Positive solutions to nonlinear p-Laplace equations with Hardy potential in exterior domains

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Abstract

We study the existence and nonexistence of positive (super) solutions to the nonlinear p-Laplace equation

$$-\Delta_p u - \frac{\mu}{|x|^p} u^{p-1} = \frac{C}{|x|^\sigma} u^q$$

in exterior domains of $\mathbb{R}^N$ $(N \geq 2)$. Here $p \in (1, +\infty)$ and $\mu \leq C_H$, where C_H is the critical Hardy constant. We provide a sharp characterization of the set of $(q, \sigma) \in \mathbb{R}^2$ such that the equation has no positive (super) solutions.

The proofs are based on the explicit construction of appropriate barriers and involve the analysis of asymptotic behavior of super-harmonic functions associated to the p-Laplace operator with Hardy-type potentials, comparison principles and an improved version of Hardy's inequality in exterior domains. In the context of the p-Laplacian we establish the existence and asymptotic behavior of the harmonic functions by means of the generalized Prüfer-Transformation.

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Keywords: p-Laplacian, Hardy potential, Liouville theorems, improved Hardy inequality, Phragmén–Lindelöf principle, Prüfer transformation.

Contents

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1 Introduction and Results

We study the problem of the existence and nonexistence of positive (super) solutions to nonlinear p-Laplace equation with Hardy potential

$$-\Delta_p u - \frac{\mu}{|x|^p} u^{p-1} = \frac{C}{|x|^\sigma} u^q \quad \text{in} \quad B_\rho^c, \tag{1.1}$$

where $-\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator, $1 < p < \infty$, $C > 0$, $\mu \in \mathbb{R}$, $(q,\sigma) \in \mathbb{R}^2$ and $B_\rho^c := \{x \in \mathbb{R}^N : |x| > \rho\}$ is the exterior of the ball in $\mathbb{R}^N$, with $N \geq 2$. We say that $u \in W^{1,p}_{loc}(G) \cap C(G)$ is a *super-solution* to equation (1.1) in a domain $G \subseteq \mathbb{R}^N$ with $0 \notin G$ if for all $0 \leq \varphi \in W^{1,p}_c(G) \cap C(G)$ the following inequality holds

$$\int_G \nabla u |\nabla u|^{p-2} \nabla\varphi \, dx - \int_G \frac{\mu}{|x|^p} u^{p-1}\varphi \, dx \geq \int_G \frac{C}{|x|^\sigma} u^q \varphi \, dx.$$

Here and below $W^{1,p}_c(G) := \{u \in W^{1,p}_{loc}(G), \mathrm{supp}(u) \Subset G\}$. The notions of a sub-solution and solution are defined similarly, by replacing " $\geq$ " with " $\leq$ " and " $=$ ", respectively. It follows from the Harnack inequality (cf. [43]) that any nontrivial nonnegative super-solution to (1.1) in G is strictly positive in G.

One of the features of equation (1.1) on unbounded domains is the nonexistence of positive solutions for certain values of the exponent q. Such Liouville type nonexistence phenomena have been known for semilinear elliptic equations ($p = 2$) at least since the celebrated works of Serrin in the earlier 70's (cf. the references in [44]) and of Gidas and Spruck [25]. One of the first Liouville-type results for the nonlinear p-Laplace equations in exterior domains is due to Bidaut–Véron [8, Theorem 1.3]. Theorem A below extends the result in [8], including the cases $p > N$ and $q < p - 1$.

Theorem A ([8, Theorem 1.3], Theorem 1.1 below). *The equation*

$$-\Delta_p u = u^q \quad \text{in} \quad B_\rho^c \tag{1.2}$$

has no positive super-solutions if and only if $q_* \leq q \leq q^*$, *where* $q^* = \frac{N(p-1)}{N-p}$ *when* $p < N$, *or* $q^* = +\infty$ *when* $p \geq N$, *and* $q_* = -\infty$ *when* $p \leq N$ *or* $q_* = \frac{N(p-1)}{N-p}$ *when* $p > N$.

Theorem A had been generalized and extended in various direction by many authors (see, e.g., [1, 9, 35, 44, 48] and references therein). The techniques in those works usually involve careful integral estimates and/or sophisticated analysis of related nonlinear ODE's. A different approach to nonlinear Liouville type theorems goes to back to an earlier paper by Kondratiev and Landis [26] and was recently developed in the context of semilinear equations ($p = 2$) in

[27, 28, 29, 30]. The approach is based on the pointwise Phragmén–Lindelöf type bounds on positive super-harmonic functions and related Hardy–type inequalities.

Recall that the classical *Hardy inequality* states that

$$\int_{B_\rho^c} |\nabla u|^p dx \geq C_H \int_{B_\rho^c} \frac{u^p}{|x|^p} dx, \qquad \forall\, u \in C_c^\infty(B_\rho^c), \tag{1.3}$$

with the sharp constant $C_H = \left|\frac{N-p}{p}\right|^p$, $p > 1$. The optimality of the Hardy constant C_H implies, via Picone's identity, the following nonexistence result.

Theorem B ([6, Corollary 2.2], Corollary 3.2 below) *The equation*

$$-\Delta_p u - \frac{\mu}{|x|^p} u^{p-1} = 0 \quad in \quad B_\rho^c \tag{1.4}$$

has no positive super-solutions if and only if $\mu > C_H$.

Let us sketch a simple proof of the nonexistence part of Theorem A in the case $p \neq N$ and $q_* < q < q^*$. Indeed, let $u > 0$ be a super-solution to (1.2). Then $-\Delta_p u \geq 0$ in B_ρ^c. A comparison principle for the p-Laplacian in exterior domains (see Theorem 2.1 and Theorems 3.4 and 3.5 below) implies that u obeys the *Phragmén–Lindelöf type bounds*

$$c|x|^{\gamma_-} \leq \inf_{|x|=r} u \leq c^{-1}|x|^{\gamma_+} \quad \text{in} \quad B_{2\rho}^c, \tag{1.5}$$

where $\gamma_- = \min\{0, \frac{p-N}{p-1}\}$ and $\gamma_+ = \max\{0, \frac{p-N}{p-1}\}$.

Assume that $q \geq p-1$ and perform a *homogenization* of equation (1.2) rewriting it in the from

$$-\Delta_p u = V u^{p-1} \quad \text{in} \quad B_\rho^c, \tag{1.6}$$

where $V(x) := u^{q-(p-1)}$. Using the lower bound from (1.5), we conclude that

$$V(x) \geq c_1 |x|^{\gamma_-(q-p+1)} \quad \text{in} \quad B_{2\rho}^c.$$

Hence, by Theorem B, equation (1.6) has no positive super-solutions provided $\gamma_-(q-p+1) > -p$. Therefore (1.2) has no positive super-solutions when $p-1 \leq q < q^*$.

Now assume that $q < p-1$. Then a standard scaling argument (see Lemma 4.4 below) shows that any super-solution $u > 0$ to nonlinear equation (1.2) obeys the lower bound

$$u \geq c|x|^{\frac{p}{(p-1)-q}} \quad \text{in} \quad B_{2\rho}^c.$$

Comparing this "nonlinear" estimate with the upper bound in (1.5), we conclude that equation (1.2) has no positive super-solutions for $q_* < q < p-1$.

The above simple proof relies only on Theorem B and pointwise Phragmén–Lindelöf type bounds (1.5). It does not cover the *critical cases* $q = q_*$ and $q = q^*$, where additional arguments are required. On the other hand, an explicit construction of radial super-solutions to (1.2) when $q \notin [q_*, q^*]$ shows that the values of the *critical exponents* q^* and q_* are sharp. Considerations of this type first appeared in [27]. They have proved to be a powerful and flexible tool for studying nonlinear Liouville phenomena for various classes of elliptic operators and domains, see [27, 28, 29, 30, 32, 33].

