CRITICALITY THEORY FOR SCHRÖDINGER OPERATORS

EXERCISES WEEK 1

If you need this exercises to be assessed submit to `v.moroz@swansea.ac.uk` by 1pm on 27 January

Exercise 1. Let

$$E_V(u) := \int_\Omega |\nabla u|^2\, dx + \int_\Omega V u^2\, dx \qquad (\varphi \in C_0^\infty(\Omega)).$$

Prove that for $u, \varphi \in C_0^\infty(\Omega)$ and $t \in \mathbb{R}$,

$$\frac{d}{dt} E_V(u + t\varphi)\Big|_{t=0} = \int_\Omega \nabla u \cdot \nabla \varphi\, dx + \int_\Omega V u \varphi\, dx.$$

Exercise 2. Explain using a result from the lectures why the equation

$$-\Delta u = \frac{u}{|x|^2} \quad \text{in } \mathbb{R}^3 \setminus \{0\}.$$

has no positive solutions.

Exercise 3. Verify for which values of $\alpha, \beta \in \mathbb{R}$,

$$|x|^\alpha \in H^1_{loc}(\mathbb{R}^N)\,? \qquad (\log|x|)^\beta \in H^1_{loc}(B_1^{(2)})\,?$$

Here and below $B_1^{(2)} := \{x \in \mathbb{R}^2 : |x| < 1\}$.

Exercise 4. Prove using a result from the lectures the following inequality,

$$\int_{B_1^{(2)}} |\nabla \varphi|^2 \geq \frac{1}{4} \int_{B_1^{(2)}} \frac{\varphi^2}{|x|^2 (\log|x|)^2} \qquad \forall \varphi \in C_c^\infty(B_1^{(2)} \setminus \{0\}).$$

Hint: Take $u_*(|x|) = -(\log|x|)^{1/2}$.