POSITIVE SOLUTIONS TO SEMILINEAR ELLIPTIC EQUATIONS WITH CRITICAL LOWER ORDER TERMS

VITALI LISKEVICH, SOFYA LYAKHOVA*AND VITALY MOROZ†

School of Mathematics, University of Bristol
Bristol BS8 1TW, United Kingdom
E-mail: V.Liskevich|S.Lyakhova|V.Moroz@bristol.ac.uk

We study the existence and nonexistence of positive (super) solutions to a semilinear elliptic equation $-\Delta u - \frac{Ax}{|x|^2} \cdot \nabla u - \frac{B}{|x|^2} u = \frac{c}{|x|^\sigma} u^p$ in cone-like domains of $\mathbb{R}^N$. On the plane $\mathbb{R}^2$ we determine the set of (p, σ) such that the equation has no positive (super) solutions, depending on the parameters $A, B \in \mathbb{R}$ and the geometry of the domain.

1. Introduction

We study the existence and nonexistence of positive (super) solutions to the semilinear elliptic equation with lower order terms of critical behavior

$$-\Delta u - \frac{Ax}{|x|^2} \cdot \nabla u - \frac{B}{|x|^2} u = \frac{c}{|x|^\sigma} u^p \quad \text{in } \mathcal{C}_\Omega^\rho. \tag{1}$$

Here $A, B \in \mathbb{R}$, $c > 0$ and $(p, \sigma) \in \mathbb{R}^2_* := \mathbb{R}^2 \setminus (1, 2)$. By $\mathcal{C}_\Omega^\rho \subset \mathbb{R}^N$ $(N \geq 2)$ we denote an exterior cone-like domain defined as

$$\mathcal{C}_\Omega^\rho = \{(r, \omega) \in \mathbb{R}^N : r > \rho,\, \omega \in \Omega\},$$

where $\rho > 0$, (r, ω) are the polar coordinates in $\mathbb{R}^N$ and $\Omega \subseteq S^{N-1}$ is a subdomain (a connected open subset) of the unit sphere S^{N-1} in $\mathbb{R}^N$. In what follows $\lambda_1 = \lambda_1(\Omega) \geq 0$ denotes the principal eigenvalue of the Dirichlet Laplace–Beltrami operator $-\Delta_\omega$ on Ω. We do not prescribe any boundary conditions to (1). A (super) solution to (1) in a domain $G \subseteq \mathbb{R}^N$ is an $u \in H^1_{loc}(G)$ such that, for all $0 \leq \varphi \in C_0^\infty(G)$,

$$\int_G \nabla u \cdot \nabla \varphi \, dx - \int_G \nabla u \cdot \frac{Ax}{|x|^2} \varphi \, dx - \int_G \frac{B}{|x|^2} u \varphi \, dx \, (\geq) = \int_G \frac{c}{|x|^\sigma} u^p \varphi \, dx.$$

*Work supported by the Overseas Research Foundation.
†Work partially supported by the Nuffield Foundation.

It has been known at least since the celebrated paper [14] that equations of type (1) admit positive (super) solutions only for some specific values of $(p, \sigma) \in \mathbb{R}^2$. The problem of the existence and nonexistence of positive (super) solutions to equations (1) under various assumptions on operators and classes of domains has been a subject of a number of publications (see, e.g. [5,8,9,10,12,14,19,20] and references therein).

Equations (1) on cone–like domains have been studied so far without lower–order terms. The superlinear case $p > 1$ has been considered in [4,6] (see also [7] for systems and [17] for uniformly elliptic equations with measurable coefficients). A new nonexistence phenomenon for the sublinear case $p < 1$ has been recently revealed in [18]. The following theorem summarizes results in [4,6] and [18].

Theorem 1. *Let $\alpha_+ \geq 0$ and $\alpha_- < 0$ be the roots of the quadratic equation $\alpha(\alpha + N - 2) = \lambda_1$. Then the equation*

$$-\Delta u = c|x|^{-\sigma} u^p \quad \text{in } \mathcal{C}^\rho_\Omega \tag{2}$$

has no positive supersolutions if and only if $1 - \frac{2-\sigma}{\alpha_+} \leq p \leq 1 - \frac{2-\sigma}{\alpha_-}$.

