ON THE MORSE CRITICAL GROUPS FOR INDEFINITE SUBLINEAR ELLIPTIC PROBLEMS

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ABSTRACT. We consider the Dirichlet problem for the equation $-\Delta u = \alpha u + m(x)u|u|^{q-2} + g(x,u)$, where $q \in (1,2)$ and m changes sign. We prove that the Morse critical groups at zero of the energy functional of the problem are trivial. As a consequence, existence and bifurcation of nontrivial solutions of the problem are established.

1. INTRODUCTION.

In this paper we are concerned with the Dirichlet problem

$$(P) \qquad \begin{cases} -\Delta u &= \alpha u + m(x)\, u|u|^{q-2} + g(x,u) \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a connected open bounded set, $\alpha \in \mathbb{R}$, $q \in (1,2)$, $m \in L^\infty(\Omega)$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function, i.e. $g(x,u)$ is measurable in x for all $u \in \mathbb{R}$ and continuous in u for almost all $x \in \Omega$.

Throughout the paper we assume that the sets

$$\Omega_+ = \{x \in \Omega : m(x) > 0\}, \qquad \Omega_- = \{x \in \Omega : m(x) < 0\}$$

are open, $\Omega_+ \neq \emptyset$ and g satisfies the condition

$$g(x,u) = o(|u|) \quad \text{as } u \to 0.$$

Thus $u \equiv 0$ is a trivial solution of (P) and the non–Lipschitz term $m(x)u|u|^{q-2}$ dominates $g(x,u)$ near zero. We are interested in the existence and bifurcation of nontrivial weak solutions of (P) for the case of indefinite weight function m, that is when both Ω_+ and Ω_- are nonempty sets. Indefinite problems of this type arise in population dynamics (see, e.g. [18]). They describes the asymptotic behaviour of population in a heterogeneous environment. The weight m represents the intrinsic growth rate of the population. It is positive on favourable habitats and negative on unfavourable ones.

In the case when m is constant or nonnegative there is extensive literature on nontrivial solutions of the problems of the type (P) (see, e.g., [4, 5, 11, 17, 14] and references therein). The typical feature of (P) in the indefinite case is that nontrivial solution of (P) may vanish on a nonempty open set within Ω_-. As a consequence of this fact, the structure of the solution set of (P) might be rather complicated. This phenomenon has been studied in details by C. Bandle, M. A. Pozio and A. Tesei [6, 21] in the case $\alpha = 0$. Existence and bifurcation of nontrivial nonnegative

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solutions for varying α has been recently studied by S. Alama [2]. The techniques used in these papers were mainly the methods of sub– and super–solutions and mountain–pass type arguments.

In the present paper we study problem (P) by methods of infinite dimensional Morse theory (see [9, 15]). Our first result here is the computation of the Morse critical groups at zero for the energy functional of (P). In the case $m \equiv 1$ this has been done in [17, 20]. The main difference and difficulty in the indefinite case is that a suitable control on the values of the parameter α is required. Roughly speaking, we prove, that if α is not too large, then the Morse critical groups at zero for the energy functional of (P) are trivial. This allows us to establish the existence of a nontrivial (possibly changing sign) solution of (P) when the nonlinearity $g(x,u)$ is asymptotically linear. We also study bifurcation of small solutions of (P) when the weight m is endowed with a small parameter.

The precise framework and statements of results are given in Section 2, while the proofs are delegated to consequent Sections 3-5.

2. Preliminaries and statements of results.

Throughout the paper, we denote by $\|\cdot\|_p$ the standard norm on the Lebesgue space $L_p(\Omega)$. By $(\cdot,\cdot)$ we denote the usual inner product on $L_2(\Omega)$. By $\|u\| = \|\nabla u\|_2$ and $\langle u, v\rangle = (\nabla u, \nabla v)$ we denote the norm and the corresponding inner product on the Sobolev space $H_0^1(\Omega)$. We denote by $\lambda_1(\Omega)$ the principal eigenvalue of the Dirichlet Laplacian on Ω and by $e_1(\Omega)$ the corresponding normalized positive eigenfunction. By a solution of problem (P) we always mean a weak solution. Also, the letter c will be used to denote various positive constants whose exact value is irrelevant.

The usual energy functional corresponding to problem (P) is defined by the formula

$$J(u) = \frac{1}{2}\|u\|^2 - \frac{\alpha}{2}\|u\|_2^2 - \frac{1}{q}\int_\Omega m(x)|u|^q\,dx - \int_\Omega G(x,u)\,dx,$$

where

$$G(x,u) = \int_0^u g(x,\xi)\,d\xi.$$

Let $2^* = \frac{2N}{N-2}$ if $N \geq 3$ or $2^* = \infty$ if $N = 2$ be the critical Sobolev exponent. It is well–known that if g satisfies the subcritical growth condition

$$(g) \qquad |g(x,u)| \leq c(1+|u|^{p-1}) \quad \text{with } \ p \in (2, 2^*),$$

then the energy J is of class $\mathcal{C}^1$ on $H_0^1(\Omega)$ and the critical points of J are weak solutions of (P). Moreover, if m, g and Ω satisfy some additional regularity assumptions, then by elliptic regularity theory weak solutions of (P) are actually classical solutions.