In this paper we are interested in nonlinear Liouville theorems for perturbations of the p-Laplace operator by the Hardy type potential. To explore the impact of the potential on the value of the critical exponents q^* and q_*, let us consider the equation of the form

$$-\Delta_p u - \frac{\mu}{|x|^{p+\epsilon}} u^{p-1} = u^q \quad \text{in} \quad B_\rho^c, \tag{1.7}$$

where $\mu \in \mathbb{R}$ and $\epsilon \in \mathbb{R}$. One can verify directly that *if* $\epsilon < 0$ *and* $\mu < 0$, *then* (1.7) *admits positive solutions for all* $q \in \mathbb{R}$, while *if* $\epsilon < 0$ *and* $\mu > 0$ *then* (1.7) *has no positive super-solutions for any* $q \in \mathbb{R}$. The latter follows immediately from Theorem B. On the other hand, one can show (see [33, Theorem 1.2]) that if $\epsilon > 0$ then (1.7) *has the same critical exponents* q^* *and* q_* *as* (1.2). This follows from the fact that positive super-solutions to

$$-\Delta_p u - \frac{\mu}{|x|^{p+\epsilon}} u^{p-1} = 0 \quad \text{in } B_\rho^c \tag{1.8}$$

satisfy the same bound (1.5) as super-solutions to $-\Delta_p u \geq 0$ in B_ρ^c.

In this paper we show that in the borderline case $\epsilon = 0$ the critical exponents for equation (1.7) explicitly depend on μ. This is a consequence of the fact that the Phragmén–Lindelöf bounds for equation (1.8) with $\epsilon = 0$ become sensitive to the value of the parameter μ. Such phenomenon and its relation to the Hardy type inequalities has been recently observed for $p = 2$ in the case of the ball as well as exterior domains in [12, 13, 19, 32, 37, 45, 47]. The main difficulty comparing with the semilinear case $p = 2$ arises when a comparison principle for the p–Laplacian has to be involved in the argument. After examples in [16, 21] (see also [23, 46]) it is known that solutions to the equation $-\Delta_p u + V u^{p-1} = 0$ may not satisfy the usual comparison principle as soon as the potential has a nontrivial negative part. The proof of the (restricted) comparison principle requires delicate arguments. We provide a new version of the comparison principle (Theorem 2.1), following the ideas from [34]. In order to use this result for obtaining sharp Phragmén–Lindelöf bounds one has to produce explicitly a radial sub-solution to a homogeneous equation in the exterior of the ball with zero data on the sphere. This has been resolved in this paper by means of the generalized Prüfer transformation (see [39] and Appendix B.3), which, up to our knowledge, has never been used before in this context. We also provide an elementary proof of an improved Hardy Inequality in exterior domains. The improved Hardy Inequality plays a crucial role in our analysis of equation (1.1) in the critical case $\mu = C_H$.

To formulate the main result of the paper we assume that $\mu \leq C_H$, otherwise (1.1) has no positive super-solutions by Theorem B. When $\mu \leq C_H$, the scalar equation

$$-\gamma|\gamma|^{p-2}\left(\gamma(p-1) + N - p\right) = \mu \tag{1.9}$$

has two real roots $\gamma_- \leq \gamma_+$. Note that if $\mu = C_H$ then $\gamma_- = \gamma_+ = \frac{p-N}{p}$. For $\mu \leq C_H$ we introduce the critical line $\Lambda^*(q,\mu)$ for equation (1.1) on the (q,σ)–plane

$$\Lambda_*(q,\mu) := \min\{\gamma_-(q-p+1)+p,\ \gamma_+(q-p+1)+p\} \qquad (q \in \mathbb{R}),$$

and the nonexistence set

$$\mathcal{N} = \{(q,\sigma) \in \mathbb{R}^2 \setminus (p-1,p) : (1.1) \textit{ has no positive supersolutions in } B_\rho^c\}.$$

Theorem 1.1. *The following assertions are valid.*

(*i*) *If* $\mu < C_H$ *then* $\mathcal{N} = \{\sigma \leq \Lambda_*(q)\}$.

(*ii*) *If* $\mu = C_H$ *then* $\mathcal{N} = \{\sigma < \Lambda_*(q)\} \cup \{\sigma = \Lambda_*(q),\ q \geq -1\}$.

Remark 1.2. (*i*) Observe that in view of the scaling invariance of (1.1) if $u(x)$ is a solution to (1.1) in B_ρ^c then $\tau^{\frac{p-\sigma}{q-(p-1)}} u(\tau y)$ is a solution to (1.1) in $B_{\rho/\tau}^c$, for any $\tau > 0$. So in what follows, for $q \neq p-1$, we confine ourselves to the study of solutions to (1.1) on B_1^c. For the same reason, for $q \neq p-1$ we may assume that $C = 1$, when convenient.

(*ii*) Using sub- and super-solutions techniques one can show that if (1.1) has a positive super-solution in B_ρ^c then it has a positive solution in B_ρ^c (see Lemma 2.6). Thus for any $(q,\sigma)\in\mathbb{R}^2\setminus\mathcal{N}$ equation (1.1) admits positive solutions.

(*iii*) Figure 1 shows the qualitative pictures of the set $\mathcal{N}$ for typical values of γ^-, γ^+ and different relations between p and the dimension $N\ge 2$.

The paper is organized as follows. Section 2 contains various preliminary results, including versions of the Comparison Principle and Weak Maximum Principle in unbounded domains. In Section 3 we give a new proof of an improved Hardy Inequality with sharp constants, which is based on Picone's identity and simplifies some arguments used in the recent papers [3, 7, 20, 24]. Section 3 also includes sharp Phragmén–Lindelöf bounds. The proof of the main result of the paper, Theorem 1.1, is contained in Section 4.

The Appendix includes various auxiliary results which are systematically used in the main part of the paper and often are of independent interest. Part A of the Appendix describes well–known Picone's identity and some of its corollaries. Parts B and C of the Appendix contain explicit constructions and estimates of radial sub- and super-solutions to homogeneous p-Laplace equations with Hardy–type potentials. Finally, in Part C of the Appendix we construct *large sub-solutions* to a homogeneous equation in the exterior of the ball with zero data on the sphere using the generalized Prüfer transformation techniques.

2 Background, framework and auxiliary facts

Here and thereafter $N\ge 2$, $1<p<\infty$, $q\in\mathbb{R}$ and $C>0$, unless specified otherwise. For $0<\rho<R\le+\infty$, we denote the exterior of the closed ball, the open annulus and the sphere of the radii ρ by

$$B_\rho^c=\left\{x\in\mathbb{R}^N\colon |x|>\rho\right\},\quad A_{\rho,R}=\left\{x\in\mathbb{R}^N\colon \rho<|x|<R\right\},\quad S_\rho=\{x\in\mathbb{R}^N\colon |x|=\rho\}.$$

For a function $u=u(x)$ we denote $u^+=\max\{u,0\}$ and $u^-=-\min\{u,0\}$ the positive and negative parts of u, respectively. By $c,c_1,c_2,\dots$ we denote various positive constants whose exact values are irrelevant.

