, which depends on the population p itself, so our population model becomes 'self–organizing'.

When population $p > 0$ is small, the rate of growth $K(p)$ is close to k, so the equation is approximately the same as in the linear Malthus' model (4.2). However, if $0 < p < M$ and p is close to the 'maximal sustainable' level M, then the rate of growth $K(p)$ is close to 0 and hence population grows very slowly. Finally, if $p > M$ then the rate of growth $K(p)$ becomes negative. In other words, when population exceeds sustainable level M, death rate becomes bigger then the birth rate and hence population decreases.

We now solve the equation explicitly. Equation (4.5) is a separable nonlinear first order ODE. Separating the variables gives

$$\frac{M}{kp(M-p)}dp = dt,$$

where we have multiplied top and bottom of the left hand side by M. Using partial fractions on the left we obtain

$$\frac{1}{k}\left(\frac{1}{p} + \frac{1}{M-p}\right) dp = dt,$$

or

$$\left(\frac{1}{p} + \frac{1}{M-p}\right) dp = k\,dt.$$

Integrating both sides,

$$\int \left(\frac{1}{p} + \frac{1}{M-p}\right) dp = \int k\,dt,$$

we obtain

$$\log|p| - \log|M-p| = kt + C,$$

or

$$\log\left(\frac{|p|}{|M-p|}\right) = kt + C.$$

Exponentiating, we get

$$\frac{|p|}{|M-p|} = Ae^{kt},$$

where $A := e^C > 0$. Rearranging, we obtain

$$p = Ae^{kt}(M-p),$$

where we can remove modulus sign on both sides by allowing A to take negative values, so that from now on $A \in \mathbb{R}$. Further rearranging, we get the *general solution* of (4.5):

$$\boxed{p(t) = M\frac{e^{kt}}{A^{-1} + e^{kt}}.}$$

Assume now that

$$p(0) = p_0 > 0.$$

Substituting, we find that

$$A = \frac{M - p_0}{p_0}$$

and therefore

$$\boxed{p(t) = \frac{M p_0 \, e^{kt}}{(M - p_0) + p_0 \, e^{kt}}}$$

is the *particular solution* of (4.5) with the initial data $p(0) = p_0$.

To understand the *asymptotic behavior* of $p(t)$ (i.e. the behavior of $p(t)$ for large t), observe that

$$\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \frac{M p_0 \, e^{kt}}{(M - p_0) + p_0 \, e^{kt}} = \lim_{t\to\infty} \frac{M}{\frac{M - p_0}{p_0 \, e^{kt}} + 1} = M.$$

so the solution $p(t)$ *asymptotically converges* to M. Moreover, if $0 < p_0 < M$ then $0 < p(t) < M$ and $p(t)$ is *monotone increasing* to M, while if $p_0 > M$ then $p(t) > M$ and $p(t)$ is *monotone decreasing* to M. Finally, note that if $p_0 = M$ then $p(t) = M$ is a *stationary* solution of (4.5). Similarly, $p_0 = 0$ then $p(t) = 0$ is another *stationary* solution of (4.5).

Analysis of Census Data, continued. We shall compare the predictions of the logistic model with the census data. To do this, we need to estimate the parameters M and k that occur in the logistic equation (4.5), using data quoted above in (4.4), and assuming that $t_0 = 0$ corresponds to the year 1801, so that, for instance, $t = 200$ corresponds to the year 2001. The calculations to find M and k are just simple algebra which can be performed using *Mathematica*, so we skip the details (see ROBINSON, p.68) and only state the result:

$$M \simeq 8.31365 \times 10^7, \qquad k \simeq 0.0140957.$$

Using these values to predict the population of the UK and the Republic of Ireland in 2001, we obtain

$$p(200) = 6.68424 \times 10^7,$$

which is remarkably close to the true figure of about 63 million!

The complete *Mathematica* code which provides such analysis of the Census Data is given below in Figure 4.

```
p[t_] = M*p0*Exp[k*t]/((M - p0) + p0*Exp[k*t])
```

$$\frac{e^{k t} M \, p0}{M - p0 + e^{k t} p0}$$

```
M = 1; k = 1; p[t]
```

$$\frac{e^{t} p0}{1 - p0 + e^{t} p0}$$

```
Plot[{M, p[t] /. {p0 → 0.1}, p[t] /. {p0 → 0.5}, p[t] /. {p0 → 1.2}, p[t] /. {p0 → 1.4}},
 {t, 0, 5}, AxesOrigin → {0, 0}]
```

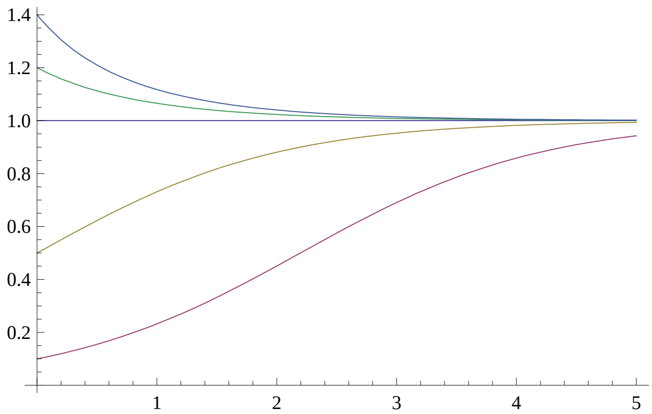

Figure 3: *Mathematica* code which plots the graph of the solution $p(t)$ of the logistic equation (with $M = 1$ and $k = 1$) for initial values $p_0 = 0.1; 0.5; 1.2; 1.5$ and $p_0 = M$.

5 Existence and uniqueness of solutions of ODEs

In all considered examples of initial value problems for ODEs we were able to find a unique particular solution. Is a general rule or there are exceptions ? Does initial value problem for an ODE always has a unique solution ? There is a very general theorem which guarantees existence and uniqueness, with hypothesis which are simple to check.

Theorem 5.1. (Existence and Uniqueness Theorem) *Assume that the functions*

$$f(x,t) \quad \text{and} \quad \frac{\partial f}{\partial x}(x,t)$$

are continuous for $a < x < b$ and $c < t < d$. Then for any $x_0 \in (a,b)$ and $t_0 \in (c,d)$, the initial value problem

$$\frac{dx}{dt} = f(x,t), \qquad x(t_0) = x_0, \tag{5.1}$$

has a unique solution $x(t)$, defined on some existence interval $[t_0, T) \subset (c,d)$.

```
ode = DSolve[{p'[t] == k * (1 - p[t] / M) * p[t], p[0] == p0}, p[t], t]
```

$$\left\{\left\{p[t] \to \frac{e^{k t} M p0}{M - p0 + e^{k t} p0}\right\}\right\}$$

```
p0 = 16 345 646
```

16 345 646

```
p[t_] = First[p[t] /. ode]
```

$$\frac{16\,345\,646\, e^{k t} M}{-16\,345\,646 + 16\,345\,646\, e^{k t} + M}$$

```
reduced = Reduce[{p[50] == 27 533 755, p[100] == 41 609 091}, {k, M}, Reals]
```

$$k == \frac{1}{50} \mathrm{Log}\left[\frac{465\,527\,045\,498\,919}{230\,070\,459\,587\,056}\right] \,\&\&\, M == \frac{6\,483\,003\,949\,633\,687\,435\,565}{77\,980\,192\,532\,239}$$

```
par = Solve[reduced, {k, M}]
```

$$\left\{\left\{k \to \frac{1}{50} \mathrm{Log}\left[\frac{465\,527\,045\,498\,919}{230\,070\,459\,587\,056}\right],\ M \to \frac{6\,483\,003\,949\,633\,687\,435\,565}{77\,980\,192\,532\,239}\right\}\right\}$$

```
k = N[First[k /. par]]
```

0.0140957

```
M = N[Last[M /. par]]
```

8.31365×10^7

```
p[t]
```

$$\frac{1.35892 \times 10^{15}\, e^{0.0140957 t}}{6.67909 \times 10^7 + 16\,345\,646\, e^{0.0140957 t}}$$

```
p[200]
```

6.68424×10^7

```
Plot[{p[t], M, p0}, {t, 0, 500}, AxesOrigin → {0, p0}]
```

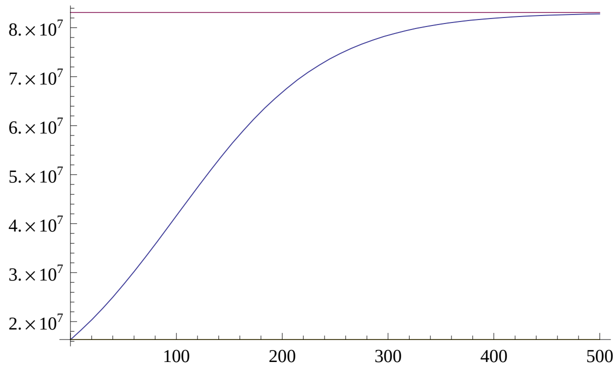

Figure 4: *Mathematica* code with the analysis of Census Data using Logistic Equation.

Essentially, the theorem says that if the function $f(x,t)$ and its derivative with respect to x are both 'sufficiently nice', the the initial value problem will have a unique solution, at least for t close to t_0.

The following examples illustrate that all of the assumptions of the theorem are essential.

Example 5.2. (Blow–Up) Consider again the initial value problem (4.1), that is

$$\frac{dx}{dt} = x^2, \qquad x(0) = x_0.$$

Here

$$f(x,t) = x^2 \quad \text{and} \quad \frac{\partial f}{\partial x}(x,t) = 2x,$$

so the function $f(x,t)$ and its derivative with respect to x are both continuous for all x and t. Thus all assumptions of Theorem 5.1 are satisfied and the initial value problem must have a unique solution. In Example 4.1, we have seen already that a particular solution of the problem is given by

$$x(t) = \frac{1}{x_0^{-1} - t}.$$

Using the uniqueness claim of Theorem 5.1, we conclude that this is in fact the only solution of the problem!

