A critical phenomenon for sublinear elliptic equations in cone–like domains

Vladimir Kondratiev
Department of Mathematics
and Mechanics
Moscow State University
Moscow 119 899, Russia
kondrat@vnmok.math.msu.su

Vitali Liskevich
School of Mathematics
University of Bristol
Bristol BS8 1TW
United Kingdom
v.liskevich@bristol.ac.uk

Vitaly Moroz
School of Mathematics
University of Bristol
Bristol BS8 1TW
United Kingdom
v.moroz@bristol.ac.uk

Zeev Sobol
Department of Mathematics
Imperial College, London
London SW7 2AZ
United Kingdom
z.sobol@imperial.ac.uk

Abstract

We study positive supersolutions to an elliptic equation $(*)$ $-\Delta u = c|x|^{-s}u^p$, $p, s \in \mathbb{R}$, in cone–like domains in $\mathbb{R}^N$ $(N \geq 2)$. We prove that in the sublinear case $p < 1$ there exists a critical exponent $p_* < 1$ such that equation $(*)$ has a positive supersolution if and only if $-\infty < p < p_*$. The value of p_* is determined explicitly by s and the geometry of the cone.

1 Introduction

We study the existence and nonexistence of positive solutions and supersolutions to the equation

$$-\Delta u = \frac{c}{|x|^s} u^p \quad \text{in } \mathcal{C}_\Omega^\rho. \tag{1}$$

Here $p \in \mathbb{R}$, $s \in \mathbb{R}$, $c > 0$ and $\mathcal{C}_\Omega^\rho \subset \mathbb{R}^N$ $(N \geq 2)$ is an unbounded cone–like domain

$$\mathcal{C}_\Omega^\rho := \{(r, \omega) \in \mathbb{R}^N : \omega \in \Omega,\ r > \rho\},$$

where (r, ω) are the polar coordinates in $\mathbb{R}^N$, $\rho > 0$ and $\Omega \subseteq S^{N-1}$ is a subdomain (a connected open subset) of the unit sphere S^{N-1} in $\mathbb{R}^N$. We say that $u \in H^1_{loc}(\mathcal{C}_\Omega^\rho)$ is a *supersolution* (*subsolution*) to equation (1) if

$$\int_{\mathcal{C}_\Omega^\rho} \nabla u \cdot \nabla \varphi \, dx \geq (\leq) \int_{\mathcal{C}_\Omega^\rho} \frac{c}{|x|^s} u^p \varphi \, dx \quad \text{for all } 0 \leq \varphi \in C_0^\infty(\mathcal{C}_\Omega^\rho).$$

If u is a sub and supersolution to (1) then u is said to be a *solution* to (1). By the weak Harnack inequality any nontrivial nonnegative supersolution to (1) is positive in $\mathcal{C}_\Omega^\rho$.

The classical Liouville theorem asserts that every positive superharmonic function on $\mathbb{R}^N$ is constant. Generalizations of this result to positive (super) solutions of semilinear elliptic equations of type (1) go back to earlier works by Serrin [14] and celebrated paper by Gidas and Spruck [6] (for recent results and a historical survey see [15]). Equation (1), known also in astrophysics as a generalized Lane–Emden equation, is a prototype model for general semilinear equations. The qualitative theory of equations (1) has been extensively studied because of the rich mathematical structure and various applications for the whole range of the parameter $p \in \mathbb{R}$, e.g in combustion theory ($p > 1$) [13], population dynamics ($0 < p < 1$) [11], pseudoplastic fluids ($p < 0$) [9]. Liouville type theorems for equation (1) have been obtained so far mainly for the superlinear case $p > 1$. In this short note we uncover a new critical phenomenon for $p < 1$, similar in nature to the phenomena known in the superlinear case. In order to describe old and new results we define *critical exponents* for equation (1) as

$$p^* = p^*(\Omega, s) = \inf\{p > 1 : (1) \text{ has a positive supersolution in } \mathcal{C}^\rho_\Omega \text{ for some } \rho > 0\},$$

$$p_* = p_*(\Omega, s) = \sup\{p < 1 : (1) \text{ has a positive supersolution in } \mathcal{C}^\rho_\Omega \text{ for some } \rho > 0\}.$$

Set $p_* = -\infty$ if (1) has no positive supersolution in $\mathcal{C}^\rho_\Omega$ for any $p < 1$.