Following [19, 3], we consider the number

$$\lambda_*(m) = \inf\{\|v\|^2 : v \in H_0^1(\Omega),\ \|v\|_2^2 = 1,\ \int_\Omega m(x)|v|^q\,dx = 0\},$$

where we set $\inf\emptyset \equiv +\infty$. In particular $\lambda_*(m) = +\infty$ provided that $mes\{x \in \Omega : m(x) \leq 0\} = 0$. It is easy to see that $\lambda_*(m) \geq \lambda_1(\Omega)$, where the equality holds if and only if

$$\int_{\Omega} m(x)|e_1(\Omega)|^q \, dx = 0. \tag{2.1}$$

It is also clear that $\lambda_*(m) \leq \lambda_1(\Omega_0)$, where $\Omega_0 = \{x \in \Omega : m(x) = 0\}$ and $\lambda_1(\Omega_0) \leq +\infty$ is the principal eigenvalue of $-\Delta$ on the (relatively) closed set Ω_0. We refer to [12] for the definition of the Dirichlet Laplacian on arbitrary subsets of Ω. One can construct an indefinite function m in such a way that $\lambda_*(m)$ becomes arbitrarily close to $\lambda_1(\Omega_0)$. A related example is given in [3, p.112].

The Morse critical groups of J at the critical point u with $J(u) = c$ are defined by

$$C_k(J,u) = H_k(J^c \cap \mathcal{U}, \{J^c \cap \mathcal{U}\} \setminus \{u\}), \quad k \in \bar{\mathbb{N}} = \mathbb{N} \cup \{0\},$$

where $J^c = \{u \in H_0^1(\Omega) : J(u) \leq c\}$ denotes the sublevel of J, $\mathcal{U}$ is a closed neighborhood of u and H_k is the k-th singular homology group, cf. [9]. Due to the excision property of homology $C_k(J,u)$ is independent of $\mathcal{U}$. We refer to [22] for the topological notions mentioned in the paper. Our main result reads as follows.

Theorem 2.1. *Let $\alpha < \lambda_*(m)$ and (g) hold. Then $C_k(J,0) = 0$ for all $k \in \bar{\mathbb{N}}$.*

Let $\sigma(\Omega)$ be the spectrum of the Dirichlet Laplacian $-\Delta$ on Ω and $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ its eigenvalues, counted with their multiplicity. Then comparing the Morse critical groups of J at zero and at infinity in the usual way (see [9, 7]), we prove the existence of nontrivial solution of (P) when g is asymptotically linear.

Theorem 2.2. *Let $\alpha < \lambda_*(m)$. Suppose that for some $l \in \mathbb{N}$ one of the following conditions holds:*

i) $g(x,u) = \beta u + o(|u|)$ as $|u| \to \infty$ with $\beta + \alpha \in (\lambda_l, \lambda_{l+1})$;
ii) $g(x,u) = \beta u + o(|u|^{q-1})$ as $|u| \to \infty$ with $\beta + \alpha = \lambda_l < \lambda_*(m)$.

Then (P) has a solution $v \neq 0$.

Next we consider the parameter dependent problem

$$(P_\mu) \qquad \begin{cases} -\Delta u = \alpha u + \mu m(x)\, u|u|^{q-2} + g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mu \in \mathbb{R}$ and the other data are as in (P). By using Theorem 2.1 and the stability of the Morse critical groups under $\mathcal{C}^1$–perturbations of the functional (cf. [10]), we prove the following bifurcation result for (P_μ).

Theorem 2.3. *Let $\alpha < \lambda_*(m)$ and $\alpha \notin \sigma(\Omega)$. Then for each $\rho > 0$ there exists $\mu_+ \in (0,\rho)$ and $\mu_- \in (-\rho, 0)$ such that $(P_{\mu_\pm})$ have nontrivial solutions $v(\mu_\pm) \neq 0$ with $\|v(\mu_\pm)\| < \rho$.*

Remarks. (1) Notice that in case $m \equiv 1$, Theorem 2.1 is a variant of results of [17, 20]. Our proof in the indefinite case relies on the ideas of S. Alama and M. Del Pino [3], where the topology of sublevels of indefinite superquadratic functionals has been studied.