Note that if $x_0 > 0$ then the solution $x(t)$ is defined for $0 \leq t < x_0^{-1}$ and 'blows-up' (has a vertical asymptote) when $T = x_0^{-1}$ (see also Figure 4.1). In other words, the *maximal existence interval* of such solution is $[0, x_0^{-1})$. This example shows, in particular, that possible restriction of the existence interval $[0,T)$ in the statement of Theorem 5.1 is indeed essential!

Exercise 5.3. Let $\alpha > 1$. Find the unique solution of the initial value problem

$$\frac{dx}{dt} = x^\alpha, \qquad x(0) = x_0,$$

and determine its maximal existence interval.

Example 5.4. (Nonuniqueness) Consider the initial value problem

$$\frac{dx}{dt} = \sqrt{x}, \qquad x(0) = 0. \tag{5.2}$$

Here

$$f(x,t) = \sqrt{x} \quad \text{and} \quad \frac{\partial f}{\partial x}(x,t) = \frac{1}{2\sqrt{x}}.$$

The function $f(x,t)$ is defined and continuous for all $x \geq 0$, however its x–partial derivative is discontinuous when $x = 0$. In particular, conditions of Theorem 5.1 fail when $x_0 = 0$.

```
x[c_, t_] = Piecewise[{{0, t < c}, {(t - c)^2 / 4, t > c}}]
```

$$\begin{cases} 0 & t < c \\ \frac{1}{4}(-c+t)^2 & t > c \end{cases}$$

```
Plot[{x[1, t], x[2, t], x[3, t]}, {t, 0, 5}]
```

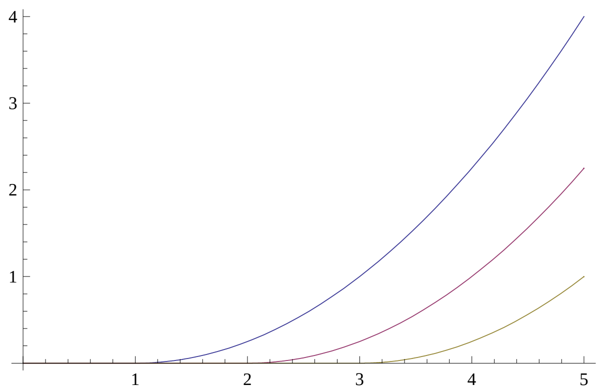

Figure 5: *Mathematica* code which plots the graph of the solution $x_c(t)$ of equation (5.4) for $c = 1; 2; 3$.

It easy to see that problem (5.2) has an obvious solution $x(t) = 0$, for all $t \geq 0$. But if we chose any value $c > 0$, then the function

$$x_c(t) = \begin{cases} 0, & t \leq c, \\ \frac{1}{4}(t-c)^2, & t > c, \end{cases} \tag{5.3}$$

also satisfies the equation and the initial data, for all $t \geq 0$. Thus initial value problem (5.2) has infinitely many solutions!

Exercise 5.5. Let $0 < \alpha < 1$. Find infinitely many solutions of the initial value problem

$$\frac{dx}{dt} = x^\alpha, \qquad x(0) = 0.$$

Hint. Use separation of variables and a construction similar to (5.3).

Example 5.6. (Nonexistence) Consider the initial value problem

$$t^2 \frac{dx}{dt} = x^2, \qquad x(0) = 1. \tag{5.4}$$

In order to fit the problem into the setting of Theorem 5.1, we rewrite (5.4) in the equivalent form

$$\frac{dx}{dt} = \frac{x^2}{t^2}, \qquad x(0) = 1,$$

so that

$$f(x,t) = \frac{x^2}{t^2}.$$

It is clear that $f(x,t)$ is discontinuous when $t = 0$ and $x \neq 0$. In particular, conditions of Theorem 5.1 fail for all $t_0 = 0$ and $x_0 = 1$.

Indeed, it is easy to see that initial problem (5.4) has no solutions when $t_0 = 0$ and $x_0 = 1$: if $t = 0$ then the left hand of the differential equation in (5.4) is zero, and we must have $x(0) = 0$.

Exercise 5.7. Initial value problem

$$t^2 \frac{dx}{dt} = x^2, \qquad x(0) = 0. \tag{5.5}$$

has an obvious solution $x(t) = 0$, for all $t \geq 0$. Show that for any $c \in \mathbb{R}$, the function

$$x_c(t) = \frac{t}{ct - 1}$$

is a solution of (5.5) for $0 \leq t < 1/c$, and hence (5.5) has many solutions.

6 Linear ODEs and integrating factor

In this section we will look at general 1st order linear inhomogeneous ODEs

$$a_1(t)\dot{x} + a_0(t)x = f(t).$$

(compare with Definition 2.2). Here $x(t)$ is the unknown function of one independent variable t, and $a_1(t)$, $a_0(t)$, $f(t)$ are coefficients. In what follows we shall assume that $a_1(t) \neq 0$ and divide by $a_1(t)$ to obtain

$$\dot{x} + p(t)x = q(t),$$

where $p(t) := a_0(t)/a_1(t)$ and $q(t) := f(t)/a_1(t)$.

6.1 Constant coefficients

First we consider the simplest case when both coefficient p and q are constant,

$$\dot{x} + px = q. \tag{6.1}$$

This is a separable ODE which could be solved by the method of separation of variables, see Problem Sheet 3, Q2(b). However, we are going to solve it by another method, which involves a certain trick, which is called "integrating factor" technique. It will be later useful for more general equations.

The key point is to observe, using the product rule, that

$$\frac{d}{dt}\big(x(t)e^{pt}\big) = \frac{dx}{dt}e^{pt} + pxe^{pt} = e^{pt}\Big(\frac{dx}{dt} + px\Big). \tag{6.2}$$

The right hand side of (6.2) is the same as the left hand side of our ODE (6.1), except that it is multiplied by a factor e^{pt}. If we multiply both sides of (6.1) by e^{pt} we obtain

$$e^{pt}\Big(\frac{dx}{dt} + px\Big) = q\, e^{pt}.$$

Using (6.2), this is simply

$$\frac{d}{dt}\big(x(t)e^{pt}\big) = q\, e^{pt}.$$

To obtain the general solution, we integrate both sides

$$\int \frac{d}{dt}\big(x(t)e^{pt}\big)\, dt = \int q\, e^{pt}\, dt,$$

to obtain

$$x(t)e^{pt} = \frac{q}{p}e^{pt} + C.$$

Hence

$$\boxed{x(t) = \frac{q}{p} + Ce^{-pt}} \tag{6.3}$$

is the general solution of (6.1).

Further, if we want to find a particular soltuion which satisfies the initial data $x(t_0) = x_0$ then we need

$$x(0) = \frac{q}{p} + Ce^{-pt_0} = x_0, \qquad \Longrightarrow \qquad C = \Big(x_0 - \frac{q}{p}\Big)e^{pt_0},$$

so this particular solution is given by the formula

$$\boxed{x(t) = \frac{q}{p} + \Big(x_0 - \frac{q}{p}\Big)e^{-p(t-t_0)}.} \tag{6.4}$$

Compare this result with the computation in Problem Sheet 3, Q2(b)!

6.2 Variable coefficients

We now use the same integrating factor trick for the more general equation with variable coefficients

$$\dot{x} + p(t)x = q(t). \tag{6.5}$$

Define the *integrating factor* for equation (6.5) by the formula

$$\boxed{I(t) = e^{\int p(t)dt}.}$$

and observe that, similarly to (6.2), by the product rule we have

$$\begin{aligned} \frac{d}{dt}\big(x(t)I(t)\big) &= \frac{d}{dt}\big(x(t)e^{\int p(t)dt}\big) = \frac{dx}{dt}e^{\int p(t)dt} + p(t)x\, e^{\int p(t)dt} \\ &= I(t)\Big(\frac{dx}{dt} + p(t)x\Big). \end{aligned} \tag{6.6}$$

If we multiply both sides of equation (6.5) by the integrating factor $I(x)$ we obtain

$$I(t)\Big(\frac{dx}{dt} + p(t)x\Big) = I(t)q(t).$$

Using (6.6), this is simply

$$\frac{d}{dt}\Big(x(t)e^{\int p(t)dt}\Big) = q(t)\, e^{\int p(t)dt}.$$

To obtain the general solution, we integrate both sides

$$\int \frac{d}{dt}\Big(x(t)e^{\int p(t)dt}\Big)\,dt = \int q(t)\,e^{\int p(t)dt}\,dt,$$

to obtain

$$x(t)e^{\int p(t)dt} = \int q(t)\,e^{\int p(t)dt}\,dt.$$

Hence

$$\boxed{x(t) = e^{-\int p(t)dt}\int q(t)\,e^{\int p(t)dt}\,dt} \tag{6.7}$$

is the general solution of (6.1). Although formula (6.7) might be looking complicated, in practice it is often not difficult to handle. Consider some examples.