Homogeneous form associated to p-Laplacian. Let $\mathcal{E}_V$ be a homogeneous form defined by

$$\mathcal{E}_V(u):=\int_G|\nabla u|^p\,dx-\int_G V|u|^p\,dx\qquad (u\in W_c^{1,p}(G)\cap C(G)),\tag{2.1}$$

where $G\subseteq\mathbb{R}^N$ is a domain (i.e. an open connected set), and $0\le V\in L^\infty_{loc}(G)$ a potential. Consider the equations associated with $\mathcal{E}_V$

$$-\Delta_p u-V|u|^{p-2}u=0\quad\text{in}\quad G,\tag{2.2}$$

$$-\Delta_p u-V|u|^{p-2}u=f\quad\text{in}\quad G,\tag{2.3}$$

where $0\le f\in L^1_{loc}(G)$. We say that $u\in W^{1,p}_{loc}(G)\cap C(G)$ is a *super-solution* to equation (2.3) in a domain $G\subseteq\mathbb{R}^N$ if for all $0\le\varphi\in W_c^{1,p}(G)\cap C(G)$ the following inequality holds

$$\int_G\nabla u|\nabla u|^{p-2}\nabla\varphi\,dx-\int_G\frac{\mu}{|x|^p}|u|^{p-2}u\varphi\,dx\ge\int_G f\varphi\,dx.$$

The notions of sub-solution and solution are defined similarly by replacing " $\ge$ " with " $\le$ " and " $=$ " respectively.

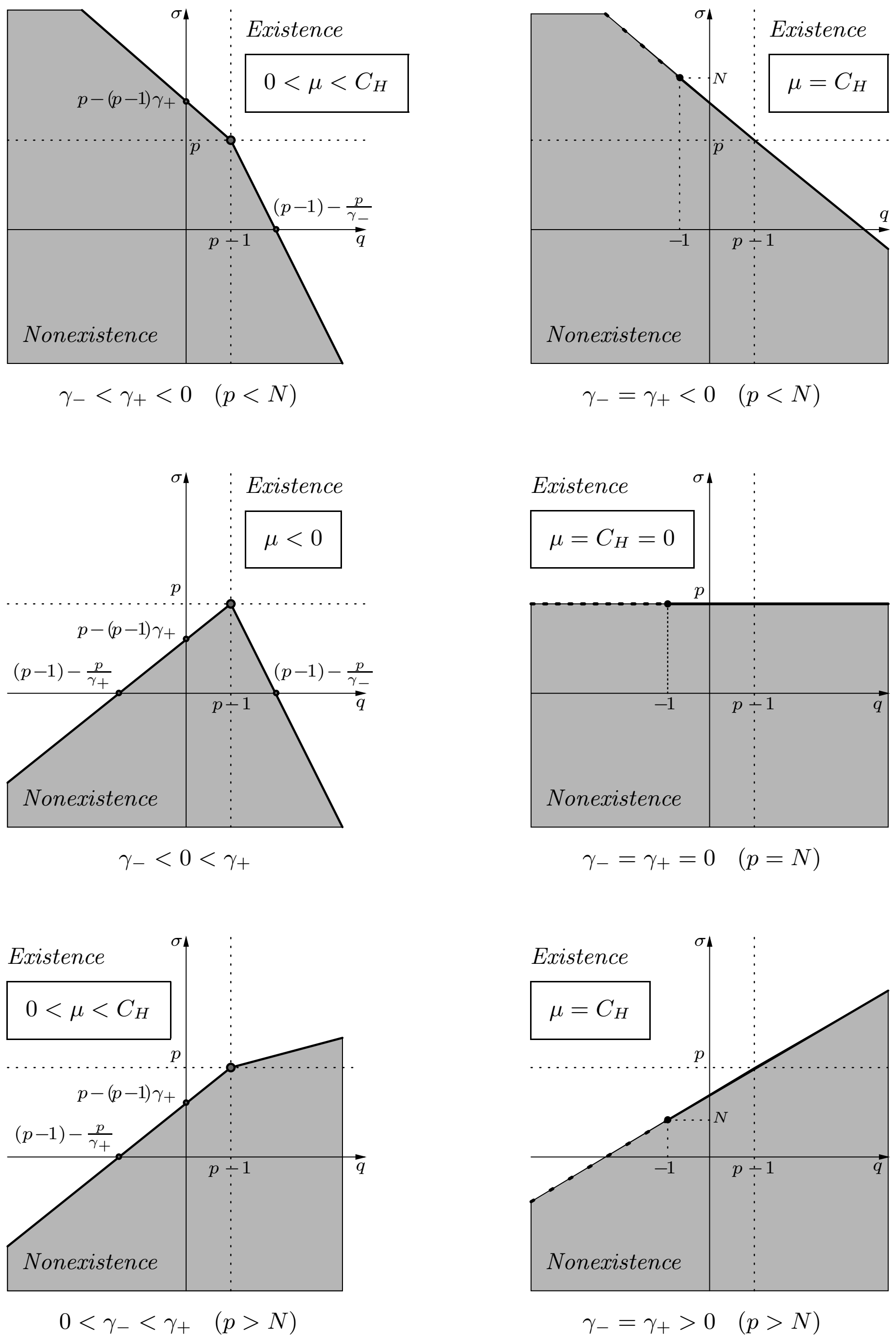

Figure 1: The nonexistence set $\mathcal{N}$ of equation (1.1) for typical values of γ_- and γ_+.

Let $u \geq 0$ be a solution to (2.2) in G and let $G' \Subset G$. Then the following strong Harnack inequality (cf. [43, Theorems 5, 6, 9]) holds

$$\sup_{G'} u \leq C_S \inf_{G'} u, \tag{2.4}$$

where the constant $C_S > 0$ depends on p, N, G', G only. The Harnack inequality and comparison principle in bounded domains [23, 46] imply that any nontrivial nonnegative super-solution to (1.1) in G is strictly positive in G.

Comparison and Maximum Principles. We say that $0 \leq w \in W^{1,p}_{loc}(G)$ satisfies condition $(\mathcal{S})$ if the following holds:

$(\mathcal{S})$ there exists $(\theta_n)_{n\in\mathbb{N}} \subset W^{1,\infty}_c(\mathbb{R}^N)$ such that $0 \leq \theta_n \to 1$ a.e. in $\mathbb{R}^N$ and

$$\int_G \mathcal{R}(\theta_n w, w)\, dx \to 0 \quad \text{as } n \to +\infty,$$

where $\mathcal{R}$ is defined in Proposition A.1. Notice that if G is bounded and $w \in W^{1,p}(G)$ then condition $(\mathcal{S})$ is trivially satisfied with $\theta = 1$ in G.

Using condition $(\mathcal{S})$, we establish a version of comparison principle in a form suitable for our framework. The proof follows with certain modifications the ideas in [34, 38, 42].

Theorem 2.1 (Comparison Principle). *Let $q < p-1$ and $0 \leq f \in L^1_{loc}(G)$. Let $0 < u \in W^{1,p}_{loc}(G) \cap C(\bar{G})$ be a super-solution and $v \in W^{1,p}_{loc}(G) \cap C(\bar{G})$ a sub-solution to equation*

$$-\Delta_p u - V|u|^{p-2}u = f|u|^{q-1}u \quad \text{in} \quad G. \tag{2.5}$$

If G is an unbounded domain, assume in addition that $\partial G \neq \emptyset$ and v^+ satisfies condition $(\mathcal{S})$. Then $u \geq v$ on ∂G implies $u \geq v$ in G.