(2) The statement of Theorem 2.2 should be compared with the results of C. Bandle, M. A. Pozio, A. Tesei [6, 21] and S. Alama [2]. In [6, 21] the existence of nonnegative nontrivial solutions of (P) has been studied for $\alpha = 0$. In [2] (under the Neumann boundary conditions and suitable assumptions on g and m) the author exhibits a number $\alpha_* \geq 0$ such that (P) has several nonnegative solutions for

$\alpha \le \alpha_*$ and no nonnegative nontrivial solutions for $\alpha > \alpha_*$. In our framework one can show that if, for instance,

$$\int_\Omega m(x)|e_1(\Omega)|^q\,dx \ge 0 \quad \text{and} \quad ug(x,u) \ge 0 \quad \text{for all } x \in \Omega, \tag{2.2}$$

then $\alpha_* = \lambda_1(\Omega)$. On the other hand, if (2.1) does not hold then $\lambda_*(m) > \lambda_1(\Omega)$. Thus $\lambda_*(m)$ could be greater then α_* at least for some specific choices of m. In these cases Theorem 2.2 complements the result from [2] for $\alpha \in (\alpha_*, \lambda_*(m))$. Moreover, if $\lambda_*(m) > \alpha_*$ then the nontrivial solution v given by Theorem 2.2 necessarily changes sign in Ω. We conjecture that v is actually a changing sign solution also for $\alpha \le \alpha_*$ and, in particular, that it does not coincide with nonnegative solutions obtained in [6, 2]. The difficulty here is that, as we observed before, nontrivial solution of (P) may vanish on a nonempty open set contained in Ω_- ([6, 21]). As a consequence, the order preserving pseudo gradient flow for$J$ (if it exists) may not be strongly order preserving. Thus the methods of critical point theory in partially ordered spaces in the spirit of [11] or [8, 14], do not apply directly to (P). We are going to return such problems in future.

(3) In Theorem 2.3 we do not assume any growth condition on g. This means that bifurcation of small solutions at $\mu = 0$ is rather a local phenomenon which does not depend on the behavior of nonlinearity for large $|u|$. This gives a non symmetric complement to the results of Z. Jin [13] and Z.–Q. Wang [23], where the local bifurcation "to the right" from $\mu = 0$ has been established for odd nonlinearities and $m \gneq 0$. Notice also that the assumption $\alpha \notin \sigma(\Omega)$ could be omitted, but then we need to impose some additional conditions on g such that critical groups of J at zero could be computed when $\mu = 0$, cf. [7].

3. Proof of Theorem 2.1.

We divide our proof in several steps. First we establish a geometrical description of the sublevels of J in a neighborhood of zero. Let

$$E^+ = \{u \in H_0^1(\Omega) : \int_\Omega m(x)|u|^q\,dx > 0\}.$$

Clearly, $E^+ \cup \{0\}$ is starshaped with respect to the origin and $H_0^1(\Omega_+) \setminus \{0\} \subset E^+$. One can easily check that for each fixed $u \in E^+$ and $\tau \in \mathbb{R}$ the point $\tau = 0$ is a strict local maximum of the scalar function $J(\tau u)$. In particular, $J^0 \cap B_\rho \cap E^+ \ne \emptyset$ for any $\rho > 0$, where B_ρ stands for the the closed ball in $H_0^1(\Omega)$ of radius $\rho > 0$ with the center at zero.

Lemma 3.1. *Let $\alpha < \lambda_*(m)$ and (g) holds. Then there exists $\rho > 0$ such that $\{J^0 \cap B_\rho\} \setminus \{0\} \subset E^+ \cap B_\rho$.*

Proof. By a direct computation we obtain

$$J(u) = \frac{1}{2}\|u\|^2 - \frac{\alpha}{2}\|u\|_2^2 - \frac{1}{q}\int_\Omega m(x)|u|^q\,dx + o(\|u\|_2^2) \quad \text{as} \quad \|u\|_2 \to 0. \tag{3.1}$$

Assume by a contradiction, that there exists a sequence $(u_n) \subset H_0^1(\Omega)$ such that

$$\|u_n\| \to 0, \quad u_n \ne 0, \quad J(u_n) \le 0 \quad \text{and} \quad \int_\Omega m(x)|u_n|^q\,dx \le 0. \tag{3.2}$$

Thus $J(u_n) \to 0$ by continuity of J. Set $v_n = u_n \|u_n\|_2^{-1}$. By (3.2) and (3.1) we obtain

$$\frac{J(u_n)}{\|u_n\|_2^2} = \frac{1}{2}\left(\|v_n\|^2 - \alpha\right) - \frac{1}{q}\|u_n\|_2^{q-2}\int_\Omega m(x)|v_n|^q\,dx + o(1) \le 0. \tag{3.3}$$

