Lecture 3, July 2, 2024

Recap of yesterday:

Assume $u \in USC(\Omega)$ visco subsol. of

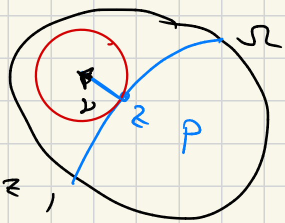
$Lu \leq 0$ in $\Omega \subseteq \mathbb{R}^n$, $Lu = -\operatorname{tr}(\sigma\sigma^T(x) D^2 u)$,

$\sigma = (\sigma^1 \dots \sigma^m)$ $\quad \sigma^j(x) \in \mathbb{R}^n$ Lip.

$x_0 \in \Omega$: $u(x_0) = M = \max\limits_{\Omega} u$

Propagation set $P = \{x \in \Omega : u(x) = M\}$

Rmk: If $P = \Omega$ we have the Strong Maximum Principle.



LEMMA If $P \subsetneq \Omega$, $z \in \partial P$, ν = Bony ext. normal to P at z,

(T) $(\sigma^j \cdot \nu)(z) = 0 \quad \forall\ j = 1, \dots, m.$

Want to use it to prove SMP, i.e. $P = \Omega$, for DEGENERATE ELLIPTIC equations

Use the tangentiality condition (T):

Thm. (Nagumo 1942, Bony, ... Viability Thm.)

$(\sigma^j \cdot \nu)(z) = 0 \quad \forall\, z \in \partial P \quad \forall\, \nu = $ normal to P at z

$\Rightarrow$ P is invariant for $\dot{y} = \sigma^j(y)$, i.e.

$y(0) \in P \Rightarrow y(t) \in P \quad \forall t > 0$

Corollary Lemma $\Rightarrow$ the propagation set P is invariant for the CONTROL SYSTEM

$$(S) \qquad \dot{y}(t) = \sum_{j=1}^{m} \sigma^j(y(t))\, \alpha_j(t)$$

↑ piecewise controls: $\sum_{j=1}^{m} \alpha_j^2(t) \leq 1$. ∎

Def: (S) is CONTROLLABLE if $\forall\, x_1, x_2 \in \Omega$ $\exists$ trajectory of (S), $s>0$, $y(0) = x_1$, $y(s) = x_2$.

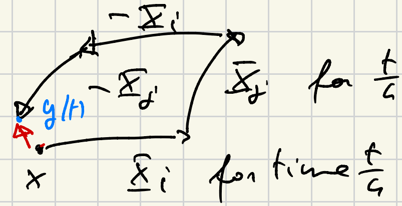
N.B.: (S) controllable in $\Omega \Rightarrow$ NO $P \subsetneq \Omega$ can be INVARIANT!!

Corollary In the ass. of Lemma, (S) controllable $\Rightarrow u(x) \equiv M$ in Ω, i.e. STRONG M.P. holds. ∎

Q: When is (S) controllable?

Can we move in more directions than $\sigma^j(x)$?

$X_j := \sigma^j \cdot \nabla$



$\nearrow \approx [X_i, X_j]$

$$y(t) = x + t^2 [X_i, X_j](x) + o(t^2) \quad \text{as } t \to 0+,$$

Lie bracket of X, Y smooth vector fields is

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \quad \forall f \in C^\infty$$

(in coordinates: $X = \sum a_i \partial_i$, $Y = \sum b_i \partial_i$

$$[X, Y] = (Da)b - (Db)a$$).

Can also ITERATE $[X_k, [X_i, X_j]]$ ... get the LIE ALGEBRA generated by $X_1, \dots, X_m$, $\mathcal{L}(X_1, \dots, X_m)$.

<u>Def.</u> RANK $\operatorname{rk} \mathcal{L}(X_1, \dots, X_m)(x_0) = \dim \operatorname{span}\{ Z(x_0) : Z \in \mathcal{L}(X_1, \dots, X_m)\}$

<u>Thm.</u> (Chow-Rashevskii 1938-9)

$$(HC) \quad \operatorname{rk} \mathcal{L}(X_1, \dots, X_n)(x) = n \quad \forall x \in \Omega$$

$\Rightarrow$ (S) is CONTROLLABLE.

<u>N.B.</u> (HC) is Hörmander condition for HYPOELLIPTICITY.

<u>Cor.</u> (Bony 1969 if $u \in C^2$) $u \in USC(\Omega)$ visc. subsol. of

$$-\operatorname{tr}(\sigma \sigma^T D^2 u) \leq 0 \quad \text{in } \Omega, \quad X_j = \sigma^j \cdot \nabla \text{ sat. (HC)}$$

$\exists\, x_0 \in \Omega \; : \; u(x_0) = \max_{\Omega} u = M \implies u \equiv M$ in M.

Examples : GRUSHIN $m = n = 2$

$\sigma^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; \sigma^2 = \begin{pmatrix} 0 \\ x_1 \end{pmatrix} \qquad -\mathrm{tr}(\sigma\sigma^T D^2 u) = u_{x_1 x_1} + x_1^2 u_{x_2 x_2}$

degenerates at $(0, x_2) \; \forall x_2$.