Proof. Let $S := \{x \in G : v > u\}$. Assume for a contradiction that $S \neq \emptyset$. Then

$$K := \sup_{x\in S} \left(\log \frac{v}{u}\right) \in (0, +\infty].$$

Fix a positive constant b such that $5b < K$. Let $\eta \in C^1(\mathbb{R})$ be a nondecreasing function such that

$$\eta(t) = 0 \text{ for } t \leq 2b, \quad \eta(t) = 1 \text{ for } t \geq 5b \quad \text{and} \quad \eta'(t) > 0 \text{ for } 3b \leq t \leq 4b.$$

Let $\xi = \eta(\log \frac{v}{u})$. Then $0 \leq \xi \in W^{1,p}_{loc}(G) \cap C(\bar{G})$ and $\operatorname{supp}(\xi) \subset S \subseteq G$. Let $\theta \in W^{1,\infty}_c(\mathbb{R}^N)$ then $\operatorname{supp}(\theta\, \xi)$ is compact in G. Later on we specify θ for the case of a bounded and unbounded G. Set

$$\phi_1 := \left(\frac{\theta^p v^p}{u^{p-1}}\right)\xi, \qquad \phi_2 := \theta^p v \xi.$$

Clearly ϕ_1, $\phi_2 \in W^{1,p}_c(G)$. Since u is a super-solution to (2.5), testing (2.5) by ϕ_1 and using Picone's Identity we infer that

$$\begin{aligned}
0 \quad &\leq \quad \int_S \xi|\nabla u|^{p-2} \cdot \nabla u \cdot \nabla\left(\frac{\theta^p v^p}{u^{p-1}}\right) dx + \int_S \theta^p v^p \, \nabla \log u \cdot |\nabla \log u|^{p-2} \cdot \nabla\xi \, dx \\
&- \quad \int_S V\theta^p v^p \xi dx - \int_S \frac{f u^q}{u^{p-1}} \theta^p v^p \xi \, dx \\
&= \quad \int_S |\nabla(\theta v)|^p \xi \, dx - \int_S \mathcal{R}(\theta v, u)\xi \, dx + \int_S \theta^p v^p \, \nabla \log u \cdot |\nabla \log u|^{p-2} \cdot \nabla\xi \, dx \\
&- \quad \int_S V\theta^p v^p \xi dx - \int_S f u^{q-(p-1)} \theta^p v^p \xi \, dx.
\end{aligned}$$

Thus from Proposition A.1 we obtain

$$\int_S |\nabla(\theta v)|^p \xi\, dx + \int_S \theta^p v^p |\nabla \log u|^{p-2} \nabla \log u \cdot \nabla \xi\, dx \quad - \quad \int_S V \theta^p v^p \xi\, dx \\ \geq \quad \int_S f u^{q-(p-1)} \theta^p v^p \xi\, dx.$$

Since v is a sub-solution to (2.5), testing (2.5) by ϕ_2 we derive

$$\int_S |\nabla v|^{p-2} \nabla v \cdot \nabla(\theta^p v) \xi dx + \int_S \theta^p v^p |\nabla \log v|^{p-2} \nabla \log v \cdot \nabla \xi dx \quad - \quad \int_S V \theta^p v^p \xi dx \\ \leq \quad \int_S f \theta^p v^{q+1} \xi dx.$$

Subtracting the former inequality from the latter one and using Picone's Identity again we obtain

$$\begin{aligned} \mathcal{I}_\theta \quad &:= \quad \int_S \theta^p v^p \, \left(|\nabla \log v|^{p-2} \cdot \nabla \log v - |\nabla \log u|^{p-2} \cdot \nabla \log u \right) \nabla \xi \, dx \\ &\leq \quad \int_S \left\{ |\nabla(\theta v)|^p - |\nabla v|^{p-2} \cdot \nabla v \cdot \nabla(\theta^p v) - f \theta^p v^p (u^{q-(p-1)} - v^{q-(p-1)}) \right\} \xi \, dx \\ &\leq \quad \int_{S \cap \operatorname{supp}(\theta \xi)} \mathcal{R}(\theta v, v) \, dx. \end{aligned} \tag{2.6}$$

We claim that

$$\mathcal{I}_* := \int_S v^p \left(|\nabla \log v|^{p-2} \cdot \nabla \log v - |\nabla \log u|^{p-2} \cdot \nabla \log u \right) \nabla \xi \, dx \leq 0 \tag{2.7}$$

implies $S = \emptyset$. Define the open subset $S' \subset S$ by

$$S' := \{x \in G : \left(\log \frac{v}{u}\right) \in (3b, 4b)\} \subset S$$

and observe that $\eta'(\log \frac{v}{u}) > 0$ on S'. There exists at least one connected component S_i of the set S' such that $\log \frac{u}{v}$ attains all values between $3b$ and $4b$ on S_i.

Since

$$\nabla \xi = (\nabla \log v - \nabla \log u) \, \eta'(\log \frac{v}{u}),$$

and

$$\left(|z_1|^{p-2} z_1 - |z_2|^{p-2} z_2 \right) (z_1 - z_2) \geq 0, \qquad \forall \, z_1, z_2 \in \mathbb{R}^N,$$

with equality if and only if $z_1 = z_2$, from (2.7) we have $\mathcal{I}_* = 0$. Therefore $\log \frac{v}{u} = c_i$ on S_i, which is a contradiction.

Below we show that (2.7) holds. Indeed, if the domain G is bounded then $\operatorname{supp}(\xi) \Subset S$ and one simply chooses $\theta \equiv 1$ on $\bar{G}$ and $\theta \equiv 0$ on $R^N \setminus \bar{G}$. Then (2.6) implies that $\mathcal{I}_* \leq 0$.

Now let G be an unbounded domain. Let θ_n satisfies condition $(\mathcal{S})$. Then $\operatorname{supp}(\theta_n) \cap \operatorname{supp}(\xi) \neq \emptyset$ for n large enough and from (2.6) we have

$$I_{\theta_n} \leq \int_G \mathcal{R}(\theta_n v, v) \, dx \to 0 \qquad \text{as} \quad n \to \infty.$$

So the assertion follows. □

The proof of the following lemma follows closely the arguments in [5, Lemma 2.9].

Lemma 2.2. *Let v be a sub-solution to* (2.2)*. Then v^+ is a sub-solution to* (2.2)*.*

Proof. For any $\epsilon > 0$ define $v_\epsilon = (v^2 + \epsilon^2)^{1/2}$. Then $0 < v_\epsilon \in W^{1,p}_{loc}(G)$ and by the Lebesgue dominated convergence theorem, v_ϵ converges to $|v|$ in $W^{1,p}_{loc}(G)$. Let $0 \le \phi \in W^{1,p}_{loc}(G) \cap C(G)$. A direct computation shows that

$$\nabla v_\epsilon \cdot \nabla \phi = \nabla v \cdot \nabla \left(\frac{v}{v_\epsilon} \phi \right) - \frac{v_\epsilon^2 - v^2}{v_\epsilon^3} \phi \, |\nabla v|^2,$$

which implies that $\nabla v_\epsilon \cdot \nabla \phi \le \nabla v \cdot \nabla \left(\frac{v}{v_\epsilon} \phi \right)$. Set $\phi_\epsilon = \frac{1}{2} \left(1 + \frac{v}{v_\epsilon} \right) \phi$. It follows that

$$\frac{1}{2} \nabla (v + v_\epsilon) \cdot \nabla \phi = \frac{1}{2} \left(\nabla v \cdot \nabla \phi + \nabla v_\epsilon \cdot \nabla \phi \right) \le \nabla v \cdot \nabla \phi_\epsilon. \tag{2.8}$$

Testing (2.2) against ϕ_ϵ and using (2.8) we derive

$$\begin{aligned} 0 \;&\ge\; \int_G |\nabla v|^{p-2} \nabla v \cdot \nabla \phi_\epsilon \, dx - \int_G V |v|^{p-2} v \phi_\epsilon \, dx \\ &\ge\; \int_G |\nabla v|^{p-2} \, \nabla \frac{1}{2} (v + v_\epsilon) \cdot \nabla \phi \, dx - \int_G V |v|^{p-2} v \phi_\epsilon \, dx. \end{aligned} \tag{2.9}$$

Notice that $\frac{1}{2}(v + v_\epsilon) \to v^+$ and $\phi_\epsilon \to \chi_{\{v^+ > 0\}} \phi$ a.e. in G as $\epsilon \to 0$. Letting $\epsilon \to 0$ in (2.9) we infer that

$$\int_G |\nabla (v^+)|^{p-2} \, \nabla (v^+) \cdot \nabla \phi \, dx - \int_G V (v^+)^{p-1} \phi \, dx \le 0,$$

which completes the proof. □

We establish the Weak Maximum Principle for super-solutions to (2.2) as a corollary of the Comparison Principle and Lemma 2.2.