Then by (3.2) it follows that

$$\frac{1}{2}\|v_n\|^2 \le \frac{1}{q}\|u_n\|_2^{q-2}\int_\Omega m(x)|v_n|^q\,dx + \frac{\alpha}{2} + o(1) \le \frac{\alpha}{2} + o(1). \tag{3.4}$$

Hence $\|v_n\|^2 \le \alpha + o(1)$ and a subsequence (which we still denote by (v_n)) converges weakly in $H_0^1(\Omega)$ to a certain $v_0 \in H_0^1(\Omega)$ such that

$$\|v_0\|^2 \le \alpha \quad \text{and} \quad \|v_0\|_2 = 1. \tag{3.5}$$

On the other hand, from (3.2) and (3.3) we derive

$$0 \ge \frac{1}{q}\|u_n\|_2^{q-2}\int_\Omega m(x)|v_n|^q\,dx \ge \frac{1}{2}\|v_n\|^2 - \frac{\alpha}{2} + o(1) \ge -\frac{\alpha}{2} + o(1).$$

Hence

$$0 \ge \int_\Omega m(x)|v_n|^q\,dx \ge \left(-\frac{\alpha q}{2} + o(1)\right)\|u_n\|_2^{2-q} = o(1).$$

Thus by weak continuity

$$\int_\Omega m(x)|v_0|^q\,dx = \lim \int_\Omega m(x)|v_n|^q\,dx = 0.$$

By definition of $\lambda_*(m)$ it follows that $\|v_0\|^2 \ge \lambda_*(m)$, in contradiction with (3.5) and with assumption $\alpha < \lambda_*(m)$. □

Lemma 3.2. *Let $\alpha < \lambda_*(m)$ and (g) hold. Then there exists $\rho > 0$ such that*

$$\frac{d}{d\tau}_{|\tau=1} J(\tau u) > 0. \tag{3.6}$$

for any $u \in M_\rho = \{u \in B_\rho \cap E^+ :\ J(u) \ge 0\}$.

Proof. By a direct computation we obtain

$$\frac{1}{q}\frac{d}{d\tau}_{|\tau=1} J(\tau u) = J(u) + \left(\frac{1}{q} - \frac{1}{2}\right)(\|u\|^2 - \alpha\|u\|_2^2) + o(\|u\|_2^2), \tag{3.7}$$

and

$$\frac{1}{2}\frac{d}{d\tau}_{|\tau=1} J(\tau u) = J(u) + \left(\frac{1}{q} - \frac{1}{2}\right)\int_\Omega m(x)|u|^q\,dx + o(\|u\|_2^2) \tag{3.8}$$

as $\|u\|_2 \to 0$. If $\alpha < \lambda_1(\Omega)$, then the statement of the lemma follows immediately from (3.7). Let $\alpha \ge \lambda_1(\Omega)$. Assume, by contradiction, that there exists a sequence $(u_n) \subset H_0^1(\Omega)$ such that

$$\|u_n\| \to 0, \quad u_n \ne 0, \quad J(u_n) \ge 0, \quad \frac{d}{d\tau}_{|\tau=1} J(\tau u_n) \le 0. \tag{3.9}$$

Then we set $v_n = u_n\|u_n\|_2^{-1}$. From (3.7) we obtain

$$0 \ge \left(\frac{1}{q} - \frac{1}{2}\right)(\|v_n\|^2 - \alpha) + o(1).$$

Thus $\|v_n\|^2 \le \alpha + o(1)$ and a subsequence (which we still denote by (v_n)) converges weakly in $H_0^1(\Omega)$ to a certain $v_0 \in H_0^1(\Omega)$ such that

$$\|v_0\|^2 \le \alpha \quad \text{and} \quad \|v_0\|_2 = 1. \tag{3.10}$$

From (3.8) we obtain

$$0 \geq (\frac{1}{q} - \frac{1}{2})\|u_n\|_2^{q-2} \int_\Omega m(x)|v_n|^q \, dx + o(1).$$

Hence

$$\int_\Omega m(x)|v_n|^q \, dx \leq \|u_n\|_2^{2-q} o(1) = o(1)$$

Thus by weak continuity we obtain

$$\int_\Omega m(x)|v_0|^q \, dx = \lim \int_\Omega m(x)|v_n|^q \, dx = 0$$

By definition of $\lambda_*(m)$ it follows that $\|v_0\|^2 \geq \lambda_*(m)$, in contradiction with (3.10) and with assumption $\alpha < \lambda_*(m)$. □

Lemma 3.3. *For all $\rho > 0$, the set $E^+ \cap B_\rho$ is contractible in itself.*

Proof. We are going to show that the embedding $i : H_0^1(\Omega_+) \setminus \{0\} \to E^+$ is a homotopy equivalence. Since $H_0^1(\Omega_+) \setminus \{0\}$ is well–known to be contractible in itself and $E^+ \cup \{0\}$ is starshaped with respect to the origin, then the assertion follows