Remark 1. (*i*) One can show that if $p < p_*$ or $p > p^*$ then (1) has a positive solution in $\mathcal{C}^\rho_\Omega$ (see [7] for the proof of the case $p > 1$ and the proofs below for the case $p < 1$). The existence (or nonexistence) of positive (super) solutions at the critical values p_* and p^* is a separate issue.

(*ii*) Observe that in view of the scaling invariance of the Laplacian the critical exponents p_* and p^* do not depend on $\rho > 0$.

(*iii*) We do not make any assumptions on the smoothness of the domain $\Omega \subseteq S^{N-1}$.

Let $\lambda_1 = \lambda_1(\Omega) \geq 0$ be the principal eigenvalue of the Dirichlet Laplace–Beltrami operator $-\Delta_\omega$ on Ω. Let $\alpha_+ \geq 0$ and $\alpha_- < 0$ be the roots of the quadratic equation

$$\alpha(\alpha + N - 2) = \lambda_1(\Omega).$$

In the *superlinear* case $p > 1$ the value of the critical exponent is $p^* = 1 - \frac{2-s}{\alpha_-}$. Moreover, if $s < 2$ then (1) has no positive supersolutions in the critical case $p = p^*$. This has been proved by Bandle and Levine [3], Bandle and Essen [2] and Berestycki, Capuzzo–Dolcetta and Nirenberg [4] (see also [7, 10, 12] for further extensions of this result and related problems).

The *sublinear* case $p < 1$ has been studied in [5, 8]. From the result of Brezis and Kamin [5] it follows that for $p \in (0, 1)$ equation (1) has a bounded positive solution in $\mathbb{R}^N$ if and only if $s > 2$. It has been proved in [8] (amongst other things) that for any $p \in (-\infty, 1)$ equation (1) has a positive supersolution outside a ball in $\mathbb{R}^N$ if and only if $s > 2$.

In this note we show that in sublinear case equation (1) exhibits a "non-trivial" critical exponent ($p_* > -\infty$) in cone–like domains. The main result of the paper reads as follows.

Theorem 1. *For $p \leq 1$, the critical exponent for equation (1) is $p_* = \min\{1 - \frac{2-s}{\alpha_+}, 1\}$. If $p_* < 1$ then (1) has no positive supersolutions in the critical case $p = p_*$. If $p < p_*$ then (1) has a positive solution.*

Remark 2. (*i*) If $\alpha_+ = 0$ then we set $p_* = -\infty$. Then the result follows from [8, Theorem 1.2].

(*ii*) If $s > 2$ then $p_* = p^* = 1$ and (1) has positive solutions for any $p \in \mathbb{R}$ [5, 8]. If $s = 2$ then $p_* = p^* = 1$. In this critical case (1) becomes a linear equation with the potential $c|x|^{-2}$, which has a positive (super) solution if and only if $c \leq \frac{(N-2)^2}{4} + \lambda_1(\Omega)$.

(*iii*) Let $S_k = \{x \in S^{N-1} : x_1 > 0, \dots x_k > 0\}$. Then $\lambda_1(S_k) = k(k+N-2)$ and $\alpha_+(S_k) = k$, $\alpha_-(S_k) = 2 - N - k$. Hence $p_*(S_k, s) = 1 - \frac{2-s}{k}$ and $p^*(S_k, s) = 1 - \frac{2-s}{2-N-k}$. In particular, in the case of the halfspace S_1 we have $p_*(S_1, s) = s - 1$ and $p^*(S_1, s) = \frac{N+1-s}{N-1}$.