The above result is stable under some classes of perturbations. One can show, e.g., that for $\epsilon > 0$ the equation

$$-\Delta u - \frac{Ax}{|x|^{2+\epsilon}} \cdot \nabla u - \frac{B}{|x|^{2+\epsilon}} u = \frac{c}{|x|^\sigma} u^p \quad \text{in } \mathcal{C}^\rho_\Omega \tag{3}$$

has the same nonexistence exponents as (2) (see, e.g. [15,16]). On the other hand it is easy to see that if $\epsilon < 0$ then (3) has no positive supersolution for any $(p, \sigma) \in \mathbb{R}^2$. In the critical case $\epsilon = 0$ nonexistence exponents become dependent on the parameters A and B. This phenomenon has been recently observed on a ball and/or exterior of a ball in [8,12] ($p > 1,\ A = 0$) and [20] ($p > 1,\ B = 0$). In this note we present a complete characterization of existence and nonexistence of positive solutions to (1) on cone–like domains over the full range of parameters $A, B \in \mathbb{R}$ and exponents $(p, \sigma) \in \mathbb{R}^2_*$.

2. Statement of results

Observe that (super) soultions to (1) are in one-to-one correspondence to (super) solutions of the singular semilinear equation

$$-\nabla \cdot (|x|^A \nabla u) - B|x|^{A-2} u = c|x|^{A-\sigma} u^p \quad \text{in } \mathcal{C}^\rho_\Omega. \tag{4}$$

Thus by the weak Harnack inequality for supersolutions any nontrivial nonnegative supersolution to (1) is positive in $\mathcal{C}^\rho_\Omega$. Following [3,13] one can derive an improved Hardy type inequality with weight $|x|^A$ for exterior cone–like domains in the form

$$\int_{\mathcal{C}_\Omega^\rho} |\nabla u|^2 |x|^A dx \geq (c_H + \lambda_1) \int_{\mathcal{C}_\Omega^\rho} u^2 |x|^{A-2} dx + c_* \int_{\mathcal{C}_\Omega^\rho} \frac{u^2}{\log^2 |x|} |x|^{A-2} dx, \quad (5)$$

for all $u \in C_0^\infty(\mathcal{C}_\Omega^\rho)$, where $c_H = \frac{(A+N-2)^2}{4}$, $c_* > 0$ and $\rho > 1$. Let γ_-, γ_+ be the roots of the equation

$$\gamma(\gamma + N - 2 + A) = \lambda_1 - B. \quad (6)$$

By a standard argument (see, e.g., [11]) inequality (5) implies that if (6) has no real roots then (4) has no positive supersolutions for any $(p, \sigma) \in \mathbb{R}^2$. Thus when $(p, \sigma) = (1, 2)$ then equation (4) has no positive supersolutions if and only if $c \geq \lambda_1 + c_H - B$, where $c > 0$ is a constant in (1). We confine ourself to the case when $\gamma_-, \gamma_+ \in \mathbb{R}$ and $(p, \sigma) \in \mathbb{R}^2_*$.

Let us introduce the critical line $l^*(p)$ on the (p, σ)–plane

$$l^*(p) := \min\{\gamma_-(p-1) + 2,\ \gamma_+(p-1) + 2\} \qquad (p \in \mathbb{R}),$$

and the nonexistence set

$$\mathcal{N} = \{(p, \sigma) \in \mathbb{R}^2_* : (1) \textit{ has no positive supersolutions}\}.$$

Our main result for exterior cone–like domains reads as follows.

Theorem 2. *Assume that $\gamma_-, \gamma_+ \in \mathbb{R}$. The following assertions are valid.*

(i) Let $\gamma_- < \gamma_+$. Then $\mathcal{N} = \{\sigma \leq l^*(p)\}$.
(ii) Let $\gamma_- = \gamma_+$. Then $\mathcal{N} = \{\sigma < l^*(p)\} \cup \{\sigma = l^*(p),\, p \geq -1\}$.