Let $\theta : \bar{\Omega} \to \mathbb{R}$ be a Lipschitz continuous function such that

$$0 < \theta \leq 1 \quad \text{on } \Omega_+, \qquad \theta = 0 \quad \text{on } \Omega \setminus \Omega_+.$$

Let ϕ be the map defined by the formula

$$\phi(u) = \theta(x)u.$$

Clearly, ϕ is a (linear) continuous map from $H_0^1(\Omega)$ into $H_0^1(\Omega)$. Moreover,

$$\phi(u) = 0 \quad \text{in } \Omega \setminus \Omega_+$$

and hence $\phi(u) \in H_0^1(\Omega_+)$ (cf. [1, Theorem 9.1.3]). Moreover, if $u \in E^+$ then $Supp(u) \cap \Omega_+ \neq \emptyset$ and hence $\phi(u) \neq 0$. Thus ϕ maps continuously E^+ into $H_0^1(\Omega_+) \setminus \{0\}$.

We now consider the linear homotopy on E^+ defined by the formula

$$\mathcal{H}_t(u) = (1-t)u + t\phi(u).$$

Clearly $\mathcal{H}_0(u) = id_{E^+}$ and $\mathcal{H}_1(u) = i \circ \phi$. Moreover, since $u\phi(u) \geq 0$ we can estimate

$$\begin{aligned}
\int_\Omega m(x)|\mathcal{H}_t(u)|^q \, dx &= \int_\Omega m(x)|(1-t)u + t\phi(u)|^q \, dx \\
&\geq \int_{\Omega_+} m(x)|(1-t)u|^q \, dx + \int_{\Omega_-} m(x)|(1-t)u|^q \, dx \\
&= (1-t)^q \int_\Omega m(x)|u|^q \, dx > 0 \quad \text{for } t \in (0,1].
\end{aligned}$$

Therefore $\mathcal{H} : [0,1] \times E^+ \to E^+$ is a homotopy between id_{E^+} and $i \circ \phi$.

In the same way we can show that $id_{H_0^1(\Omega_+)\setminus\{0\}}$ is (linearly) homotopic to $\phi \circ i$. Thus ϕ is the homotopy inverse to i. □

Remark. A version of Lemma 3.3 has been proved in [3, Lemma 2.4] under the additional assumption $\bar{\Omega}_+ \cap \bar{\Omega}_- = \emptyset$. We note that the statement of Lemma 3.3 remains true for any $q \in [1, 2^*]$.

Proof of Theorem 2.1. Let us fix $\rho > 0$ such that Lemmas 3.1, 3.2 hold. To prove the theorem it is enough to verify the following two conditions

(a) $J^0 \cap B_\rho$ is contractible in itself;
(b) $\{J^0 \cap B_\rho\} \setminus \{0\}$ is contractible in itself.

Thus the result follows by the standard exactness properties of homology groups, cf. [22] and [17] for details.

Claim (a). We will show that the set $J^0 \cap B_\rho$ is starshaped with respect to the origin, i.e. that $u \in J^0 \cap B_\rho$ implies that $\tau u \in J^0 \cap B_\rho$ for all $\tau \in [0,1]$. Then the claim follows.

Assume, by a contradiction, that there exists $u_0 \in J^0 \cap B_\rho$ and $\tau_0 \in (0,1)$ such that $J(\tau_0 u_0) > 0$. Then from (3.6) it follows that

$$\frac{d}{d\tau} J(\tau_0 u_0) > 0.$$

By the monotonicity arguments this implies that

$$J(\tau u_0) > 0 \quad \text{for all } \ \tau \in [\tau_0, 1].$$

This contradicts the definition of u_0.

Claim (b). By Lemma 3.1 we know that $\{J^0 \cap B_\rho\} \setminus \{0\} \subseteq E^+$. By Lemma 3.3, the set $E^+ \cap B_\rho$ is contractible in itself. Moreover, by [22] the retract of the set which is contractible in itself is also contractible in itself. Thus it is suffices to show that $\{J^0 \cap B_\rho\} \setminus \{0\}$ is a retract of $E^+ \cap B_\rho$, which we now will prove.

As above, we set

$$M_\rho = \{u \in E^+ \cap B_\rho : J(u) \geq 0\}.$$

From Lemma 3.2 it follows that for each $u \in M_\rho$ there exists a positive solution $\tau(u) \in (0,1]$ of the equation

$$J(\tau u) = 0.$$

From the starshapeness of $J^0 \cap B_\rho$ it follows that such a solution $\tau(u)$ is unique. Fix $u \in M_\rho$. By (3.6) we have

$$\frac{d}{d\tau} J(\tau(u)u) > 0.$$

Hence the Implicit Function Theorem implies the continuity of the function $\tau(u)$ in a neighborhood of u in M. Therefore $\tau : M_\rho \to (0,1]$ is a continuous function.