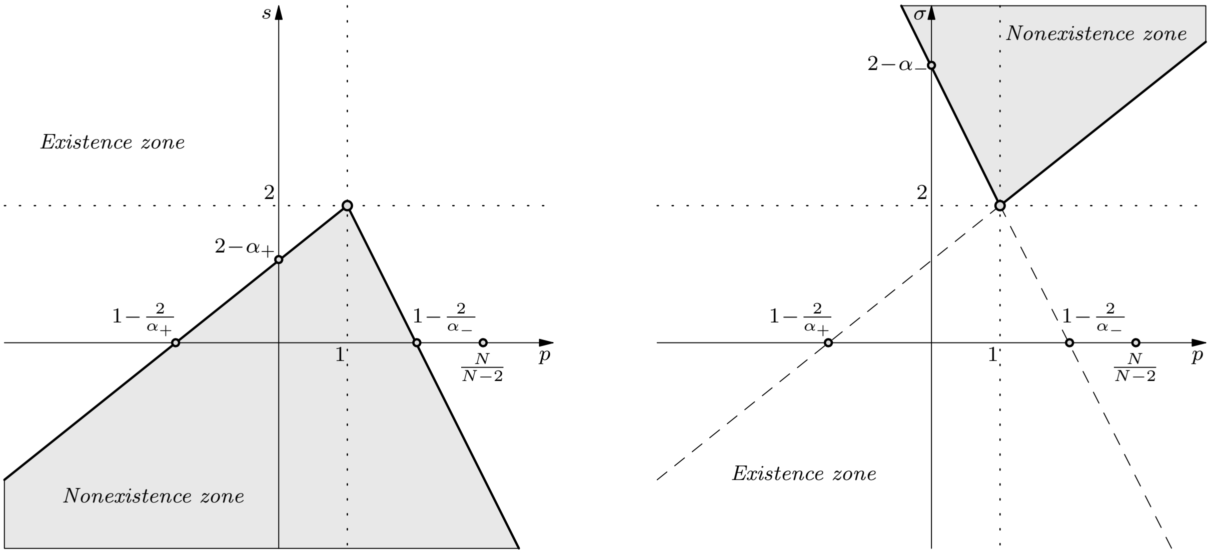

Figure 1: Existence and nonexistence zones for equations (1) (left) and (2) (right).

Applying the Kelvin transformation $y = y(x) = \frac{x}{|x|^2}$ we see that if u is a positive solution to (1) in $\mathcal{C}^1_\Omega$ then $\hat{u}(y) = |y|^{2-N}u(x(y))$ is a positive solution to

$$-\Delta\hat{u} = \frac{c}{|y|^\sigma}\hat{u}^p \quad \text{in } \widehat{\mathcal{C}}^1_\Omega, \tag{2}$$

where $\sigma = (N+2) - p(N-2) - s$ and $\widehat{\mathcal{C}}^1_\Omega := \{(r,\omega) \in \mathbb{R}^N : \omega \in \Omega,\ 0 < r < 1\}$. We define the critical exponents $\widehat{p}^* = \widehat{p}^*(\Omega, s)$ and $\widehat{p}_* = \widehat{p}_*(\Omega, s)$ for equation (2) similarly to $p^*(\Omega, s)$ and $p_*(\Omega, s)$. In the superlinear case $p > 1$, Bandle and Essen [2] proved that if $\sigma > 2$ then $\widehat{p}^* = 1 - \frac{2-\sigma}{\alpha_+}$ and (2) has no positive supersolutions when $p = \widehat{p}^*(\Omega)$. In the sublinear case $p < 1$ by an easy computation we derive from Theorem 1 the following result.

Theorem 2. *For $p \le 1$, the critical exponent for equation (2) is $\widehat{p}_* = \min\{1 - \frac{2-\sigma}{\alpha_-}, 1\}$. If $\widehat{p}_* < 1$ then (2) has no positive supersolutions in the critical case $p = \widehat{p}_*$. If $p < \widehat{p}_*$ then (2) has a positive solution.*

In the remaining part of the paper we prove Theorem 1.

2 Proof of Theorem 1

From now on we assume that $\lambda_1(\Omega) > 0$.