Proposition 2.3 (Weak Maximum Principle)**.** *Let $\partial G \ne \emptyset$. Assume that* (2.2) *admits a positive super-solution $0 < \phi \in W^{1,p}_{loc}(G) \cap C(\bar{G})$. Let $u \in W^{1,p}_{loc}(G) \cap C(\bar{G})$ be a super-solution to equation* (2.2) *such that $u \ge 0$ on ∂G. For an unbounded G assume in addition that u^- satisfies condition* ($\mathcal{S}$)*. Then $u \ge 0$ in G.*

Proof. By Proposition 2.2 observe that $u^- \in W^{1,p}_{loc}(G) \cap C(\bar{G})$ is a sub-solution to (2.2) and $u^- = 0$ on ∂G. Thus $u^- \le \varepsilon \phi$ on ∂G, for any $\varepsilon > 0$. By Theorem 2.1, we conclude that $u^- \le \varepsilon \phi$ in G for an arbitrary small $\varepsilon > 0$. Hence $u^- = 0$ in G. □

Remark 2.4. After examples constructed in [16, 21] (see also discussions in [23, 46]) it is known that the form $\mathcal{E}_V$ is nonconvex as soon as $p \ne 2$ and the potential V has a nontrivial 'negative' part V^+, even if $\mathcal{E}_V$ is nonnegative and admits representation (A.1) with respect to a positive super-solution of (2.3). One of consequences of this fact is that the assumption $f \ge 0$ in Theorem 2.1 can not be removed, otherwise the comparison principle fails.

Positive solution between sub- and super-solutions. We show that the existence of a positive super-solution to nonlinear equation (1.1) implies the existence of a positive *solution* to (1.1). The following result on bounded domains is standard.

Lemma 2.5. *Let $\mu \le C_H$ and $G \subset B_1^c$ be a bounded smooth domain. Let $v, u \in W^{1,p}(G) \cap C(\bar{G})$ be a sub- and super-solution to (1.1) in G, respectively. Assume that $0 < v \le u$ in $\bar{G}$. Then there exists a solution $w \in W^{1,p}(G) \cap C(\bar{G})$ to (1.1) in G, so that $v \le w \le u$ in G and $w = v$ on ∂G.*

Proof. The proof is a standard consequence of the comparison principle and monotone iterations scheme (cf. [18, 46] for similar results). We omit the details. □

By means of the standard digitalization techniques Lemma 2.5 extends to the following.

Proposition 2.6. *Let $\mu \le C_H$. Assume that (1.1) has a positive super-solution in B_1^c. Then (1.1) has a positive solution in B_1^c.*

Proof. Let $u > 0$ be a super-solution to (1.1). Set $v = cr^{\gamma_-}$ and observe that

$$-\Delta_p v - \frac{\mu}{|x|^p} v^{p-1} = 0 \quad \text{in} \quad B_1^c,$$

so $v > 0$ is a sub-solution to nonlinear equation (1.1) in B_1^c. By Proposition C.1, v satisfies condition $(\mathcal{S})$. Choose c in such a way that $u \ge cv$ for $|x| = 2$. Thus Theorem 2.1 implies that $u \ge v$ in B_2^c. By Lemma 2.5, for each $n \ge 3$ there exists a solution $w_n \in W^{1,p}(A_{2,n})$ to (1.1) in $A_{2,n}$ such that

$$v \le w_n \le u \quad \text{in} \quad A_{2,n}, \qquad w_n = v \quad \text{on} \quad \partial A_{2,n}. \tag{2.10}$$

By Corollary A.6, we conclude that there exists a constant $M_n > 0$ such that

$$||\nabla w_{n+1}||_{L^p(A_{3,n})} \le M_n, \qquad \forall\, n \ge 4. \tag{2.11}$$

Using (2.11) and (2.10), one can proceed following the standard digitalization techniques in order to construct a solution to (1.1) with the required properties. □

3 Hardy inequalities and positive super-solutions

One of the crucial components in our proof of Theorem 1.1 is an improved Hardy inequality on exterior domains. Inequalities of this type were recently obtained by several authors using various techniques, see [2, 3, 7, 20, 24]. Here we give a simple proof of an improved Hardy inequality on exterior domains for all $p > 1$ and $N \ge 2$, which is based on the explicit construction of appropriate super- and sub-solution and inequality (A.1).

Throughout the paper we use the notation $\gamma_* := \frac{p-N}{p}$ and

$$C_H := \left|\frac{p-N}{p}\right|^p, \quad C_* := \begin{cases} \frac{p-1}{2p}\left|\frac{N-p}{p}\right|^{p-2}, & N \ne p, \\ \left(\frac{N-1}{N}\right)^N, & N = p, \end{cases} \qquad m_* := \begin{cases} 2, & N \ne p, \\ N, & N = p. \end{cases} \tag{3.1}$$

Recall that, according to Proposition A.2 the existence of a positive super-solution to the equation

$$-\Delta_p u - \frac{\mu}{|x|^p} u^{p-1} - \frac{\epsilon}{|x|^p \log^{m_*}|x|} u^{p-1} = 0 \quad \text{in} \quad B_\rho^c, \tag{3.2}$$

with some $\rho > 1$, implies that the form

$$\mathcal{E}_{\mu,\varepsilon}(u) = \int_G |\nabla u|^p dx - \mu \int_G \frac{u^p}{|x|^p} dx - \epsilon \int_G \frac{|u|^p}{|x|^p \log^{m_*}|x|} dx,$$

is nonnegative for all $u \in W^{1,p}_c(B^c_\rho) \cap C(B^c_\rho))$. Thus, in order to prove an improved Hardy inequality it is sufficient to find a super-solution for the corresponding equation. The idea to use Picone's identity for proving Hardy type inequalities related to p-Laplace operator goes back to [6], see also [1, 2]. However, as discovered in [22], such a technique can be in fact attributed as far as to an 1907's paper by Boggio [11].

Theorem 3.1 (Improved Hardy Inequality). *For every $p > 1$ there exists $\rho \geq 1$ such that*

$$\int_{B^c_\rho} |\nabla v|^p dx \geq C_H \int_{B^c_\rho} \frac{|v|^p}{|x|^p} dx + C_* \int_{B^c_\rho} \frac{|v|^p}{|x|^p \log^{m_*}|x|} dx, \tag{3.3}$$

for all $v \in W^{1,p}_c(B^c_\rho) \cap C(B^c_\rho)$. The constants C_H and C_* are sharp in the sense that the inequality

$$\mathcal{E}_{\mu,\epsilon}(v) \geq 0, \qquad \forall\, v \in W^{1,p}_c(B^c_\rho) \cap C(B^c_\rho),$$

fails in any of the following two cases:

(i) $\mu = C_H$, $\epsilon > C_*$,

(ii) $\mu > C_H$, $\epsilon \in \mathbb{R}$.

Proof. Lemma B.1 for $p \neq N$ and a direct computation for $p = N$ show that the function

$$\phi(r) = r^{\gamma_*}(\log r)^\beta (\log\log r)^\tau, \qquad \begin{cases} \beta = \frac{1}{p}, & \tau \in \left(0, \frac{2}{p}\right) \quad \text{for } p \neq N, \\ \beta = \frac{N-1}{N}, & \tau = 0 \qquad \text{for } p = N. \end{cases}$$

is a super-solution to equation (3.2) with $\mu = C_H$ and $\epsilon = C_*$ in B^c_ρ with some $\rho > 1$. Thus (3.3) follows immediately from Proposition A.2.