Example 6.1. Find the particular solution of the initial value problem

$$\dot{x} + 3x = t, \qquad x(0) = 8/9.$$

Solution. The integrating factor for this ODE is

$$I(t) = e^{\int 3\,dt} = e^{3t}.$$

Multiplying the equation by $I(t)$ we obtain

$$e^{3t}\Big(\frac{dx}{dt} + 3x\Big) = te^{3t},$$

or, by (6.6) this is equivalent to

$$\frac{d}{dt}\left(e^{3t}x\right) = te^{3t}.$$

Integrating both sides

$$\int \frac{d}{dt}\left(e^{3t}x\right)dt = \int te^{3t}dt$$

and taking into account that

$$\int te^{3t}dt = \frac{t}{3}e^{3t} - \frac{1}{9}e^{3t},$$

we obtain

$$e^{3t}x(t) = \frac{t}{3}e^{3t} - \frac{1}{9}e^{3t} + C,$$

so the general solution is

$$x(t) = \frac{t}{3} - \frac{1}{9} + Ce^{-3t}.$$

Since $x(0) = 8/9$ and hence

$$x(0) = -\frac{1}{9} + C = 8/9,$$

we conclude that $C = 1$. Therefore

$$x(t) = \frac{t}{3} - \frac{1}{9} + e^{-3t}$$

is the required particular solution of the initial value problem.

Example 6.2. Find the general solution of the equation

$$(x^2+1)\frac{dy}{dx} + 4xy = 12x.$$

Solution. If we divide both sides by $x^2 + 1$ then we obtain equation in the form (6.5):

$$\frac{dy}{dx} + \underbrace{\frac{4x}{x^2+1}}_{p(x)} y = \underbrace{\frac{12x}{x^2+1}}_{q(x)}.$$

The integrating factor for this ODE is

$$I(x) = e^{\int \frac{4x}{x^2+1}\,dx} = e^{2\log(x^2+1)} = (x^2+1)^2.$$

Multiplying the equation by $I(x)$ we obtain

$$(x^2+1)^2\Big(\frac{dy}{dx} + \frac{4x}{x^2+1}y\Big) = 12x(x^2+1),$$

or, by (6.6) this is equivalent to

$$\frac{d}{dx}\big((x^2+1)^2 y\big) = 12x(x^2+1).$$

Integrating both sides

$$\int \frac{d}{dx}\big((x^2+1)^2 y\big)\,dx = \int 12x(x^2+1)dx,$$

we obtain

$$(x^2+1)^2 y(x) = 3(x^2+1)^2 + C,$$

so the required general solution is

$$y(x) = 3 + \frac{C}{(x^2+1)^2}.$$

Note that $y(x) \to 3$ as $x \to \infty$, for any value of the arbitrary constant C!

Example 6.3. Find the general solution of the equation

$$\frac{dy}{dx} + \frac{y}{x} = x^2.$$

Solution. The integrating factor for this ODE is

$$I(x) = e^{\int \frac{1}{x}\,dx} = e^{\log(x)} = x.$$

Multiplying the equation by $I(x)$ we obtain

$$x\frac{dy}{dx} + y = x^3,$$

or, by (6.6) this is equivalent to

$$\frac{d}{dx}(xy) = x^3.$$

Integrating both sides

$$\int \frac{d}{dx}(xy)\,dx = \int x^3 dx,$$

we obtain

$$xy(x) = \frac{x^4}{4} + C,$$

so the required general solution is

$$y(x) = \frac{x^3}{4} + \frac{C}{x}.$$

Note that the only solution which is finite when $x = 0$ is the solution obtained by taking $C = 0$! [4]

6.3 Newton's law of cooling

An important example of a linear equation arises from *Newton's law of cooling*, which states that the temperature $T(t)$ of an object in surroundings of temperature $A(t)$ is governed by the ODE

$$\dot{T} = -k(T - A(t)), \tag{6.8}$$

where $k > 0$ measures the rate that heat is absorbed (or emitted) by the object. Equation (6.8) can be rewritten in the form

$$\dot{T} + kT = kA(t),$$

so it is a 1st order linear inhomogeneous ODE which could be solved by the integrating factor method.

Newton's law of cooling plays important role in many applications.

[4] Compare with Example 5.6.

Example 6.4. (ESTIMATING TIME OF DEATH.) A forensic method for estimating the time of death of a body is based on Newton's law of cooling. The idea is to take temperature of the body at two different different times in order to give an estimate of the constant k to be used in (6.8).

To consider a specific example, suppose that a body is found in a room, which is kept at a constant temperature $24°C$. At 8 a.m. in the morning its temperature is $28°C$, while an hour later it is $26°C$.

With the time t measured in hours, we need to solve the equation

$$\dot{T} + kT = 24k.$$

This is a linear 1st order ODE with constant coefficients, so according to (6.4) its particular solution $T(t)$ which satisfies the initial condition $T(t_0) = T_0$ is given by the formula

$$T(t) = 24 + \Big(T_0 - 24\Big)e^{-k(t-t_0)}. \tag{6.9}$$

Using the available data, we can set $t_0 = 8$ so that $T_0 = T(8) = 28$ and hence

$$T(t) = 24 + \Big(28 - 24\Big)e^{-k(t-8)}.$$

We also know that $T(9) = 26$ and therefore

$$T(9) = 24 + \Big(28 - 24\Big)e^{-k(9-8)} = 24 + 4e^{-k} = 26 \quad \Longrightarrow \quad e^{-k} = 1/2,$$

which gives $k = \log(2)$.

Now we can use the value of k to estimate the actual time of death. If the time of death was t_0 then $T_0 = T(t_0) = 37$ (unless the deceased had a fever at the time of death). Using (6.9) and the measurement $T(8) = 28$ again we obtain

$$T(8) = 24 + \Big(37 - 24\Big)e^{-\log(2)(8-t_0)} = 28 \quad \Longrightarrow \quad e^{-\log(2)(8-t_0)} = 4/13.$$

Taking the logarithms gives

$$-\log(2)(8 - t_0) = \log(4) - \log(13) \quad \Longrightarrow \quad t_0 = 8 - \frac{\log(13) - \log(4)}{\log(2)} \simeq 6.3,$$

putting time of death approximately at 6.20 a.m.

Exercise 6.5. Use Newtons law of cooling to solve the following problem. At 7 a.m. I made a cup of tea; after adding some milk it is about 90 °C. When I left at 7.30 a.m. the tea is still drinkable at about 40 °C. When I get back home at 8 a.m. the tea has cooled to 30 °C.

Assuming that the temperature of the house is constant, write down the differential equation for the temperature of tea. What is the temperature of my house ?

7 Some tricks for solving ODEs

In this section we discuss several tricks which could be used to solve some particular classes of ODEs.

7.1 Substitution method

In some cases an equation could be simplified considerably by making an appropriate substitution. Below we consider two particular classes of equations which could be solved in this way.

7.1.1 Homogeneous equations

A 1st order ODE is called homogeneous if it can be written in the form

$$\frac{dy}{dx} = F\Big(\frac{y}{x}\Big). \tag{7.1}$$

In this case we make the substitution

$$\boxed{u = \frac{y}{x}.}$$

Then

$$y = ux \qquad \text{and} \qquad \frac{dy}{dx} = x\frac{du}{dx} + u.$$

Substituting into equation (7.1), we obtain the new equation

$$x\frac{du}{dx} + u = F(u), \tag{7.2}$$

which is a *separable* equation and could be integrated using the methods developed in Section 4!

Example 7.1. Find the general solution of the equation

$$xy\frac{dy}{dx} = 2x^2 + 3y^2.$$

Solution. Divide the equation by xy to obtain

$$\frac{dy}{dx} = \underbrace{2\frac{x}{y} + 3\frac{y}{x}}_{F\left(\frac{x}{y}\right)}.$$

This is a homogeneous equation. Making the substitution

$$u = \frac{y}{x},$$

we obtain

$$x\frac{du}{dx} + u = \frac{2}{u} + 3u,$$

or

$$x\frac{du}{dx} = 2\frac{1+u^2}{u},$$

which is a separable equation. Separating the variables, we obtain

$$\frac{u}{1+u^2}du = 2\frac{dx}{x}.$$

After integration on both sides we obtain the general solution

$$u = \pm\sqrt{C^2x^4 - 1},$$

and after 'resubstitution' $u = \frac{y}{x}$, we finally obtain the required general solution

$$y(x) = \pm x\sqrt{C^2x^4 - 1}.$$

7.1.2 Bernoulli equations

Bernoulli equation is an equation of the type

$$\frac{dy}{dx} + p(x)y = q(x)y^n.$$

When $n = 0$ or 1 this is just linear equation. For other, possibly negative values of n this equation is nonlinear. It appears, that the substitution

$$\boxed{u = y^{1-n}}$$

turns it into a linear equation. Indeed,

$$\begin{aligned}
\frac{du}{dx} &= (1-n)y^{-n}\frac{dy}{dx} \\
&= (1-n)y^{-n}\left(q(x)y^n - p(x)y\right) \\
&= (1-n)\left(q(x) - p(x)y^{1-n}\right) \\
&= (1-n)\left(q(x) - p(x)u\right),
\end{aligned}$$

so we obtain the linear equation

$$\frac{du}{dx} + (1-n)p(x)u = (1-n)q(x),$$

which could be integrated using the integrating factor method of Section 6.

8 2nd order linear ODE

An ODE of the form

$$a(t)\ddot{x} + b(t)\dot{x} + c(t)x = f(t), \tag{8.1}$$

where $a(t)$, $b(t)$, $c(t)$ and $f(t)$ are continuous functions with $a(t) \neq 0$ is called a *homogeneous 2nd order linear ODE*. If $f(t) = 0$ then the equation is said to be *homogeneous*, otherwise the equation is said to be *inhomogeneous*. Equation (8.1), coupled with *two* initial conditions

$$x(t_0) = x_0, \quad \dot{x}(t_0) = x_1$$

is called a *2nd order initial value problem* for equation (8.1). Note that in order to determine the solution uniquely, two initial conditions are required for the 2nd order equations.

In this section we discuss methods of solving some simple 2nd order ODEs.