$[X_1, X_2](0, x_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \implies \mathrm{rk}\, \mathcal{L}(X_1, X_2) = 2 = n$

- HEISENBERG sublaplacian $m = 2 < n = 3$

$\sigma^1 = \begin{pmatrix} 1 \\ 0 \\ 2x_2 \end{pmatrix}, \; \sigma^2 = \begin{pmatrix} 0 \\ 1 \\ -2x_1 \end{pmatrix}$ degenerate on x_3 axis $x_1 = x_2 = 0$

$[X_1, X_2] = (D\sigma_2)\sigma_1 - (D\sigma_1)\sigma_2 = \ldots \begin{pmatrix} 0 \\ 0 \\ -4 \end{pmatrix}$

$\implies \mathrm{rk}\, \mathcal{L}(X_1, X_2) = 3 = n$. $\blacksquare$

Q : what to do for fully nonlinear eqs.

(E) $F(x, u, Du, D^2u) = 0$?

What replaces X_j ? [M.B. - A. Goffi 2019]

SUBUNIT VECTORS for F

Def $Z(x) \in \mathbb{R}^n$ is S.V. Field at $x \in \Omega$ if

$$\sup_{\gamma > 0} F(x, 0, p, I - \gamma\, p \otimes p) > 0 \qquad \forall p : Z(x) \cdot p \neq 0$$

$[(p \otimes p)_{ij} = p_i p_j]$

Motivation & exemple: $F = -\mathrm{tr}(A(x)D^2u)$

Fefferman-Phong 1983: $Z(x)$ SUBUNIT for $A(x)$ if

$$\xi^T A(x)\xi \geq (Z(x)\cdot\xi)^2 \quad \forall \xi\in\mathbb{R}^n.$$

In particular: $A(x) = \sigma\sigma^T(x)$, $\sigma = (\sigma^1 \dots \sigma^m)$

$\Rightarrow$ $\sigma^j(x)$, $j=1,\dots,m$, are SVF.

SCALING CONDITION on F

(replacing 1-homogeneity ....)

(Sc) $\exists\ \varphi : (0,1] \to [0,+\infty)$:

$$F(x, \tau r, \tau p, \tau X) \geq \varphi(\tau) F(x, r, p, X) \qquad \forall \tau\in(0,1], \ r\leq 0$$

$\Rightarrow$ $\uparrow$ and have the same sign.

Thm. F sats. (Sc) & $\exists$ Lip. SVF $Z(\cdot)$.

$u\in USC(\Omega)$, $F[u]\leq 0$ visco sol., $x_0\in\Omega$:

$u(x_0) = M = \max_{\overline{\Omega}} u \geq 0$. $\Longrightarrow$

$u(x) = M \quad \forall x\in y(s), s\geq 0$ $\quad \begin{cases} \dot{y}(t) = Z(y(t)) \\ y(0) = x_0 \end{cases}$.

i.e.: the max propagates on the trajectories of $\uparrow$.

Corollary: If $\exists\ Z_1, \dots, Z_m$ SVF with Hörmander rank condition (HC) $\Rightarrow$ SMP holds.

Pf of thm.: adaptation of the proof of Lemma proved yesterday. $\square$

Examples ① "EUCLIDEAN": choose $Z_j = (0, \dots, 0, \overset{j\text{th}}{1}, 0, \dots, 0)$

they are SVF if

$$\sup_{\gamma > 0} F(x, 0, p, I - \gamma\, p \otimes p) > 0 \quad \forall p \neq 0,$$

ok for

- Bellman ops. $\sup_\alpha L^\alpha u$ or $\inf L^\alpha u$,

$L^\alpha u = -\operatorname{tr}(A^\alpha D^2 u) + \dots \ \exists \lambda > 0 : A^\alpha \xi_i \xi_j \geq \lambda |\xi|^2 \ \forall \xi$

- PUCCI EXTREMAL OPERATORS: $0 < \lambda \leq \Lambda$

$X \in S_n$

$$\mathcal{M}^+(X) = -\lambda \sum_{ev_k > 0} ev_k - \Lambda \sum_{ev_k < 0} ev_k \qquad ev_k \text{ eigenv. of } X$$

$$\mathcal{M}^-(X) = -\Lambda \sum_{ev_k > 0} ev_k - \lambda \sum_{ev_k < 0} ev_k$$

- Many UNIFORMLY ELLIPTIC F, using

$$\mathcal{M}^-(X) \leq F(x, r, p, X) - F(x, r, p, 0) \leq \mathcal{M}^+(X)$$

② SUBELLIPTIC : $(X_1, \dots, X_m) = \mathcal{X}$

satg (HC) $\operatorname{rk} \mathcal{L}(X_1, \dots, X_m)(x) = n \quad \forall x$

$$F(x, u, Du, D^2u) = G(x, u, D_{\mathcal{X}} u, (D^2_{\mathcal{X}} u)^*)$$

horizontal grad horizontal Hess.

Ex for $0 < \lambda \le \Lambda$ use Pucci in $\mathcal{S}_m$

$$\mathcal{M}^+((D^2_{\mathcal{X}} u)^*) + H(x, u, D_{\mathcal{X}} u) \le 0$$

with $H(x, \tau s, \tau p) = \tau H(x, s, p) \quad \forall \tau > 0$

or

$$\mathcal{M}^-((D^2_{\mathcal{X}} u)^*) + H(x, u, D_{\mathcal{X}} u) \le 0$$

satisfy Strong. Max Princ. ∎