Sharpness of the constants. (i) Define

$$\phi(r) = r^{\gamma_*}(\log r)^\beta (\log\log r)^\tau, \qquad \begin{cases} \beta = \frac{1}{p}, & \tau \in \left(-\frac{1}{p}, 0\right) \quad \text{for } p \neq N, \\ \beta = \frac{N-1}{N}, & \tau = 0 \qquad \text{for } p = N. \end{cases}$$

By Lemma B.1 (ii) one can choose $\rho > 11$ such that ϕ is a sub-solution with $\mu = C_H$ and $\epsilon = C_*$ in B^c_ρ. Let $R > \rho$. Following [3], we define the cut-off function

$$\theta_R(t) := \begin{cases} 2t/\rho - 3, & \frac{3}{2}\rho \leq t \leq 2\rho, \\ 1 & 2\rho \leq t \leq R, \\ \frac{\log \frac{R^2}{t}}{\log R}, & R \leq t \leq R^2. \end{cases} \tag{3.4}$$

Below we show that for any $\varepsilon > 0$,

$$\mathcal{E}_{C_H, C_*+\varepsilon}(\phi\, \theta_R^\alpha) \to -\infty \quad \text{as} \quad R \to \infty,$$

where $\alpha = 1$ if $p \geq 2$, and $\alpha > \frac{2}{p}$ if $p < 2$. By Proposition A.2 and using (C.5), (C.6), (C.7) we obtain

$$\mathcal{E}_{C_H,C_*}(\phi\,\theta_R^\alpha) \leq c_1 + c_2 \int_{A_{R,R^2}} \mathcal{R}(\theta_R^\alpha\,\phi,\phi)\,dx \leq c_3. \tag{3.5}$$

Further, it is easy to see that

$$\int_{A_{\frac{3}{2}\rho,R^2}} \frac{|\phi\,\theta_R^\alpha|^p}{|x|^p \log^{m_*}|x|}dx \;\;\geq\;\; \int_{2\rho}^{R} \frac{(\log\log r)^{\tau p}}{r\log r}dr = c_4(\log\log R)^{\tau p+1} - c_5.$$

Thus for any $\varepsilon > 0$ we arrive at

$$\mathcal{E}_{C_H,C_*+\varepsilon}(\phi\,\theta_R^\alpha) = \mathcal{E}_{C_H,C_*}(\phi\,\theta_R^\alpha) - \varepsilon \int_{B^c_{\frac{3}{2}\rho}} \frac{|\phi\,\theta_R^\alpha|^p}{|x|^p\log^2|x|}dx \to -\infty \quad \text{as} \quad R \to \infty.$$

(ii) Choosing $\phi(r) = r^{\gamma_*}$ as a sub-solution to (3.2) with $\mu = C_H$ and $\epsilon = C_*$ in B_2^c, one can verify that (3.3) with $\mu > C_H$ and any $\epsilon \in \mathbb{R}$ fails on the family of functions $\phi\theta_R$ defined as above. □

As a consequence of the last theorem we obtain the following nonexistence result, which is crucial in our proofs of nonexistence of positive super-solutions to nonlinear equation (1.1).

Corollary 3.2. *Equation* (3.2) *admits positive super-solutions in* B_ρ^c *with some* $\rho > 1$ *if and only if* $\mu < C_H$ *and* $\epsilon \in \mathbb{R}$*, or* $\mu = C_H$ *and* $\epsilon \leq C_*$*.*

Remark 3.3. Equation (3.2) with $\epsilon \neq 0$ is not homogeneous with respect to scaling, i.e. the existence of a positive (super) solution in B_ρ^c with $\rho > 1$ does not imply the existence of positive (super) solution in B_1^c and so the value of the radius $\rho > 1$ becomes essential.

Next we describe the behavior at infinity of positive super-solutions to equation (3.2) in the case when $\mu \leq C_H$ and $\epsilon \in [0, C_*)$. For $\epsilon \in [0, C_*)$, denote by $\beta_- < \beta_+$ the real roots of the equation

$$\begin{array}{lcll} \frac{1}{2}|\gamma_*|^{p-2}(p-1)(2-\beta p)\beta & = & \epsilon & \text{if} \quad p \neq N, \\ (N-1)(1-\beta)\beta^{N-1} & = & \epsilon & \text{if} \quad p = N. \end{array} \tag{3.6}$$

Notice that $0 \leq \beta_- < \frac{1}{p} < \beta_+ \leq \frac{2}{p}$ if $p \neq N$ and $0 \leq \beta_- < \frac{N-1}{N} < \beta_+ \leq 1$ if $p = N$.

Theorem 3.4 (Lower bound)**.** *Let* $u > 0$ *be a super-solution to* (3.2) *in* B_ρ^c*. The following assertions are valid.*

(i) *Let* $\mu \leq C_H$, $\epsilon = 0$*. There exists* $c > 0$ *such that*

$$u \geq c|x|^{\gamma_-}, \quad x \in B^c_{2\rho}.$$

(ii) *Let* $\mu = C_H$, $\epsilon = 0$*. There exists* $c > 0$ *such that*

$$u \geq c|x|^{\gamma_*}, \quad x \in B^c_{2\rho}.$$

(iii) *Let* $\mu = C_H$, $\epsilon \in (0, C_*)$*. For every* $\tau < 0$ *there exists* $c > 0$ *such that*

$$u \geq c|x|^{\gamma_*}(\log|x|)^{\beta_-}(\log\log|x|)^{\tau}, \quad x \in B^c_{2\rho}.$$

Proof. Follows from Theorem 2.1 and small sub-solutions estimates in Proposition C.1. □

The next lemma establishes a Phragmén–Lindelöf type upper bound on super-solutions.

Theorem 3.5 (Upper bound)**.** *Let $u > 0$ be a super-solution to (3.2) in B_ρ^c. The following assertions are valid.*

(i) Let $\mu < C_H$, $\epsilon = 0$. There exists $c > 0$ such that

$$\inf_{S_R} u \le cR^{\gamma_+}, \quad R > 2\rho.$$

(ii) Let $\mu = C_H$, $\epsilon = 0$. There exists $c > 0$ such that

$$\inf_{S_R} u \le cR^{\gamma_*}(\log R)^{\beta^*}, \quad R > 2\rho,$$

where $\beta^* = \frac{2}{p}$ for $p \ne N$ or $\beta^* = 1$ for $p = N$.

(iii) Let $\mu = C_H, 0 < \epsilon < C_*$. For every $\beta \in (\beta_+, \beta_*)$ there exists $c > 0$ such that

$$\inf_{S_R} u \le cR^{\gamma_*}(\log R)^{\beta}, \quad R > 2\rho.$$

Proof. Let $v > 0$ be a large sub-solution to (3.2), that is a positive sub-solution to (3.2) that satisfies the boundary condition $v = 0$ on S_ρ, as constructed in Appendix D. We are going to show that

$$\inf_{S_R} u \le c \sup_{S_R} v, \quad R > 2\rho. \tag{3.7}$$

For a contradiction, assume that for an arbitrary large $c > 0$ there exists $R > 2\rho$ so that $u \ge cv$ on S_R. Thus

$$u - cv \ge 0 \quad \text{on } \partial A_{\rho,R}.$$

Then Theorem 2.1, applied on $A_{\rho,R}$ yields

$$u - cv \ge 0 \quad \text{on } A_{\rho,R}.$$

In particular, this implies that

$$u(x) \ge cv(x), \quad x \in S_{2\rho}.$$

But this contradicts to the continuity of u.

Now the assertions (i)-(iii) follow from (3.7) via Theorems D.1 and D.2. □

4 Proof Theorem 1.1

First, we prove the nonexistence of positive super-solutions to (1.1) in the super-homogeneous case $q \ge p - 1$ and sub-homogeneous case $q < p - 1$. After this we show sharpness of our nonexistence results by constructing explicit super-solutions in all complementary cases.