8.1 Constant coefficients

An ODE of the form

$$a\ddot{x} + b\dot{x} + cx = 0, \tag{8.2}$$

where a, b and c are real numbers with $a \neq 0$, is called a *homogeneous 2nd order linear ODE with constant coefficients*.

Our first important observation about linear homogeneous ODEs is the following important principle.

Superposition Principle. If $x_1(t)$ and $x_2(t)$ are solutions of the equation (8.2) then for any real numbers A and B the *superposition*

$$Ax_1(t) + Bx_2(t)$$

is also a solutions of the equation (8.2).

Definition 8.1. The quadratic equation

$$\boxed{ak^2 + bk + c = 0}$$

is called *characteristic equation* of the ODE (8.2).

We are going to give explicit formulas for the general solution of (8.2), which will depend on the number of roots of the characteristic equation.

The roots of the characteristic equation are determined by the formula

$$k = \frac{-b \pm \sqrt{D}}{2a},$$

where the quantity

$$D = b^2 - 4ac$$

is often called the *discriminant* of the characteristic equation.

- **Case** $D > 0$ **– distinct roots.** If $D > 0$ then

$$k_{1,2} = \frac{-b \pm \sqrt{D}}{2a},$$

are two distinct roots of the characteristic equation. In this case the general solution of the ODE (8.2) is

$$\boxed{x(t) = Ae^{k_1 t} + Be^{k_2 t},}$$

where A and B are arbitrary real constants.

Example 8.2. Find the solution of the initial value problem

$$\ddot{x} + \dot{x} - 6x = 0, \qquad x(0) = 0, \quad \dot{x}(0) = 5.$$

Solution. The characteristic equation is

$$k^2 + k - 6 = 0.$$

The discriminant $D = 25 > 0$ and the roots of the characteristic equation are $k_1 = -3$ and $k_2 = 2$. Then the general solution of the equation is

$$x(t) = Ae^{-3t} + Be^{2t}.$$

To find the solution of the initial value problem we first compute the derivative of the general solution:

$$\dot{x}(t) = -3Ae^{-3t} + 2Be^{2t}.$$

Using the initial conditions at $t = 0$ we obtain

$$x(0) = Ae^0 + Be^0 = A + B = 0,$$

$$\dot{x}(0) = -3Ae^0 + 2Be^0 = -3A + 2B = 2.$$

Therefore $A = -1$, $B = 1$ and the required solution of the initial value problem is

$$\boxed{x(t) = -e^{-3t} + e^{2t}.}$$

Example 8.3. Find the general solution of the equation

$$\ddot{x} - 2\dot{x} = 0,$$

and a solution which satisfies two *boundary conditions* conditions $x(0) = 0$ and $x(1) = 1$.

Solution. The characteristic equation is

$$k^2 - 2k = 0.$$

It has two roots $k_1 = 0$ and $k_2 = 2$. Then the general solution of the equation is

$$\boxed{x(t) = A + Be^{2t}.}$$

To satisfy the boundary conditions, we first substitute $t = 0$ and $t = 1$ into the general solution to obtain

$$\underbrace{x(0)}_{=0} = A + Be^0 = A + B = 0,$$

$$\underbrace{x(1)}_{=1} = A + Be^2 = 1.$$

Therefore $A = -\frac{1}{e^2-1}$, $B = \frac{1}{e^2-1}$ and hence the required solution is

$$\boxed{x(t) = -\frac{1}{e^2-1} + \frac{1}{e^2-1}e^{2t}.}$$

- **Case $D = 0$ – repeated root.** If $D = 0$ then

$$k = -\frac{b}{2a}$$

is the only (repeated) root of the characteristic equation. In this case the general solution of the ODE (8.2) is

$$\boxed{x(t) = Ae^{kt} + Bte^{kt},}$$

where A and B are arbitrary real constants.

Example 8.4. Find the solution of the initial value problem

$$\ddot{x} + 2\dot{x} + x = 0, \qquad x(0) = 0, \quad \dot{x}(0) = 1.$$

Solution. The characteristic equation is

$$k^2 + 2k + 1 = 0.$$

The discriminant $D = 0$ and the equation has repeated root $k = -1$. Then the general solution of the equation is

$$x(t) = Ae^{-t} + Bte^{-t}.$$

To find the solution of the initial value problem we first compute the derivative of the general solution:

$$\dot{x}(t) = (B - A)e^{-t} - Bte^{-t}.$$

Using the initial conditions at $t = 0$ we obtain

$$x(0) = A = 0,$$

$$\dot{x}(0) = B - A = 1,$$

so that $A = 0$, $B = 1$ and the required solution of the initial value problem is

$$\boxed{x(t) = te^{-t}.}$$

- **Case** $D < 0$ **– complex roots.** If $D < 0$ then

$$k_{1,2} = \underbrace{-\frac{b}{2a}}_{\rho} \pm i\, \underbrace{\frac{\sqrt{-D}}{2a}}_{\omega},$$

are two complex roots of the characteristic equation. In this case the general solution of the ODE (8.2) is

$$\boxed{x(t) = e^{\rho t}(A\cos(\omega t) + B\sin(\omega t)),}$$

where A and B are arbitrary real constants.

Example 8.5. Find the solution of the initial value problem

$$\ddot{x} + 2\dot{x} + 5x = 0, \qquad x(0) = 1, \quad \dot{x}(0) = 0.$$

Solution. The characteristic equation is

$$k^2 + 2k + 5 = 0.$$

The discriminant $D = -16 < 0$ and the characteristic equation has two complex roots

$$k_{1,2} = -1 \pm \sqrt{-4} = -1 \pm 2i.$$

Then the general solution of the equation is

$$\boxed{x(t) = e^{-t}\big(A\cos(2t) + B\sin(2t)\big).}$$

To find the solution of the initial value problem we first compute the derivative of the general solution:

$$\dot{x}(t) = e^{-t}\big((2B - A)\cos(2t) - (2A + B)\sin(2t)\big).$$

Using the initial conditions at $t = 0$ we obtain

$$x(0) = A = 1,$$

$$\dot{x}(0) = 2B - A = 0.$$

Therefore $A = 1$, $B = 1/2$ and the required solution of the initial value problem is

$$\boxed{x(t) = e^{-t}\big(\cos(2t) + \frac{1}{2}\sin(2t)\big).}$$

```
ode = DSolve[{x''[t] + 2*x'[t] + 5*x[t] == 0, x[0] == 1, x'[0] == 0}, x[t], t]
```

$$\left\{\left\{x[t] \to \frac{1}{2} e^{-t} (2\,\text{Cos}[2\,t] + \text{Sin}[2\,t])\right\}\right\}$$

```
Plot[Evaluate[x[t] /. ode], {t, 0, 2*Pi}, PlotRange -> All]
```

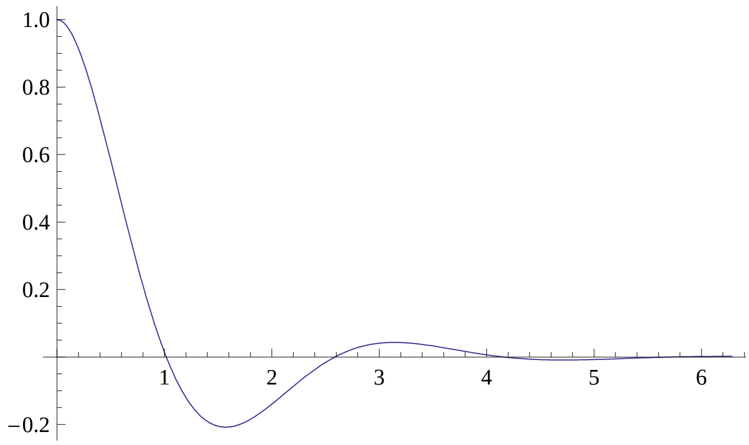

Figure 6: *Mathematica* code which plots the graph of the oscillating solution in Example 8.5.

8.2 Inhomogeneous equations

An ODE of the form

$$a\ddot{x} + b\dot{x} + cx = f(t), \tag{8.3}$$

where a, b and c are real numbers with $a \neq 0$, and $f(t)$ is a nonzero continuous function, is called an *inhomogeneous 2nd order linear ODE with constant coefficients*.

Complementary function. The *complementary function* of equation (8.3) is the general solution of the homogeneous equation

$$a\ddot{x} + b\dot{x} + cx = 0.$$

Particular integral. The *particular integral* is any particular solution of inhomogeneous equation (8.3).

General solution of inhomogeneous equation (8.3)**.** The general solution of inhomogeneous equation (8.3) is given by the formula

$$\boxed{x(t) = x_c(t) + x_p(t),}$$

where $x_c(t)$ is the complementary function and $x_p(t)$ a particular integral.

There is no general methods for finding particular integrals of inhomogeneous equations. Below we consider several specific types of examples, where a particular integral could be found.

8.2.1 Polynomial $f(t)$.

When $f(t)$ is a polynomial $f(t) = c_n t^n + \cdots + c_1 t + c_0$ of degree n then a particular integral usually can be found in the form of a polynomial of the same degree n. If however, $f(t)$ or its derivative is a solution of the homogeneous equation then one has to multiply our guess solution by an additional factor t. Consider two examples.