Let $r : E^+ \to \{J^0 \cap B_\rho\} \setminus \{0\}$ be the map defined by the formula

$$r(u) = \begin{cases} \tau(u)u, & u \in M_\rho, \\ u, & u \in \{J^0 \cap B_\rho\} \setminus \{0\}. \end{cases}$$

The continuity of r follows from the continuity of τ and by the fact that by the definition

$$\tau(u) = 1 \quad \text{as} \quad u \in M_\rho \quad \text{and} \quad J(u) = 0.$$

Moreover

$$r(u) = u \quad \text{for } \ u \in \{J^0 \cap B_\rho\} \setminus \{0\}.$$

Thus r is a retraction of $\{E^+ \cap B_\rho\} \setminus \{0\}$ into $\{J^0 \cap B_\rho\} \setminus \{0\}$. □

4. Proof of Theorem 2.2.

We will only deal with case (ii), where $\alpha+\beta=\lambda_l$ for some $l\in\mathbb{N}$. Case (i) is simple and can be treated following arguments used in [17, p.394–395].

Our hypotheses on g implies that J is of class $\mathcal{C}^1$ on $H_0^1(\Omega)$, is asymptotically quadratic and has the form

$$J(u) = \frac{1}{2}\left(\|u\|^2-\omega\|u\|_2^2\right)-\frac{1}{q}\int_\Omega m(x)|u|^q\,dx+o(\|u\|_2^q) \tag{4.1}$$

$$= \frac{1}{2}\left(\|u\|^2-\omega\|u\|_2^2\right)+o(\|u\|_2^2)\quad \text{as } \|u\|_2\to\infty, \tag{4.2}$$

where $\omega=\alpha+\beta$. We claim also that J satisfies Palais–Smale condition (PS). We are going to show that one can split $H_0^1(\Omega)$ into the direct sum $H_0^1(\Omega)=V^+\oplus V^-$ with $\dim(V^-)=k\in\mathbb{N}$ such that

$$J \text{ is bounded from below on } V^+, \tag{4.3}$$

$$J(u)\to-\infty\quad \text{as } u\in V^-,\ \|u\|\to\infty. \tag{4.4}$$

Then J has a critical point v such that $C_k(J,v)\neq 0$ by the version of the Saddle Point Theorem with information on Morse critical groups. (See [15, Theorem 8.11] for the $\mathcal{C}^2$–statement of the theorem. The proof for $\mathcal{C}^1$–functionals is the same if one uses the $\mathcal{C}^1$–Deformation Lemma, see [9, p.21].)

We decompose $H_0^1(\Omega)=H^+\oplus H^-\oplus H^0$ according to the eigenvalues of $-\Delta-\omega$. That is, H^+ and H^- are the eigenspaces of $-\Delta$ spanned by the eigenfunctions corresponding to the eigenvalues $\lambda>\omega$ and $\lambda<\omega$, H^0 is the eigenspace of $-\Delta$ corresponding to eigenvalues $\lambda=\omega$. The subspaces H^- and H^0 are finite–dimensional and H^- is possibly trivial. Set

$$\Lambda_*=\int_\Omega m(x)|e_1(\Omega)|^q\,dx.$$

We shall distinguish the cases $\Lambda_*>0$ and $\Lambda_*<0$. Note that $\Lambda_*=0$ implies $\lambda_*(m)=\lambda_1(\Omega)$, see (2.1). This contradicts assumptions of the theorem.

Case $\Lambda_*>0$. Set $V^+=H^+$ and $V^-=H^0\oplus H^-$. First we observe that

$$\int_\Omega m(x)|u|^q\,dx>0\quad \text{for all } u\in V^-,\ u\neq 0. \tag{4.5}$$

Indeed, assume that

$$\int_\Omega m(x)|u|^q\,dx\le 0\quad \text{for some } u\in V^-. \tag{4.6}$$

Note that $e_1(\Omega)\in V^-$ and that $V^-\setminus\{0\}$ is arcwise connected. Since $\Lambda_*>0$ then by (4.6) and continuity there exists $\tilde u\in V^-\setminus\{0\}$ such that

$$\int_\Omega m(x)|\tilde u|^q\,dx=0.$$

Thus $\|\tilde u\|^2\ge\lambda^*(m)\|\tilde u\|_2^2$ by definition of $\lambda_*(m)$. On the other hand $\|\tilde u\|^2\le\omega\|\tilde u\|_2^2$ by definition of V^-, in contradiction with assumption $\omega<\lambda_*(m)$. This proves (4.5).