4.1 Nonexistence: super-homogeneous case $q \ge p - 1$

We distinguish between the cases $\mu < C_H$ and $\mu = C_H$.

Case $\mu < C_H$. First we prove the nonexistence of super-solutions in the subcritical case, i.e. when (q,σ) is below the critical line Λ^*.

Proposition 4.1. *Let $\sigma < \gamma_-(q-p+1)+p$. Then* (1.1) *has no positive super-solution in B_1^c.*

Proof. Let $u>0$ be a super-solution to (1.1) in B_1^c. Then u is a super-solution to the homogeneous equation

$$-\Delta_p u - \frac{\mu}{|x|^p} u^{p-1} = 0 \quad \text{in} \quad B_1^c. \tag{4.1}$$

By Theorem 3.4(i) we conclude that $u \geq c_1|x|^{\gamma_-}$ in B_2^c. Thus from equation (1.1) it follows that $u>0$ is a super-solution to

$$-\Delta_p u - \frac{\mu + W(x)}{|x|^p} u^{p-1} = 0 \quad \text{in} \quad B_2^c, \tag{4.2}$$

where

$$W(x) := C|x|^{p-\sigma} u^{q-(p-1)} \geq C c_1^{q-(p-1)} |x|^{\gamma_-(q-p+1)+p-\sigma},$$

with $\gamma_-(q-p+1)+p-\sigma > 0$. Then the assertion follows by Corollary 3.2. □

Next we prove the nonexistence in the critical case, i.e. when (q,σ) belongs to the critical line Λ^*.

Proposition 4.2. *Let $\sigma = \gamma_-(q-p+1)+p$. Then* (1.1) *has no positive super-solution in B_1^c.*

Proof. Let $u>0$ be a super-solution to (1.1) in B_1^c. Arguing as in the proof above, we conclude that u is a super-solution to (4.2), where

$$W(x) := C|x|^{p-\sigma} u^{q-(p-1)} \geq C c_1^{q-(p-1)} |x|^{\gamma_-(q-p+1)+p-\sigma} = c.$$

Thus $u>0$ is a super-solution to the homogeneous equation

$$-\Delta_p u - \frac{\tilde{\mu}}{|x|^p} u^{p-1} = 0 \quad \text{in} \quad B_2^c, \tag{4.3}$$

where $\tilde{\mu} = \mu + c$. Without loss of generality, we may assume that $\tilde{\mu} < C_H$. Then by Theorem 3.4(i) we conclude that $u \geq c_2|x|^{\tilde{\gamma}_-}$ in B_2^c, with $\tilde{\gamma}_- \in (\gamma_-, \gamma_*)$. Therefore u is a super-solution to (4.2) with

$$W(x) \geq C c_2^{q-(p-1)} |x|^{\tilde{\gamma}_-(q-p+1)+p-\sigma}$$

and $\tilde{\gamma}_-(q-p+1)+p-\sigma > 0$. Then the assertion follows by Corollary 3.2. □

Case $\mu = C_H$. In this case the proof of the nonexistence can be performed in one step for both subcritical and critical cases.

Proposition 4.3. *Let $\sigma \leq \gamma_*(q-p+1)+p$. Then* (1.1) *has no positive super-solution in B_1^c.*

Proof. Let $u>0$ be a super-solution to (1.1) in B_1^c. Then u is a super-solution to

$$-\Delta_p u - \frac{C_H}{|x|^p} u^{p-1} = 0 \quad \text{in} \quad B_1^c. \tag{4.4}$$

By Theorem 3.4(ii) we conclude that $u \geq c|x|^{\gamma_*}$ in B_2^c. So u is a super-solution to

$$-\Delta_p u - \frac{C_H + W(x)}{|x|^p} u^{p-1} = 0 \quad \text{in} \quad B_2^c, \tag{4.5}$$

where

$$W(x) := C|x|^{p-\sigma} u^{q-(p-1)} \geq C c^{q-(p-1)} |x|^{\gamma_*(q-p+1)+p-\sigma},$$

with $\gamma_*(q-p+1)+p-\sigma \geq 0$. Then the assertion follows by Corollary 3.2. □

4.2 A nonlinear lower bound

We will use the comparison principle (Theorem 2.1 in order to establish the following lower bound on positive solutions to nonlinear equation (1.1) in the sub-homogeneous case $q < p-1$.

Lemma 4.4. *Let $q < p-1$. Let $u > 0$ be a solution to (1.1) in B_1^c. Then there exists $c > 0$ such that*

$$u \geq c|x|^{\frac{\sigma-p}{q-(p-1)}} \quad \text{in } B_2^c. \tag{4.6}$$

Proof. Let $u > 0$ be a solution to (1.1) in B_1^c. Let $x = Ry$ with $y \in A_{2,R}$ and $R \geq 1$. Set

$$v_R(y) := R^{-\frac{\sigma-p}{q-(p-1)}} u(Ry).$$

Then $v_R(y)$ satisfies

$$-\Delta_p v_R - \frac{\mu}{|y|^p} {v_R}^{p-1} = \frac{C}{|y|^\sigma} {v_R}^q \quad \text{in} \quad A_{2,4}. \tag{4.7}$$

Let $\lambda_1 > 0$ be the principal eigenvalue and $\phi_1 > 0$ be the principal eigenfunction to the eigenvalue problem

$$-\Delta_p \phi - \frac{\mu}{|y|^p} \phi^{p-1} = \lambda \phi, \qquad \phi \in W_0^{1,p}(A_{2,4}),$$

see [23]. By the direct computation, $\tau_0 \phi_1$ is a sub-solution to (1.1) for a sufficiently small $\tau_0 > 0$. Therefore, Theorem 2.1 implies that

$$v_R \geq \tau_0 \phi_1 \quad \text{in } A_{2,4}.$$

So, lower bound (4.6) follows. □

4.3 Nonexistence: sub-homogeneous case $q < p-1$

As before, we distinguish the cases $\mu < C_H$ and $\mu = C_H$.

Case $\mu < C_H$. First we consider the subcritical case, when (q, σ) is below to the critical line Λ_*.

Proposition 4.5. *Let $\sigma < \gamma_+(q-p+1)+p$. Then (1.1) has no positive super-solution in B_1^c.*

Proof. Let $u > 0$ be a super-solution to (1.1) in B_1^c. According to Lemma 2.6, we may assume that u is a solution to (1.1) in B_1^c. Then u is a super-solution to the homogeneous equation

$$-\Delta_p u - \frac{\mu}{|x|^p} u^{p-1} = 0 \quad \text{in} \quad B_1^c. \tag{4.8}$$

By Theorem 3.5(i) we conclude that

$$\inf_{S_R} u \leq c_1 R^{\gamma_+}, \qquad R > 2. \tag{4.9}$$

Since $\gamma_+ < \frac{\sigma-p}{q-(p-1)}$ this contradicts to lower bound (4.6). □

Next we prove the nonexistence in the critical case. When (q, σ) belongs to the critical line Λ_*, (4.9) is no longer incompatible with (4.6), so we need to improve estimate (4.9).