Example 8.6. Find the general solution of the equation

$$\ddot{x} - 4x = t^2.$$

Solution. The complementary function $x_c(t)$ is the general solution of the homogeneous ODE

$$\ddot{x} - 4x = 0,$$

which is

$$x_c(t) = Ae^{-2t} + Be^{2t}.$$

To solve the inhomogeneous ODE we need to find a particular integral $x_p(t)$. The function on the right hand side, $f(t) = t^2$, is a 2nd order polynomial. Neither $f(t)$ nor any of its derivative are solutions of the homogeneous equation. So we try to find a particular integral in the form of a 2nd order polynomial

$$x_p(t) = at^2 + bt + c.$$

Then

$$\dot{x}_p(t) = 2at + b, \qquad \ddot{x}_p(t) = 2a.$$

Substituting into the equation we get

$$(2a) - 4(at^2 + bt + c) = t^2.$$

This requires $a = -1/4$, $b = 0$ and $c = -1/8$. So the particular integral is

$$x_p(t) = -\frac{1}{4}t^2 - \frac{1}{8}.$$

Hence the general solution of the inhomogeneous equation is

$$\boxed{x(t) = Ae^{-2t} + Be^{2t} - \frac{1}{4}t^2 - \frac{1}{8}.}$$

Example 8.7. Find the general solution of the equation

$$\ddot{x} - 2\dot{x} = 4.$$

Solution. Example 8.3 The complementary function $x_c(t)$ is the general solution of the homogeneous ODE

$$\ddot{x} - 2x = 0,$$

which is

$$x_c(t) = A + Be^{2t},$$

see Example 8.3.

To solve the inhomogeneous ODE we need to find a particular integral $x_p(t)$. The function on the right hand side, $f(t) = 4$, can be seen as a zero order polynomial. Note that $f(t)$ is a solution of the homogeneous equation. If we try

$$x_p(t) = a$$

as a particular integral then after substitution into the inhomogeneous equation we will get

$$0 = 4,$$

so $x_p(t) = a$ is a wrong choice.

Instead of $x_p(t) = a$ we multiply our 'guess solution' by an extra factor t, and try a particular integral in the form

$$x_p(t) = at.$$

Then

$$\dot{x}_p(t) = a, \qquad \ddot{x}_p(t) = 0.$$

Substituting into the equation we get

$$0 - 2a = 4.$$

This requires $a = -2$, and so the particular integral is

$$x_p(t) = -2t.$$

Hence the general solution of the inhomogeneous equation is

$$\boxed{x_c(t) = A + Be^{2t} - 2t.}$$

8.2.2 Exponential $f(t)$.

When $f(t)$ is an exponential function of the form ce^{kt} then a particular integral usually can be found in the form of the same exponential function with a different coefficient, e.g. ae^{kt}. However, if e^{kt} is a solution of the homogeneous equation then we have to try ate^{kt}, or, if k is the repeated root of the characteristic equation then we try at^2e^{kt}. Consider two examples.

Example 8.8. Find the general solution of the equation

$$\ddot{x} + \dot{x} - 6x = e^{-2t}.$$

Solution. The complementary function $x_c(t)$ is the general solution of the homogeneous ODE

$$\ddot{x} + \dot{x} - 6x = 0,$$

which is

$$x_c(t) = Ae^{-3t} + Be^{2t},$$

see Example 8.2.

To solve the inhomogeneous ODE we need to find a particular integral $x_p(t)$. The function on the right hand side, $f(t) = e^{-2t}$ is an exponential function, which is not a solution of the homogeneous equation. We try to find a particular integral in the form

$$x_p(t) = ae^{-2t}.$$

Then

$$\dot{x}_p(t) = -2ae^{-2t}, \qquad \ddot{x}_p(t) = 4ae^{-2t}.$$

Substituting into the equation we get

$$(4a - 2a - 6a)e^{-2t} = e^{-2t}.$$

This requires $a = -1/4$. So the particular integral is

$$x_p(t) = -\frac{1}{4}e^{-2t}.$$

Hence the general solution of the inhomogeneous equation is

$$\boxed{x(t) = Ae^{-3t} + Be^{2t} - \frac{1}{4}e^{-2t}.}$$

Example 8.9. Find the general solution of the equation

$$\ddot{x} + \dot{x} - 6x = 5e^{-3t}.$$

Solution. The complementary function $x_c(t)$ here is the same as in the previous example:

$$x_c(t) = Ae^{-3t} + Be^{2t}.$$

To solve the inhomogeneous ODE we need to find a particular integral $x_p(t)$. The function on the right hand side, $f(t) = e^{-3t}$ is an exponential function, which is a solution of the homogeneous equation. Therefore we try to find a particular integral in the form

$$x_p(t) = ate^{-3t}.$$

Then

$$\dot{x}_p(t) = -3ate^{-3t} + ae^{-3t}, \qquad \ddot{x}_p(t) = 9ate^{-3t} - 6ae^{-3t}.$$

Substituting into the equation we get

$$(9ate^{-3t} - 6ae^{-3t}) + (-3ate^{-3t} + ae^{-3t}) - 6(ate^{-3t}) = 5e^{-3t}.$$

The terms te^{-3t} cancel, so we get

$$-5ae^{-3t} = 5e^{-3t}.$$

This requires $a = -1$. So the particular integral is

$$x_p(t) = -te^{-3t}.$$

Hence the general solution of the inhomogeneous equation is

$$\boxed{x(t) = Ae^{-3t} + Be^{2t} - te^{-3t}.}$$

8.2.3 Trigonometric $f(t)$.

When $f(t)$ is a trigonometric function such as $\sin(kt)$ or $\cos(kt)$ then a particular integral usually can be found in the form

$$x_p(t) = a\sin(kt) + b\cos(kt).$$

Note that $x_p(t)$ has to include *both* sin and cos components even $f(t)$ involves only one of those two. However, if $\sin(kt)$ and $\cos(kt)$ satisfy the homogeneous equation then we have to try

$$x_p(t) = at\sin(kt) + bt\cos(kt),$$

with an extra factor of t. Consider examples.

Example 8.10. Find the general solution of the equation

$$\ddot{x} + 2\dot{x} + x = 100\cos(2t).$$

Solution. The complementary function $x_c(t)$ is the general solution of the homogeneous ODE

$$\ddot{x} + 2\dot{x} + x = 0,$$

which is

$$x_c(t) = (A + Bt)e^{-t},$$

see Example 8.4.

To solve the inhomogeneous ODE we need to find a particular integral $x_p(t)$. The function on the right hand side, $f(t) = 100\cos(2t)$ is clearly not a solution of the homogeneous equation. We then try to find a particular integral in the form

$$x_p(t) = a\cos(2t) + b\sin(2t).$$

Then

$$\dot{x}_p(t) = -2a\sin(2t) + 2b\cos(t), \qquad \ddot{x}_p(t) = -4a\cos(2t) - 4b\sin(2t).$$

Substituting into the equation we get

$$\big(-4a\cos(2t) - 4b\sin(2t)\big) + 2\big(-2a\sin(2t) + 2b\cos(t)\big) + \big(a\cos(2t) + b\sin(2t)\big) = 100\cos(2t),$$

or

$$(4b - 3a)\cos(2t) - (3b + 4a)\sin(2t) = 100\cos(2t).$$

This requires

$$4b - 3a = 100, \qquad 3b + 4a = 0.$$

Therefore we get $b = 16$ and $a = -12$. So the particular integral is

$$x_p(t) = -12\cos(2t) + 16\sin(2t).$$

Hence the general solution of the inhomogeneous equation is

$$\boxed{x(t) = (A + Bt)e^{-t} - 12\cos(2t) + 16\sin(2t).}$$

Example 8.11. Find the general solution of the equation

$$\ddot{x} + x = 8\cos(t).$$

Solution. The complementary function $x_c(t)$ is the general solution of the homogeneous ODE

$$\ddot{x} + x = 0,$$

which is

$$x_c(t) = A\sin(t) + B\cos(t).$$

To solve the inhomogeneous ODE we need to find a particular integral $x_p(t)$. The right hand side, $f(t) = 8\cos(t)$, is a solution of the homogeneous equation. We then try to find a particular integral multiplying out the original guess by t, in the form

$$x_p(t) = at\cos(t) + bt\sin(t).$$

Then

$$\dot{x}_p(t) = b\sin(t) + bt\cos(t) + a\cos(t) - at\sin(t),$$
$$\ddot{x}_p(t) = 2b\cos(t) - bt\sin(t) - 2a\sin(t) - at\cos(t).$$

Substituting into the equation we get

$$2b\cos(t) - 2a\sin(t) = 8\cos(t).$$

This requires $b = 4$ and $a = 0$. So the particular integral is

$$x_p(t) = 4t\sin(t).$$

Hence the general solution of the inhomogeneous equation is

$$\boxed{x(t) = A\sin(t) + B\cos(t) + 4t\sin(t).}$$

8.2.4 Rule of thumb and more general $f(t)$.

As a rule of thumb for finding a particular integrals for 2nd order equations:

- The standard guess is a general version of the right hand side $f(t)$, e.g. if $f(t)$ is an n–thy order polynomial then try a *general* n–th order polynomial as a "prototype" solution
- If the standard guess contains a term that satisfy the homogeneous equation then multiply the standard guess by t and try again. Repeat until your guess no longer contains any term that solve homogeneous equation.

More general $f(t)$. Similar consideration could be used for other choices of right hand sides, e.g. when $f(t)$ is a trigonometric function. The following rule could be used to construct a particular solution of inhomogeneous equations with "composite" right hand sides $f(t)$.

Exercise 8.12. Let $x_1(t)$ be a particular solution of the equation

$$a\ddot{x} + b\dot{x} + cx = f_1(t),$$

and $x_2(t)$ be a particular solution of the equation

$$a\ddot{x} + b\dot{x} + cx = f_2(t).$$

Verify that then $\alpha x_1(t) + \beta x_2(t)$ is a particular solution of the equation

$$a\ddot{x} + b\dot{x} + cx = \alpha f_1(t) + \beta f_2(t).$$

Example 8.13. Find the general solution of the equation

$$\ddot{x} + \dot{x} - 6x = 5e^{-3t} - 2e^{-2t}.$$

Solution. Using Examples 8.8 and 8.9, we conclude that the general solution is given by

$$\boxed{x(t) = Ae^{-3t} + Be^{2t} - te^{-3t} + 2e^{-2t}.}$$

8.3 Oscillations of the Millennium Bridge.

When first opened, the Millennium Bridge in London wobbled from side to side as people crossed.[5] Footfalls created small side-to-side movements of the bridge, which were then enhanced by the tendency of people to adjust their steps to compensate for the wobbling. With more then a critical number of pedestrians the bridge began to wobble violently.