Remarks. (*i*) Observe that the set $\mathcal{N}$ does not depend on the value of the parameter $c > 0$ in (1). In view of the scaling properties of (1) the set $\mathcal{N}$ also does not depend on the value of $\rho > 0$.
(*ii*) Using sub and supersolutions techniques one can show that if (1) has a positive supersolution in $\mathcal{C}_\Omega^\rho$ then it has a positive solution in $\mathcal{C}_\Omega^\rho$. Thus for any $(p, \sigma) \in \mathbb{R}^2_* \setminus \mathcal{N}$ equation (1) has positive solutions.
(*iii*) Figure 1 shows the qualitative pictures of the set $\mathcal{N}$ for typical values of γ_-, γ_+. The case (*a*) is typical for $A, B = 0$. The case (*b*) occurs, e.g., when $A, B = 0$ and $N = 2$. The cases (c, d, e, f) could never be realized for equation (1) without critical lower order terms.

Applying the Kelvin transformation $y = y(x) = \frac{x}{|x|^2}$ we see that if u is a positive (super) solution to equation (1) then $\check{u}(y) = |y|^{2-N} u(x(y))$ is a

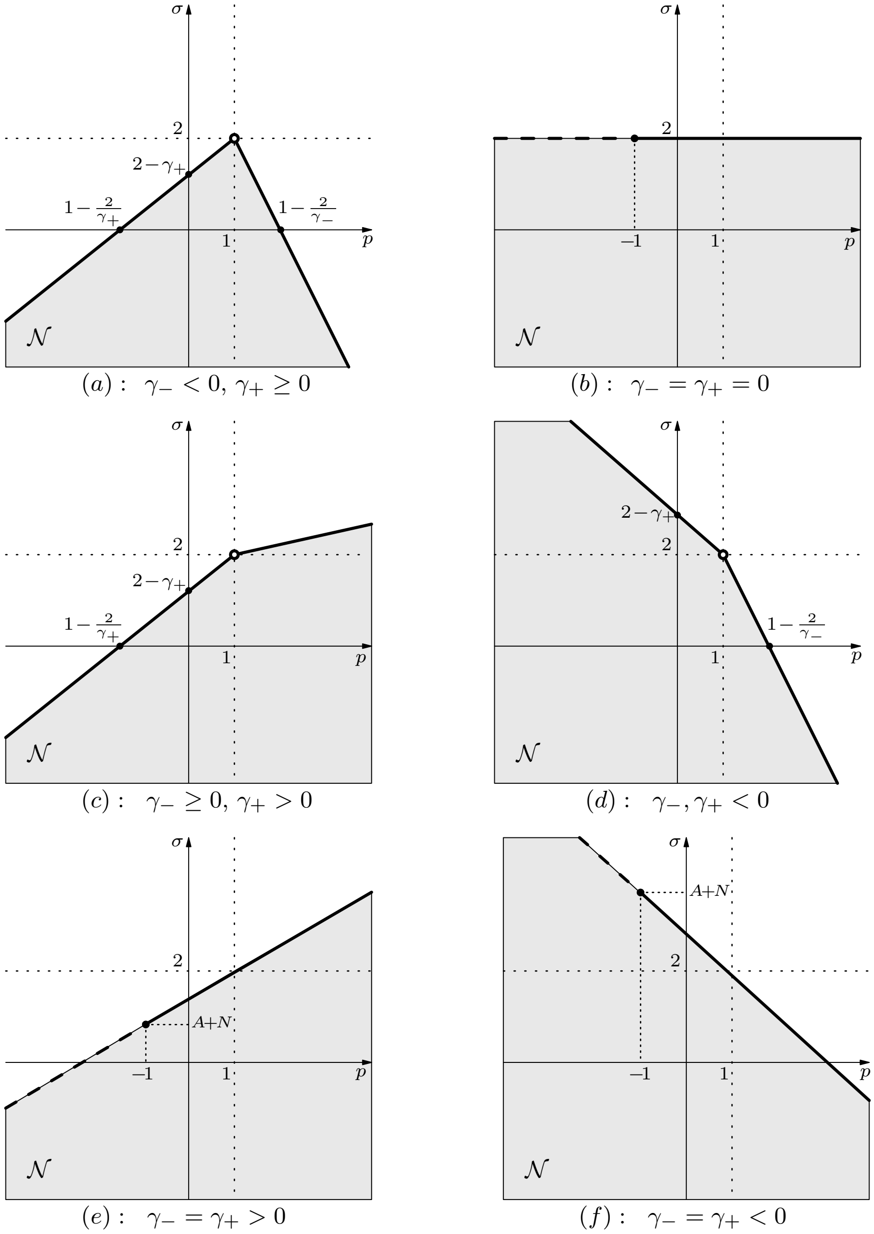

Figure 1. The nonexistence set $\mathcal{N}$ of equation (1) for typical values of γ_- and γ_+.