Existence. In the polar coordinates equation (1) reads as follows

$$-u_{rr} - \frac{N-1}{r}u_r - \frac{1}{r^2}\Delta_\omega u = \frac{c}{r^s}u^p \quad \text{in } \mathcal{C}^1_\Omega. \tag{3}$$

Let $s \le 2$, $p < 1 - \frac{2-s}{\alpha_+}$. Let $0 < \psi \in H^1_{loc}(\Omega)$ be a positive solution to the equation

$$-\Delta_\omega\psi - \alpha(\alpha + N - 2)\psi = \psi^p \quad \text{in } \Omega, \tag{4}$$

where $\alpha := \frac{2-s}{1-p}$. Then it is readily seen that $u := c^{\frac{1}{1-p}} r^\alpha \psi \in H^1_{loc}(\mathcal{C}^1_\Omega)$ is a positive solution to (3) in $\mathcal{C}^1_\Omega$. Thus the problem reduces to the existence of positive solutions to (4).

Note that $0 < \alpha(\alpha + N - 2) < \lambda_1(\Omega)$. Hence the operator $-\Delta_\omega - \alpha(\alpha + N - 2)$ is coercive on $H_0^1(\Omega)$ and satisfies the maximum principle. We consider separately the cases $p \in [0, 1)$ and $p < 0$.

Case $p \in [0, 1)$. Let $\phi_1 > 0$ be the principal Dirichlet eigenfunction of $-\Delta_\omega$ on Ω. Let $\overline{\phi} > 0$ be the unique solution to the problem

$$-\Delta_\omega \phi - \alpha(\alpha + N - 2)\phi = 1, \qquad \phi \in H_0^1(\Omega).$$

Observe that $\phi_1, \overline{\phi} \in L^\infty$. Hence $\tau\overline{\phi}$ is a supersolution to (4) for a large $\tau > 0$, and $\epsilon\phi_1$ is a subsolution to (4) for a small $\epsilon > 0$. Thus by the sub and supersolutions argument equation (4) has a solution $\psi \in H_0^1(\Omega)$ such that $\epsilon\phi_1 < \psi \leq \tau\overline{\phi}$.

Case $p < 0$. Consider the problem

$$-\Delta_\omega \phi - \alpha(\alpha + N - 2)(\phi + 1) = (\phi + 1)^p, \qquad \phi \in H_0^1(\Omega). \tag{5}$$

Let $\overline{\phi} > 0$ be the unique solution to the problem

$$-\Delta \phi - \alpha(\alpha + N - 2)(\phi + 1) = 1, \qquad \phi \in H_0^1(\Omega).$$

It is clear that $\overline{\phi}$ is a supersolution to (5) and $\underline{\phi} \equiv 0$ is a subsolution to (5). We conclude that (5) has a positive solution $\phi \in H_0^1(\Omega)$ such that $0 < \phi \leq \overline{\phi}$. Then $\psi := \phi + 1 \in H^1_{loc}(\Omega)$ is a positive solution to (4). This completes the proof of the existence part of Theorem 1.

Nonexistence. In what follows we set $\delta := 1$ if $p < 0$ and $\delta := 0$ if $p \in [0, 1)$. Let $G \subset \mathbb{R}^N$ be a domain, $0 \notin G$. Observe that equation (1) has a positive supersolution in G if and only if the equation

$$-\Delta w = \frac{c}{|x|^s}(w + \delta)^p \quad \text{in } G \tag{6}$$

has a positive supersolution. Indeed, if $u > 0$ is a supersolution to (1) in G then u is a supersolution to (6). If $w > 0$ is a supersolution to (6) then $u = w + \delta$ is a supersolution to (1). The main argument of the proof nonexistence rests upon the following two lemmas.

The next lemma is an adaptation a comparison principle by Ambrosetti, Brezis and Cerami [1, Lemma 3.3].