Without any pedestrians, the displacement x of a representative point on the bridge away from its normal position would satisfy

$$M\ddot{x} + \mu\dot{x} + \lambda x = 0, \tag{8.4}$$

where

$$M \approx 4 \times 10^5 \text{ kg}, \quad \mu \approx 5 \times 10^4 \text{ kg/s}, \quad \lambda \approx 10^7 \text{ kg/s}^2.$$

Since $D = \mu^2 - 4M\lambda \approx -1.6 \times 10^{13} < 0$, we find that the characteristic equation

$$Mk^2 + \mu k + \lambda = 0$$

has two complex roots

$$k_{1,2} = -\frac{\mu}{2M} \pm i\frac{\sqrt{-D}}{2M} \approx -0.006 - 5.0i,$$

and hence the general solution of (8.4) is given by

$$x(t) = e^{-0.006t}(A\cos(5t) + B\sin(5t)),$$

with exponentially decaying amplitude $e^{-0.006t}$ and frequency $\omega = 5$.

The effective forcing from each pedestrian was found by experiment (which involved varying numbers of people walking across the bridge). If there are N pedestrians on the bridge then the displacement of the bridge satisfies

$$M\ddot{x} + \big(\mu - 300N\big)\dot{x} + \lambda x = 0. \tag{8.5}$$

Now the characteristic equation has the form

$$Mk^2 + (\mu - 300N)k + \lambda = 0.$$

[5] See a video at `www.arup.com/MillenniumBridge`

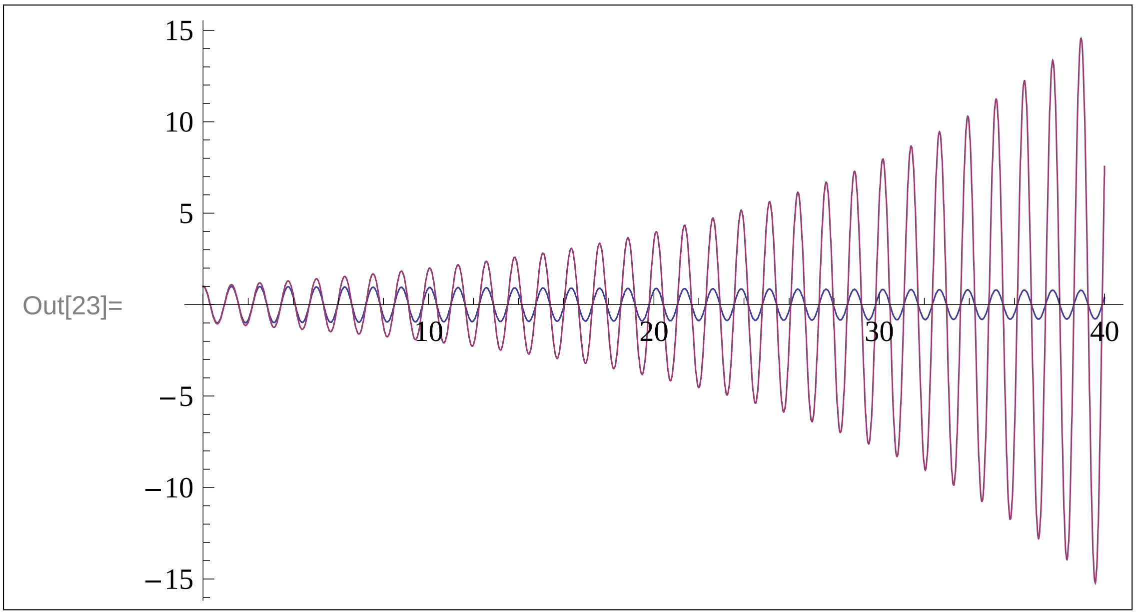

Figure 7: Plots of the decaying solution $x(t)$ (blue color) and exponentially growing solution $x_{200}(t)$ (red color).

We find that $D = (\mu - 300N)^2 - 4M\lambda < 0$ provided that $N < 13350$. In this case the characteristic equation has two complex roots

$$k_{1,2} = -\frac{\mu - 300N}{2M} \pm i\frac{\sqrt{-D}}{2M}.$$

The general solution of (8.5) is given by

$$x_N(t) = e^{\rho}(A\cos(\omega t) + B\sin(\omega t)),$$

where

$$\rho = -\frac{\mu - 300N}{2M}, \qquad \omega = \frac{\sqrt{-D}}{2M}.$$

It is easy to see that $\rho > 0$ for $N > 167$ and then the general solution of (8.5) will have exponentially *increasing* amplitude of oscillations, which means in practice that the bridge will wobble violently!

For example, if the number of pedestrians $N = 200$ then $\rho \approx 0.01$, $\omega \approx 5$ and the general solution of (8.4) is given by

$$x_{200}(t) = e^{0.012t}(A\cos(5t) + B\sin(5t)),$$

with exponentially increasing amplitude $e^{0.01t}$ and frequency $\omega = 5$.

8.4 Resonance phenomena.

Suppose that $x(t)$ represents the distance of some system from its equilibrium position at time t, and that without any external forcing the system would ascillate around this equilibrium position satisfying

$$\ddot{x} + \omega^2 x = 0. \tag{8.6}$$

We know that this equation has the general solution

$$x(t) = A\cos(\omega t) + B\sin(\omega t),$$

where the quantity $2\pi/\omega$ is often known as the natural frequency, this is how the system oscillates with no external forces applied.

Now consider what happens if we apply to the system an external force that is also oscillating,

$$\ddot{x} + \omega^2 x = \underbrace{\cos(\alpha t)}_{\text{external force}}. \tag{8.7}$$

We consider separately the cases $\alpha \neq \omega$ and $\alpha = \omega$.

Beating phenomenon: case α is close to ω, but $\alpha \neq \omega$. First consider what happens when the system is forced at a frequency that differs from the natural frequency, i.e. when $\alpha \neq \omega$. We can try the particular integral in the form

$$x_p(t) = C\cos(\alpha t),$$

where the sin–term can be omitted because the equation does not contain $\dot{x}$. Substituting $x_p(t)$ into the equation we get

$$-C\alpha^2\cos(\alpha t) + \omega^2 C\cos(\alpha t) = \cos(\alpha t),$$

and then

$$C = \frac{a}{\omega^2 - \alpha^2}.$$

Thus the general solution of (8.7) is

$$x(t) = A\cos(\omega t) + B\sin(\omega t) + \frac{1}{\omega^2 - \alpha^2}\cos(\alpha t),$$

and $x(t)$ combines oscillations at two frequencies: the 'natural frequency' ω and the external force frequency α.

To understand this phenomenon better consider (8.7) with initial data:

$$\ddot{x} + \omega^2 x = \cos(\alpha t), \qquad x(t) = \dot{x}(t) = 0. \tag{8.8}$$

The solution of such initial value problem is

$$x(t) = \frac{1}{\omega^2 - \alpha^2}\big(\cos(\alpha t) - \cos(\omega t)\big).$$

Using the double angle formula

$$\cos(\theta \pm \phi) = \cos(\theta)\cos(\phi) \mp \sin(\theta)\sin(\phi),$$

$x(t)$ could be rewritten in the 'product' form

$$x(t) = \frac{2}{\omega^2 - \alpha^2} \sin\big(\frac{1}{2}(\omega + \alpha)\, t\big) \sin\big(\frac{1}{2}(\omega - \alpha)\, t\big).$$

If α is close to ω then $|\omega + \alpha|$ is much larger then $|\omega - \alpha|$, which means that the graph of such solutions combines two oscillations of very different frequencies, and the amplitude of the oscillations growth proportionally to the inverse of $\omega^2 - \alpha^2$. This phenomenon is known as *beating*, see Figure 8.

```
ode = DSolve[{x''[t] + x[t] == Cos[alpha * t], x[0] == 0, x'[0] == 0}, x[t], t]
```

$$\left\{\left\{x[t] \to \frac{\text{Cos}[t] - \text{Cos}[t]^2\,\text{Cos}[\text{alpha } t] - \text{Cos}[\text{alpha } t]\,\text{Sin}[t]^2}{-1 + \text{alpha}^2}\right\}\right\}$$

```
alpha = 1.05;

Plot[x[t] /. ode, {t, 0, 100 * Pi}]
```

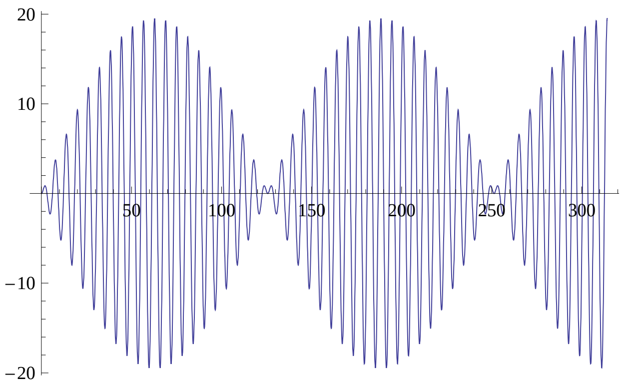

Figure 8: *Mathematica* code which plots the graph of the 'beating' solution of (8.8) in the interval $t \in [0, 100\pi]$: case $\omega = 1$, $\alpha = 1.05$.

