CRITICALITY THEORY FOR SCHRÖDINGER OPERATORS

EXERCISES WEEK 1

If you need this exercises to be assessed submit to `v.moroz@swansea.ac.uk` by 1pm on 3 February

Exercise 1. Revisit carefully the arguments in the lecture to show that $C_H = \frac{(N-2)^2}{4}$ $(N \geq 3)$ is the *best constant* in the Hardy inequality

$$\int_{\mathbb{R}^N} |\nabla\varphi|^2 \geq C_H \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} \qquad \forall\varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\}),$$

i.e. the inequality fails for any $c > C_H$.

Exercise 2. For $R > 0$ and $B_R \subset \mathbb{R}^N$, prove the inequality

$$\int_{B_R} |\nabla\varphi|^2 \geq 2N \int_{B_R} \frac{\varphi^2}{R^2 - |x|^2} \qquad \forall\varphi \in C_c^\infty(B_R).$$

Exercise 3. Assume that E_V satisfies the λ–property and denote

$$\|\varphi\|_{D_V^1} := \sqrt{E_V(\varphi)}.$$

Let $(\varphi_n) \in C_0^\infty(\Omega)$ be a Cauchy sequence w.r.t $\|\cdot\|_{D_V^1}$ and $\varphi_n \to \varphi_0$ in $L^2(\Omega, \lambda(x)dx)$.

(i) Prove that $E_V(\varphi_n)$ is a Cauchy sequence of real numbers, and hence $E_V(\varphi_n) \to E_0 \geq 0$, up to a subsequence.

(ii) Show that if $\varphi_0 \neq 0$ then $E_0 > 0$.