Lemma 3. *Let $G \subset \mathbb{R}^N$ be a domain, $0 \notin G$. Let $0 \leq \underline{w} \in H_0^1(G)$ be a subsolution and $0 \leq \overline{w} \in H^1_{loc}(G)$ a supersolution to (6). Then $\underline{w} \leq \overline{w}$ in G.*

Proof. In [1, Lemma 3.3] the result was proved for a smooth bounded domain G and $\underline{w}, \overline{w} \in H_0^1(G)$ (and more general nonlinearities). The proof given in [1] carries over literally to the case of an arbitrary bounded domain G and $\underline{w}, \overline{w} \in H_0^1(G)$, or a smooth domain G, $\underline{w} \in H_0^1(G)$ and $0 \leq \overline{w} \in H^1(G)$. Thus we only need to extend the lemma to an arbitrary bounded domain G and $\overline{w} \in H^1_{loc}(G)$.

Let $\overline{w} \in H^1_{loc}(G)$ be a supersolution to (6) in G. Let $(G_n)_{n\in\mathbb{N}}$ be an exhaustion of G, that is a sequence of bounded smooth domains such that $\overline{G}_n \subset G_{n+1} \subset G$ and $\cup_{n\in\mathbb{N}} G_n = G$. Analogously to the argument given above in the existence part of the proof, one can readily see that, for each $n \in \mathbb{N}$, there exists a solution $0 < w_n \in H_0^1(G_n)$ to (6) (e.g., by constructing appropriate sub and supersolutions). Moreover, $w_n \leq w_{n+1}$. Observe that $w_n \leq \overline{w}$ in G_n by [1, Lemma 3.3].

We claim that $\sup \|\nabla w_n\|_{L^2} < \infty$. This is clear for $p < 0$, since $(u+1)^p \le 1$. For $p \in [0,1)$, using w_n as a test function in (6), we have

$$\int_G |\nabla w_n|^2 dx = \int_G \frac{c}{|x|^s} w_n^{p+1}\, dx \le c_1 \left(\int_G |\nabla w_n|^2 dx \right)^{(p+1)/2},$$

which implies the claim. It follows that w_n converges pointwise in G, strongly in $L^2(G)$ and weakly in $H_0^1(G)$ to a positive $w_* \in H_0^1(G)$. Clearly $w_* > 0$ is a solution to (6) in G and $0 < w_* \le \overline{w}$ in G.

Now let $0 \le \underline{w} \in H_0^1(G)$ be a subsolution to (6) in G. By [1, Lemma 3.3] we conclude that $\underline{w} \le w_*$ in G. □

Next, consider the initial value problem

$$(7) \qquad -v_{rr} - \frac{N-1}{r} v_r + \frac{\lambda_1}{r^2} v = \frac{c}{r^s} v^p \quad \text{for } r > 1; \qquad v(1) = \delta, \quad v_r(1) = K;$$

where $p < 1$, $s \in \mathbb{R}$, $c > 0$, $K > 1$ and δ as above. Let $(1, R)$, $R = R(\delta, K) \le \infty$, be the maximal right interval of existence of the solution v to (7) in the region $\{(r,v) \in (1,+\infty) \times (\delta, +\infty)\}$.

Lemma 4. *Let $s < 2$ and $p \in [1 - \frac{2-s}{\alpha_+}, 1)$. Then for any interval $[r_*, r^*] \subset (1, +\infty)$ there exists $K_0 > 1$ such that*

i) for all $K > K_0$ one has $r^* < R < +\infty$ and $v(r) \to \delta$ as $r \nearrow R$;

ii) for any $M > \delta$ there exists $K > K_0$ such that $\min_{[r_*, r^*]} v \ge M$.