positive (super) solution to the equation

$$-\Delta\check{u} - \frac{Ay}{|y|^2}\nabla\check{u} - \frac{B}{|y|^2}\check{u} = \frac{c}{|y|^s}\check{u}^p \quad \text{in} \quad \check{\mathcal{C}}^1_\Omega, \tag{7}$$

where $s = (N+2)-p(N-2)-\sigma$ and $\check{\mathcal{C}}^1_\Omega := \{(r,\omega) \in \mathbb{R}^N : 0 < r < 1,\, \omega \in \Omega\}$ is the interior cone–like domain. For equation (7) we define the critical line

$$\check{l}^*(p) := \max\{\gamma_-(p-1)+2,\ \gamma_+(p-1)+2\} \qquad (p \in \mathbb{R}),$$

and the set $\check{\mathcal{N}} = \{(p,\sigma) \in \mathbb{R}^2_* : (7)$ *has no positive supersolutions*$\}$. By an easy computation we derive from Theorem 2 the following result.

Theorem 3. *Assume that $\gamma_-, \gamma_+ \in \mathbb{R}$. The following assertions are valid.*

(*i*) *Let $\gamma_- < \gamma_+$. Then $\check{\mathcal{N}} = \{\sigma \geq \check{l}^*(p)\}$.*
(*ii*) *Let $\gamma_- = \gamma_+$. Then $\check{\mathcal{N}} = \{\sigma > \check{l}^*(p)\} \cup \{\sigma = \check{l}^*(p),\, p \geq -1\}$.*

Our approach to the problem employs and extends new techniques developed in [15,16,17,18] and is quite different from that used in the quoted papers [4,6,8,12,20]. In the following we sketch the steps for proving Theorem 2. The complete proofs will be published elsewhere.

3. Sketch of the proof of Theorem 2

To show the existence one constructs radial supersolutions explicitly. The proof of the nonexistence is based upon comparison principles and asymptotics of harmonic functions of linear operators associated to (1). Observe that the transformation $\tilde{u} := |x|^{-\frac{A}{2}} u$ reduces (1) to an equation of type (1) with $A = 0$. We shall distinguish the cases $\gamma_- < \gamma_+$ and $\gamma_- = \gamma_+$.

$\gamma_- < \gamma_+$. In this case the proof is carried out following the ideas in [17,18] with minor modifications.

$\gamma_- = \gamma_+$. We only outline the ideas for the subcritical situation $\sigma < l^*(p)$. Let $u > 0$ be a positive supersolution to (1) in $\mathcal{C}^1_\Omega$. Then, following [1,17], one can prove the lower bound $u \geq c_1|x|^{\gamma_-}$ in $\mathcal{C}^2_{\Omega'}$, where Ω' is a subdomain of Ω. This bound in combination with a scaling argument as in [15,17] yields a contradiction if $\sigma < l^*(p)$ and $p \geq 1$.

When $p < 1$ rescaling of equation (4) together with a nonlinear comparison principle from [2,18] shows that any supersolution $u > 0$ to (4) obeys the lower bound $u \geq c_3|x|^{\frac{2-\sigma}{1-p}}$ in $\mathcal{C}^2_{\Omega'}$. An application of a Phragmèn–Lindelöf type comparison principle (see, e.g., [15]) allows one to conclude that $\liminf_{|x|\to\infty} \frac{u}{\log|x||x|^{\gamma_-}} < +\infty$ in $\mathcal{C}^2_{\Omega'}$. Comparison with the lower bound yields a contradiction for $\sigma < l^*(p)$ and $p < 1$.

More delicate analysis is needed on the critical line $\sigma = l^*(p)$ when $p > -1$ and especially at the critical point $(p, \sigma) = (-1, A + N)$. Along with the arguments similar to those used in the subcrtical case, the use of improved Hardy inequality (5) is required.

Acknowledgments

The authors are grateful to Vladimir Kondratiev and Zeev Sobol for interesting stimulating discussions.

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