Later on, since E is homogeneous of order q and V^- is finite–dimensional, then (4.5) implies that

$$\int_\Omega m(x)|u|^q\,dx \ge c\|u\|^q \quad \text{for all } \ u \in V^-. \tag{4.7}$$

Using (4.1), (4.7) by direct computations we obtain

$$\begin{aligned} J(u) &\le -\frac{1}{q}\int_\Omega m(x)|u|^q\,dx + o(\|u\|^q) \\ &\le -\frac{c}{q}\|u\|^q + o(\|u\|^q) \to -\infty \quad \text{as } \ u \in V^-, \ \|u\| \to \infty. \end{aligned}$$

Also one can easily check that J is bounded from below on V^+. Thus (4.4) and (4.3) hold. Therefore J has a critical point v with $C_k(J,v) \neq 0$ where $k = dim(V^-)$. But $C_k(J,0) = 0$ by Theorem 2.1 and hence $v \neq 0$.

Case $\Lambda_* < 0$. Set $V^+ = H^0 \oplus H^+$ and $V^- = H^-$. It is easy to see that J satisfies (4.4) on V^-. We are going to check (4.3). Assume, by a contradiction, that there exists a sequence $(u_n) \subset H_0^1(\Omega)$ such that

$$(u_n) \subset V^+ \quad \text{and} \quad J(u_n) \le -n. \tag{4.8}$$

Thus by (4.2) we have

$$\frac{\omega}{2}\|u_n\|_2^2 \ge n + \frac{1}{2}\|u_n\|^2 + o(\|u_n\|_2^2) \ge n + o(\|u_n\|_2^2).$$

We conclude that $\|u_n\|_2$ is unbounded.

Set $v_n = u_n\|u_n\|_2^{-1}$. Thus by (4.2) we obtain

$$\frac{J(u_n)}{\|u_n\|_2^2} = \frac{1}{2}\left(\|v_n\|^2 - \omega\right) + o(1) \le 0. \tag{4.9}$$

It follows that $\|v_n\|^2 \le \omega + o(1)$. Since $\|v_n\|^2 \ge \omega$ on V^+ we conclude that a subsequence (which we still denote by (v_n)) converges weakly in $H_0^1(\Omega)$ to a certain $v_0 \in H_0^1(\Omega)$ such that

$$v_0 \in H^0 \quad \text{and} \quad \|v_0\|_2 = 1. \tag{4.10}$$

Later on, we observe that

$$\frac{1}{q}\int_\Omega m(x)|u_n|^q\,dx + o(\|u_n\|_2^q) \ge 0$$

in view of (4.8) and (4.1). By dividing by $\|u_n\|_2^q$ we obtain

$$\int_\Omega m(x)|v_n|^q\,dx + o(1) \ge 0.$$

Thus by weak continuity

$$\int_\Omega m(x)|v_0|^q\,dx = \lim \int_\Omega m(x)|v_n|^q\,dx \ge 0. \tag{4.11}$$

Set $\tilde V = H^- \oplus H^0$. Thus $v_0, e_1(\Omega) \in \tilde V$. Note that $\tilde V \setminus \{0\}$ is arcwise connected. Since $\Lambda_* < 0$ then by (4.11) and continuity one can find $\tilde u \in \tilde V \setminus \{0\}$ such that

$$\int_\Omega m(x)|\tilde u|^q\,dx = 0.$$

Therefore $\|\tilde{u}\|^2 \geq \lambda^*(m)\|\tilde{u}\|_2^2$ by definition of $\lambda_*(m)$. On the other hand $\|\tilde{u}\|^2 \leq \omega\|\tilde{u}\|_2^2$ by definition of $\tilde{V}$, in contradiction with assumption $\omega < \lambda_*(m)$.

Thus (4.4) and (4.3) hold. Therefore J has a critical point v with $C_k(J,v) \neq 0$ where $k = dim(V^-)$. But $C_k(J,0) = 0$ by Theorem 2.1 and hence $v \neq 0$.

(PS)*–condition.* To complete the proof we have to verify (PS). Let $(u_n) \in H_0^1(\Omega)$ be a (PS)–sequence for J, i.e.

$$J(u_n) = O(1), \qquad \|\nabla J(u_n)\| = o(1). \tag{4.12}$$

We claim that $\|u_n\|_2$ is bounded. Suppose the contrary and set $v_n = u_n\|u_n\|_2^{-1}$. Using (4.2) and (4.12) we obtain

$$o(1) = \frac{J(u_n)}{\|u_n\|_2^2} = \frac{1}{2}\left(\|v_n\|^2 - \omega\right) + o(1).$$