Resonance phenomenon: case $\alpha = \omega$. Now consider the case when $\alpha = \omega$. Then the standard 'guess' $C\cos(\omega t) + D\sin(\omega t)$ solves the homogeneous equation (8.6) and we have to try

$$x_p(t) = Ct\sin(\omega t) + Dt\cos(\omega t)$$

as a 'guess' solution. We then get

$$\dot{x}_p(t) = C\sin(\omega t) + Ct\omega\cos(\omega t) + D\cos(\omega t) - D\omega t\sin(\omega t),$$

$$\ddot{x}_p(t) = 2C\omega\cos(\omega t) - 2D\omega\sin(\omega t) - \omega^2\underbrace{\left[Ct\sin(\omega t) + Dt\cos(\omega t)\right]}_{x(t)}.$$

Therefore

$$\ddot{x}_p + \omega^2 x_p = 2C\omega\cos(\omega t) - 2D\omega\sin(\omega t),$$

and since we require the right hand side to be equal to $\cos(\omega t)$, we choose

$$C = \frac{1}{2\omega}, \qquad D = 0,$$

so

$$x_p(t) = \frac{1}{2\omega}\,t\sin(\omega t)$$

is the particular solution, and the general solution of (8.7) when $\alpha = \omega$ is

$$x_p(t) = A\cos(\omega t) + B\sin(\omega t) + \frac{a}{2\omega}\,t\sin(\omega t).$$

We can now see that the amplitude of the resulting oscillations growth linearly with t, and after a time the amplitude of oscillations becomes arbitrary large. This phenomenon is known as the *resonance*, see Figure 9.

```
ode = DSolve[{x''[t] + x[t] == Cos[t], x[0] == 0, x'[0] == 0}, x[t], t]

{{x[t] → 1/4 (-2 Cos[t] + 2 Cos[t]^3 + 2 t Sin[t] + Sin[t] Sin[2 t])}}

Plot[x[t] /. ode, {t, 0, 100*Pi}]
```

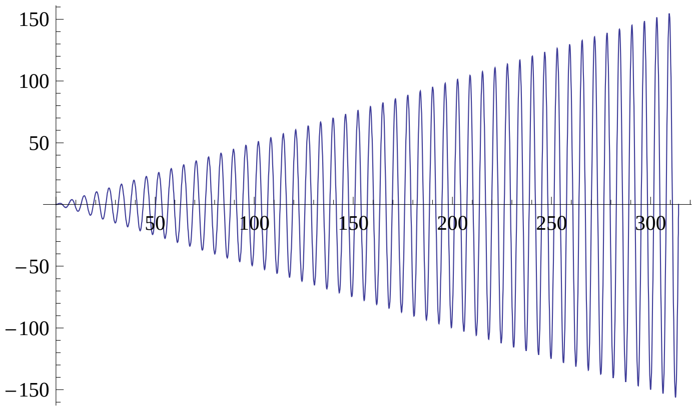

Figure 9: *Mathematica* code which plots the graph of the 'resonance' solution of (8.8) in the interval $t \in [0, 100\pi]$: case $\omega = \alpha = 1$.

9 Linear systems of ODEs

9.1 Reducing a 2nd order equation to a system.

Consider a 2nd order linear ODE with constant coefficients

$$\ddot{x} + p\dot{x} + qx = f(t). \tag{9.1}$$

Introduce a new function $y = y(t)$ defined by

$$y := \dot{x}, \qquad \text{so that} \quad \dot{y} = \ddot{x}.$$

Substituting into (9.1) we obtain

$$\dot{y} + py + qx = f(t).$$

Combining with the definition of y and rearranging we obtain a *system* of 1st order ODEs

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= -py - qx + f(t). \end{cases} \tag{9.2}$$

This system could be also written in the equivalent *matrix form* as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -p & -q \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}. \tag{9.3}$$

In such a way 2nd order equation (9.1) is reduced to the 1st order system (9.3).

9.2 Reducing a system to a 2nd order equation.

Consider a *system* of 1st order ODEs

$$\begin{cases} \dot{x} &= ax + by + f(t), \\ \dot{y} &= cx + dy + g(t). \end{cases} \tag{9.4}$$

This system could be also written in the equivalent *matrix form* as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}. \tag{9.5}$$

Assume that $b \neq 0$. Then from the 1st equation of (9.5) we represent

$$y = \frac{1}{b}\left(\dot{x} - x - f(t)\right).$$

Substituting into the 2nd equation of (9.5) we obtain

$$\frac{1}{b}\left(\ddot{x} - \dot{x} - \dot{f}(t)\right) = dx + \frac{d}{b}\left(\dot{x} - x - f(t)\right) + g(t).$$

After rearrangement, we obtain a linear 2nd order ODE

$$\ddot{x} - (a+d)\dot{x} + (ad-bc)x = \dot{f}(t) - df(t) + bg(t). \tag{9.6}$$

In such a way system of 1st order ODEs (9.5) is reduced to the 2nd order equation (9.6), which could be solved by methods described in the previous Section. In particular, note that characteristic equation of (9.6) is given by

$$k^2 - (a+d)k + (ad-bc) = 0. \tag{9.7}$$

Example 9.1. Consider an initial value problem for a system of 1st order ODEs

$$\left\{\begin{array}{lcll} \dot{x} & = & x+y, & \qquad x(0)=1, \\ \dot{y} & = & 4x-2y, & \qquad y(0)=0. \end{array}\right. \tag{9.8}$$

This system could be also written in the equivalent *matrix form* as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{9.9}$$

From the 1st equation of (9.9) we represent

$$y = \dot{x} - x.$$

Substituting into the 2nd equation of (9.5) we obtain

$$\ddot{x} - \dot{x} = 4x - 2\left(\dot{x} - x\right).$$

After rearrangement (or using (9.5)) we obtain a linear 2nd order ODE

$$\ddot{x} + \dot{x} - 6x = 0. \tag{9.10}$$

The characteristic equation of (9.10) is given by

$$k^2 + k - 6 = 0, \tag{9.11}$$

with the roots $k_1 = 2$, $k_2 = -3$. Thus the general solution of (9.10) is

$$x(t) = Ae^{2t} + Be^{-3t},$$

where $A, B \in \mathbb{R}$ are arbitrary constants. Substituting into $y(t)$ we also obtain

$$y(t) = \dot{x}(t) - x(t) = Ae^{2t} - 4Be^{-3t}.$$

The general solution of the system (9.8) is therefore

$$\left\{\begin{array}{lcl} x(t) & = & Ae^{2t} + Be^{-3t}, \\ y(t) & = & Ae^{2t} - 4Be^{-3t}. \end{array}\right. \tag{9.12}$$

This could be also written in the equivalent *matrix form* as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = A \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + B \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}. \tag{9.13}$$

Substituting the initial data

$$\begin{cases} x(0) = 1, \\ y(0) = 0, \end{cases} \tag{9.14}$$

into (9.12) we obtain a scalar system

$$\begin{cases} A + B &= 1, \\ A - 4B &= 0. \end{cases} \tag{9.15}$$

Solving this system we conclude that

$$\begin{cases} A &= 4/5, \\ B &= 1/5, \end{cases} \tag{9.16}$$

so finally we find that the solution (in the vector form) of the initial value problem (9.8) is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{4}{5} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \frac{1}{5} \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}. \tag{9.17}$$

9.3 Matrix approach to linear systems

We now reconsider the homogeneous linear system

$$\begin{cases} \dot{x_1} &= ax_1 + bx_2, \\ \dot{x_2} &= cx_1 + dx_2. \end{cases} \tag{9.18}$$

Because of the important geometric properties of systems of type (9.18), we are going to study such systems directly, without employing the reduction to 2nd order equation (9.1).

Our first observation that there is a compact way to write (9.18) by fully exploring vector and matrix notations. If we denote

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \qquad \dot{\mathbf{x}}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix}, \qquad \mathbb{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then we can rewrite (9.18) as

$$\dot{\mathbf{x}} = \mathbb{A}\mathbf{x}. \tag{9.19}$$

Our first important observation about linear system (9.19) is the following important principle.

Superposition Principle. If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are solutions of the system (9.19) then for any real numbers A and B the *superposition*

$$A\mathbf{x}_1(t) + B\mathbf{x}_2(t)$$

is also a solutions of the system (9.19).

In order to formulate explicit formulae for solutions of (9.19) we need several notion from the theory of matrices.

Eigenvalues and eigenvectors of a matrix. In this course we adopt an approach which is independent of your knowledge of Linear Algebra. Consider a matrix

$$\mathbb{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Definition 9.2. The quadratic equation

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0. \tag{9.20}$$

is called *characteristic equation* of matrix $\mathbb{A}$.

Remark 9.3. It is clear that $\mathbb{A}$ has precisely two eigenvalues λ_1 and λ_2, if we allow $\lambda_1 = \lambda_2$ when (9.20) has a repeated root. In what follows we will distinguish 3 different cases, according to the properties of λ_1 and λ_2:

- distinct real eigenvalues;
- complex eigenvalues;
- repeated real eigenvalue.

Remark 9.4. Characteristic equation of matrix $\mathbb{A}$ could be also written in the matrix form

$$\det(\mathbb{A} - \lambda\mathbb{I}) = 0, \tag{9.21}$$

where det stands for the matrix determinant and

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

denotes the *identity matrix.*

Definition 9.5. Roots of the characteristic equation of matrix $\mathbb{A}$ are called *eigenvalues* of matrix $\mathbb{A}$.

Definition 9.6. If λ is an eigenvalue of matrix $\mathbb{A}$ then solutions $\mathbf{v}$ of the matrix equation

$$\mathbb{A}\mathbf{v} = \lambda\mathbf{v}. \tag{9.22}$$

are called *eigenvectors* of matrix $\mathbb{A}$ which correspond to λ.