Proof. Set $\alpha := \alpha_+$, $v := w r^\alpha$, $t = r^{2-N-2\alpha}$. Then w solves the following problem

$$w_{tt} + c_1 t^{-\sigma} w^p = 0 \quad \text{for } t \in (T, 1); \qquad w(1) = \delta, \quad w_t(1) = -L,$$

where $\sigma = \frac{2N-2+\alpha(p+3)-s}{N-2+2\alpha} \ge 2$, $c_1 > 0$, $0 \le T = R^{2-N-2\alpha} < 1$ and $L = \frac{K - \alpha\delta}{N-2+2\alpha} \to \infty$ as $K \to \infty$. Choose K_0 such that $L > \delta$. Observe that $w(t)$ is concave, hence

$$\delta < w(t) \le w(1) - w_t(1)(1-t) \le \delta + L \quad \text{for } t \in (T, 1).$$

To see that $T > 0$ let $\tilde{w} := w$ for $p < 0$, otherwise let $\tilde{w} := w^{1-p}$. Then $\tilde{w}$ satisfies the inequality

$$\tilde{w}_{tt} + c_2 t^{-2} \tilde{w}^q \le 0 \quad \text{for } t \in (T, 1),$$

with $c_2 > 0$ and $q := \min\{p, 0\}$. Integrating $\tilde{w}_{tt}$ twice one can easily see that such inequality has no positive solutions in any neighborhood of zero. Thus we conclude that $T > 0$, hence $w(t) \to \delta$ as $t \searrow T$. In particular, $w(t)$ attains its maximum on $(T, 1)$.

Let $T_0 \in (T, 1)$ be such that $w_t(T_0) = -\frac{L-\delta}{2}$. Since $\delta \le w(t) \le \delta + L$ for $t \in (T_0, 1)$, it follows that

$$\frac{L+\delta}{2} = w_t(T_0) - w_t(1) = -\int_{T_0}^1 w_{tt} d\tau = c_1 \int_{T_0}^1 \frac{w^p}{\tau^\sigma} d\tau \le c_3 \left(\frac{1}{T_0^{\sigma-1}} - 1 \right) \quad \text{for } t \in (T_0, 1).$$

Hence $T_0 \to 0$ as $L \to +\infty$. Therefore for any given $t^* < 1$ there exists $L_0 > 1$ such that for any $L > L_0$ one has $0 < T < T_0 < t^*$. Thus, (i) follows with $r^* = (t^*)^{\frac{1}{N-2+2\alpha}}$.

Observe now that for any $L > L_0$ we have

$$-\frac{L-\delta}{2} \ge w_t(t) \ge -L \quad \text{for } t \in (t^*, 1),$$

since w is concave. Hence for any $t \in (t^*, 1)$ we obtain

$$w(t) = w(1) - \int_t^1 w_t\, d\tau \ge \delta + (1-t)\frac{L-\delta}{2} \to \infty \quad \text{as } L \to \infty.$$

Thus (ii) follows. □

Nonexistence – completed. Let $p \in [1 - \frac{2-s}{\alpha_+}, 1)$. Fix a compact $K \subset \mathcal{C}^1_\Omega$ and $M > 1$. There exists an interval $[r_*, r^*] \subset (1, +\infty)$ such that $K \subset \mathcal{C}^{(r_*, r^*)}_\Omega$, where $\mathcal{C}^{(r_1, r_2)}_\Omega$ denotes the set $\{x \in \mathcal{C}^1_\Omega \,|\, r_1 \le |x| \le r_2\}$. Then by Lemma 4 there exists $v : (1, R) \to (\delta, +\infty)$ solving (7) such that $R > r^*$ and $\inf_{[r_*, r^*]} v \ge M + \delta$.

Let $\phi_1 > 0$ be the principal Dirichlet eigenvalue of $-\Delta_\omega$ on Ω with $\|\phi_1\|_\infty = 1$. Set $w_M := (v - \delta)\phi_1$. Then $0 < w_M \in H^1_0(\mathcal{C}^{(1,R)}_\Omega)$, and direct computation shows that w_M is a subsolution to (6) in $\mathcal{C}^{(1,R)}_\Omega$. Now assume that $w > 0$ is a supersolution to (6) in $\mathcal{C}^1_\Omega$. By Lemma 3 it follows that that $w \ge w_M$ in $\mathcal{C}^{(1,R)}_\Omega$. By the weak Harnack inequality we have

$$\inf_K w \ge c_H \int_K w \, dx \ge c_H \int_K w_M \, dx \ge c_2 M.$$

Since M was arbitrary, we conclude that $w \equiv +\infty$ in K. $\square$

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