Hence $\|v_n\|^2 = \omega + o(1)$ and a subsequence (which we still denote by (v_n)) converges weakly to $v_0 \in H_0^1(\Omega)$ such that

$$\|v_0\|^2 \leq \omega \quad \text{and} \quad \|v_0\|_2 = 1. \tag{4.13}$$

Then by a direct computation and exploiting (4.12) we obtain

$$\begin{aligned} O(1) + o(1)\|u_n\| &= J(u_n) - \frac{1}{2}\langle \nabla J(u_n), u_n\rangle \\ &= (\frac{1}{q} - \frac{1}{2})\int_\Omega m(x)|u_n|^q\,dx + o(\|u_n\|_2^q). \end{aligned}$$

By dividing by $\|u_n\|_2^q$ we obtain

$$o(1) + o(1)\|v_n\| = (\frac{1}{q} - \frac{1}{2})\int_\Omega m(x)|v_n|^q\,dx + o(1).$$

Since $\|v_n\|$ is bounded then by weak continuity

$$\int_\Omega m(x)|v_0|^q\,dx = \lim \int_\Omega m(x)|v_n|^q\,dx = 0.$$

Also, by definition of $\lambda_*(m)$, we have $\|v_0\|^2 \geq \lambda_*(m)$, in contradiction with (4.13). We conclude that $\|u_n\|_2$ is bounded.

Finally, from (4.2) and (4.12), we obtain

$$\begin{aligned} O(1) = J(u_n) &= \frac{1}{2}\left(\|u_n\|^2 - \omega\|u_n\|_2^2\right) + o(\|u_n\|^2) \\ &= \frac{1}{2}\|u_n\|^2 + O(1). \end{aligned}$$

Therefore (u_n) is bounded in $H_0^1(\Omega)$ and the subcritical growth of g ensures that there is a strongly convergent subsequence. □

5. Proof of Theorem 2.3.

Let $r > 0$ be fixed and $\tilde{g} : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that

$$\tilde{g}(x,u) = g(x,u) \quad \text{for all } |u| \leq r, \tag{5.1}$$

$$\lim_{|u|\to\infty} \frac{\tilde{g}(x,u)}{u} = \beta \quad \text{with } \beta + \alpha < \lambda_1(\Omega). \tag{5.2}$$

We define the truncated energy for (P_μ) by

$$\tilde{J}_\mu(u) = \frac{1}{2}\|u\|^2 - \frac{\alpha}{2}\|u\|_2^2 - \frac{\mu}{q}\int_\Omega m(x)|u|^q\,dx - \int_\Omega \tilde{G}(x,u)\,dx,$$

where

$$\tilde{G}(x,u) = \int_0^u \tilde{g}(x,\xi)\,d\xi.$$

Due to (5.2) the functional $\tilde{J}_\mu$ is of class $\mathcal{C}^1$, is coercive and satisfies (PS) for each fixed $\mu \in \mathbb{R}$. Furthermore, for each $\rho > 0$ the map $\mu \mapsto \tilde{J}_\mu$ is continuous from $[0,\rho]$ into $\mathcal{C}^1(B_\rho)$ and $\tilde{J}_\mu$ is uniformly bounded on $[0,\rho] \times B_\rho$.

By contradiction, we assume that there exists $\rho > 0$ such that for each fixed $\mu \in [0,\rho]$ zero is the unique critical point of $\tilde{J}_\mu$ in B_ρ. Then by the $\mathcal{C}^1$–homotopy invariance of the Morse critical groups (see Theorems IV.4 and IV.3 in [10]) we can conclude that $C_k(\tilde{J}_0, 0) = C_k(\tilde{J}_\rho, 0)$.

On the other hand, we have $\alpha \notin \sigma(\Omega)$ by the assumption of the theorem. Then zero is an (isolated) nondegenerate critical point of $\tilde{J}_0$. Therefore $C_k(\tilde{J}_0, 0) = \delta_{lk}\mathcal{F}$ for some $l \in \mathbb{N}$, cf. [16]. Since $\lambda_*(\rho m) = \lambda_*(m)$, the assumption $\alpha < \lambda_*(m)$ and Theorem 2.1 ensure that $C_k(\tilde{J}_\rho, 0) = 0$ for all $k \in \bar{\mathbb{N}}$, a contradiction. Due to the arbitrariness of $\rho > 0$ this means that there exists a sequence $\mu_n \downarrow 0$ such that $\tilde{J}_{\mu_n}$ has critical points $v(\mu_n) \neq 0$ with $\|v(\mu_n)\| \to 0$.

By using (5.2) it can be shown by standard arguments that $\|v(\mu_n)\|_\infty < c$ for some $c > 0$. By elliptic regularity theory this implies that the sequence $(v(\mu_n))$ is precompact in $L^\infty(\Omega)$ and hence that $\|v(\mu_n)\|_\infty \to 0$. Thus, for n large $\|v(\mu_n)\|_\infty < r$ and $v(\mu_n)$ are solutions of the original problems (P_{μ_n}).

The same arguments apply in the left neighborhoods of $\mu = 0$. □

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