Remark 9.7. Eigenvectors which correspond to the eigenvalue λ are not uniquely defined. Indeed, if vector $\mathbf{v}$ is a solution of

$$\mathbb{A}\mathbf{v} = \lambda\mathbf{v},$$

then vector $\alpha\mathbf{v}$ is also a solution of the same equation, for any scalar α, so eigenvectors are defined up to a multiplication by a scalar. For practical purposes, it is sufficient to find one particular eigenvector that correspond to the eigenvalue λ, all other eigenvectors will be then scalar multiples of that given one.

Example 9.8. Find eigenvalues and eigenvectors of the matrix

$$\mathbb{A} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}.$$

Solution. The characteristic equation of matrix $\mathbb{A}$ is

$$\lambda^2 + \lambda - 6 = 0.$$

Its roots

$$\lambda_1 = 2, \quad \lambda_2 = -3.$$

are two eigenvalues of matrix $\mathbb{A}$.

To find an eigenvector v_1 that corresponds to $\lambda_1 = 2$ we need to solve the matrix equation

$$\mathbb{A}\mathbf{v}_1 = \lambda_1\mathbf{v}_1,$$

or equivalently,

$$(\mathbb{A} - \lambda_1\mathbb{I})\mathbf{v}_1 = 0,$$

which is

$$\begin{cases} -v_1 + v_2 &= 0, \\ 4v_1 - 4v_2 &= 0. \end{cases} \tag{9.23}$$

Using the first equation we conclude that any vector $\mathbf{v}$ with $v_1 = v_2$ satisfies this system. (Note that the 2nd equation is a multiple of the first.) For instance,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is an eigenvector of matrix $\mathbb{A}$ that corresponds to $\lambda_1 = 2$.

Similarly, to find an eigenvector v_2 that corresponds to $\lambda_2 = -3$ we need to solve the matrix equation

$$\mathbb{A}\mathbf{v}_2 = \lambda_2\mathbf{v}_2,$$

which is

$$\begin{cases} 4v_1 + v_2 &= 0, \\ 4v_1 + v_2 &= 0. \end{cases} \tag{9.24}$$

Using the first equation we conclude that any vector $\mathbf{v}$ with $v_2 = -4v_1$ satisfies this system. (Note that in this case the 2nd equation is equal to the first.) For instance, we can take

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

is an eigenvector of matrix $\mathbb{A}$ that corresponds to $\lambda_2 = -3$.

Example 9.9. Find eigenvalues and eigenvectors of the matrix

$$\mathbb{A} = \begin{pmatrix} 2 & 5 \\ -2 & 0 \end{pmatrix}.$$

Solution. The characteristic equation of matrix $\mathbb{A}$ is

$$\lambda^2 - 2\lambda + 10 = 0.$$

The roots of this equation are the complex conjugate

$$\lambda_1 = 1 + 3i, \quad \lambda_2 = 1 - 3i.$$

so matrix $\mathbb{A}$ has two complex conjugate eigenvalues.

To find the eigenvector v_1 that corresponds to λ_1 we need to solve the matrix equation

$$\mathbb{A}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1,$$

that is

$$\left\{ \begin{array}{rcl} (1-3i)v_1 + 5v_2 & = & 0, \\ -2v_1 - (1+3i)v_2 & = & 0. \end{array} \right. \tag{9.25}$$

From the first equation, we must have $5v_2 = (3i - 1)v_1$. (As usual, the 2nd equation is a multiple of the first, although its not so obvious in this case.) Thus we can take,

$$\mathbf{v}_1 = \begin{pmatrix} 5 \\ -1 + 3i \end{pmatrix}$$

as an eigenvector of matrix $\mathbb{A}$ that corresponds to $\lambda_1 = 1 + 3i$. The eigenvector of the *conjugate* eigenvalue $\lambda_2 = 1 - 3i$ will be the complex conjugate of the eigenvector $\mathbf{v}_1$,

$$\mathbf{v}_2 = \begin{pmatrix} 5 \\ -1 - 3i \end{pmatrix}.$$

It is convenient to write eigenvalues and eigenvectors of $\mathbb{A}$ could in the *conjugate* form:

$$\lambda_\pm = 1 \pm 3i,$$
$$\mathbf{v}_\pm = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ -3 \end{pmatrix}.$$

Now we are in the position to formulate explicit formulae for solutions of the linear system of differential equations

$$\dot{\mathbf{x}} = \mathbb{A}\mathbf{x}. \tag{9.26}$$

We will distinguish several cases, according to the properties of eigenvalues of the matrix $\mathbb{A}$.

Distinct real eigenvalues of $\mathbb{A}$. Assume that matrix $\mathbb{A}$ has two distinct real eigenvalues λ_1, λ_2 and $\mathbf{v}_1$, $\mathbf{v}_2$ are the corresponding eigenvectors. Then the general solution of (9.26) is given by the formula

$$\boxed{\mathbf{x}(t) = Ae^{\lambda_1 t}\mathbf{v}_1 + Be^{\lambda_2 t}\mathbf{v}_2,} \tag{9.27}$$

where $A, B \in \mathbb{R}$ are arbitrary constants.

Example 9.10. We know already eigenvalues and eigenvectors of the matrix

$$\mathbb{A} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$$

from Example 9.8. Using formula (9.27) we conclude that the general solution of

$$\dot{\mathbf{x}} = \mathbb{A}\mathbf{x}$$

is given by

$$\mathbf{x}(t) = Ae^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + Be^{-3t}\begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

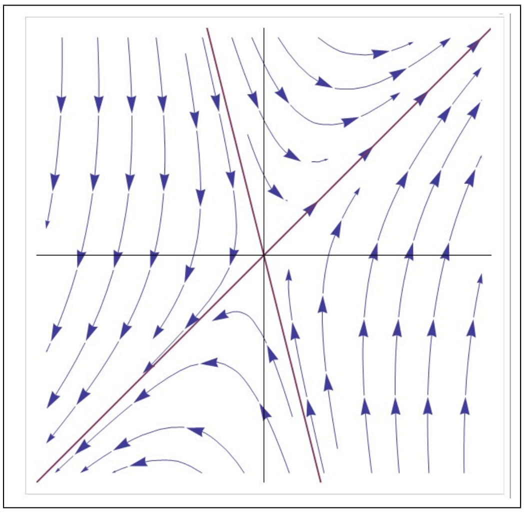

Figure 10: Phase portrait of the system in Example 9.10.

Complex eigenvalues of $\mathbb{A}$. Assume that matrix $\mathbb{A}$ has two complex conjugate eigenvalues

$$\lambda_\pm = \rho \pm i\omega,$$

and

$$\mathbf{v}_\pm = \mathbf{u} \pm i\mathbf{w}$$

denotes the corresponding conjugate eigenvectors. Then the general solution of (9.26) is given by the formula

$$\boxed{\mathbf{x}(t) = e^{\rho t}\big([A\cos(\omega t) + B\sin(\omega t)]\mathbf{u} + [B\cos(\omega t) - A\sin(\omega t)]\mathbf{w}\big),} \tag{9.28}$$

where $A, B \in \mathbb{R}$ are arbitrary constants.

Example 9.11. We know already eigenvalues and eigenvectors of the matrix

$$\mathbb{A} = \begin{pmatrix} 2 & 5 \\ -2 & 0 \end{pmatrix}$$

from Example 9.9. Using formula (9.28) we conclude that the general solution of

$$\dot{\mathbf{x}} = \mathbb{A}\mathbf{x}$$

is given by

$$\mathbf{x}(t) = e^t[A\cos(3t) + B\sin(3t)]\begin{pmatrix} 5 \\ -1 \end{pmatrix} + e^t[B\cos(3t) - A\sin(3t)]\begin{pmatrix} 0 \\ -3 \end{pmatrix}.$$

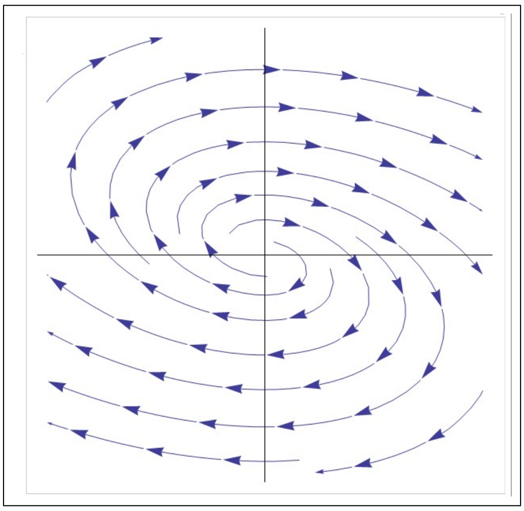

Figure 11: Phase portrait of the system in Example 9.11.

Repeated real eigenvalues of $\mathbb{A}$. There are two possibilities in that case. The first one is that $\mathbb{A}$ is a multiple of identity,

$$\mathbb{A} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

In this case equation $\dot{\mathbf{x}} = \mathbb{A}\mathbf{x}$ *decouples* into two simple independent equations

$$\dot{x_1} = \lambda x_1, \qquad \dot{x_2} = \lambda x_2,$$

each of which could be solved individually.

The second case is when $\lambda_1 = \lambda_2$ but the matrix $\mathbb{A}$ is not a multiple of the identity. For example, the matrix

$$\mathbb{A} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

has repeated eigenvalue $\lambda_1 = \lambda_2 = \lambda$. In this case the situation is more difficult because we will be able to find only one eigenvector of $\mathbb{A}$. The construction of the 2nd eigenvector would require some additional linear algebra consideration. We do not discuss this case here.