Proposition 4.6. *Let $\sigma = \gamma_+(q-p+1)+p$. Then (1.1) has no positive super-solution in B_1^c.*

Proof. Let $u > 0$ be a super-solution to (1.1) in B_1^c. According to Lemma 2.6, we may assume that u is a solution to (1.1) in B_1^c. Using (4.6) we conclude that $u > 0$ is a solution to

$$-\Delta_p u - \frac{\mu + W(x)}{|x|^p} u^{p-1} = 0 \quad \text{in} \quad B_1^c, \tag{4.10}$$

where $W(x) := C|x|^{p-\sigma} u^{q-(p-1)} \in L^\infty(B_2^c)$. Thus the strong Harnack Inequality (2.4) combined with upper bound (4.9) implies that

$$\sup_{A_{R/2,R}} u \le C_S \inf_{A_{R/2,R}} u \le cR^{\gamma_+}, \qquad R > 4,$$

and hence $W(x) \ge c_1$ in B_4^c, for some $c_1 > 0$. Therefore $u > 0$ is a super-solution to

$$-\Delta_p u - \frac{\tilde{\mu}}{|x|^p} u^{p-1} = 0 \quad \text{in} \quad B_4^c, \tag{4.11}$$

where $\tilde{\mu} = \mu + \epsilon$ with $0 < \epsilon < c_1$ small enough. Without loss of generality we may assume that $\tilde{\mu} < C_H$. Then by Theorem 3.4(i) we conclude that $\inf_{S_R} u \le cR^{\tilde{\gamma}_+}$ for all $R > 4$, where $\tilde{\gamma}_+ \in (\gamma_*, \gamma_+)$ is the largest root of the equation (B.3) with $\tilde{\mu}$ in place of μ. This improved estimate contradicts to lower bound (4.6). □

Case $\mu = C_H$. First we prove the nonexistence in the subcritical case, when (q, σ) is below to the critical line Λ_*.

Proposition 4.7. *Let $\sigma < \gamma_*(q - p + 1) + p$. Then* (1.1) *has no positive super-solution in B_1^c.*

Proof. We start as in the proof of Proposition 4.5 with C_H in place of μ in (4.8). By Theorem 3.5(ii) we conclude that

$$\inf_{S_R} u \le cR^{\gamma_*} (\log R)^{\beta^*}, \quad R > 2, \tag{4.12}$$

where $\beta^* = 1$ for $p = N$ and $\beta^* = \frac{2}{p}$ for $p \ne N$. This contradicts to lower bound (4.6). □

Now we consider the critical case, i.e. when (q, σ) belongs to the critical line Λ_*. We need to distinguish between the cases $q > -1$ and $q = 1$.

Proposition 4.8. *Let $q \in (-1, p - 1)$ and $\sigma = \gamma_*(q - p + 1) + p$. Then* (1.1) *has no positive super-solution in B_1^c.*

Proof. We start as in Proposition 4.6 with C_H in place of μ in (4.10). The strong Harnack Inequality (2.4) and upper bound (4.12) imply that

$$\sup_{A_{R/2,R}} u \le C_S \inf_{A_{R/2,R}} u \le cR^{\gamma_*} (\log R)^{\beta^*}, \quad R > 4. \tag{4.13}$$

We conclude that

$$W(x) \ge \epsilon (\log |x|)^{\beta^*(q-p+1)} \quad \text{in} \ B_4^c,$$

for some $\epsilon > 0$. Hence $u > 0$ is a super-solution to the equation

$$-\Delta_p u - \frac{C_H}{|x|^p} u^{p-1} - \frac{\epsilon (\log |x|)^t}{|x|^p \log^{m_*} |x|} u^{p-1} = 0 \quad \text{in} \quad B_4^c, \tag{4.14}$$

where $t := \beta^*(q - p + 1) + m > 0$. So, the assertion follows by Corollary 3.2. □

In the 'double critical' case $q = -1$ equation (4.14) does not directly lead to the nonexistence, because $t = 0$. So we need to improve estimate (4.13).

Proposition 4.9. *Let $q = -1$ and $\sigma = \gamma_*(q - p + 1) + p$. Then* (1.1) *has no positive super-solution in B_1^c.*

Proof. Arguing as in the proof of Proposition 4.8, we conclude that $u > 0$ is a super-solution to the equation (4.14) with t=0. We may assume that $\epsilon_1 < C_*$. Then using Lemma 3.5(iii) and applying the strong Harnack inequality to equation (4.14), we conclude that

$$\sup_{A_{R/2,R}} u \le C_S \inf_{A_{R/2,R}} u \le cR^{\gamma_*}(\log R)^{\beta}, \quad R > 2\rho, \tag{4.15}$$

where $\beta \in (\beta_+, \beta_*)$ and $\rho > 4$. Therefore $u > 0$ is a super-solution to the equation

$$-\Delta_p u - \frac{C_H}{|x|^p} u^{p-1} - \frac{W(x)}{|x|^p \log^{m_*}|x|} u^{p-1} = 0 \quad \text{in} \quad B_{2\rho}^c,$$

where

$$W(x) := C|x|^{p-N} \log^{m_*}|x|\, u^{-p} \ge c(\log|x|)^{2-\beta p} \quad \text{in} \ B_{2\rho}^c.$$

Hence, the assertion follows by Corollary 3.2. □

This completes the description of the nonexistence region $\mathcal{N}$ and the proof of the nonexistence part of Theorem 1.1. Next we show that the established nonexistence results are sharp.

4.4 Existence

As soon as the nonexistence region $\mathcal{N}$ is described, the construction of explicit super-solutions in its complement is straightforward.

Case $\mu < C_H$. Let $(q, \sigma) \in \mathbb{R}^2 \setminus \mathcal{N}$. Choose $\gamma \in (\gamma_-, \gamma_+)$ such that

$$\begin{cases} \gamma_- < \gamma < \frac{\sigma - p}{q - p + 1} & \text{if} \quad q > p - 1, \\ \frac{\sigma - p}{q - p + 1} < \gamma < \gamma_+ & \text{if} \quad q < p - 1, \\ \gamma_- < \gamma < \gamma_+ & \text{if} \quad q = p - 1. \end{cases}$$

Then one can verify directly that the functions $u = \tau r^{\gamma}$ are super-solutions to (1.1) in B_ρ^c for an appropriate choice of $\tau > 0$ and $\rho \ge 1$.

Case $\mu = C_H$. Let $(q, \sigma) \in \mathbb{R}^2 \setminus \mathcal{N}$. For $p = N$, choose $\beta \in (0, 1)$ such that

$$\begin{cases} 0 < \beta < 1 & \text{if} \quad \sigma > N, \quad q \in \mathbb{R}, \\ \frac{N}{N - 1 - q} < \beta < 1 & \text{if} \quad \sigma = N, \quad q < -1. \end{cases}$$

Then one can verify directly that the functions $u = \tau \log^{\beta} r$ are super-solutions to (1.1) in B_ρ^c for an appropriate choice of $\tau > 0$ and $\rho > 1$.

For $p \neq N$, choose $\beta \in (0, 2/p)$ such that

$$\begin{cases} 0 < \beta < \frac{2}{p} & \text{if} \quad \sigma > \Lambda_*(q), \quad q \in \mathbb{R}, \\ -\frac{2}{q - p + 1} < \beta < \frac{2}{p} & \text{if} \quad \sigma = \gamma_*(q - p + 1) + p, \quad q < -1. \end{cases}$$

Then (B.6) implies that the function $u = \tau r^{\gamma_*}(\log r)^\beta$ satisfies

$$-\Delta_p u - \frac{C_H}{|x|^p}u^{p-1} \geq \frac{\epsilon}{|x|^p \log^2|x|}u^{p-1} \geq \frac{C}{|x|^\sigma}u^q \quad \text{in } B_\rho^c, \tag{4.16}$$

where $\epsilon = \beta(p-1)(2-\beta p)/2 \in (0, C_*)$ and $\tau > 0$, $\rho > 1$ are chosen appropriately. This completes the proof of Theorem 1.1.

A Picone's identity and corollaries

We say that the form $\mathcal{E}_V$ is *positive definite* if

$$\mathcal{E}_V(u) > 0, \qquad \forall\, u \in W_c^{1,p}(G) \cap C(G), \quad u \neq 0.$$

Below we describe the relation between the positivity of the form $\mathcal{E}_V$ and the existence of positive super-solutions to the equation (2.3). In the linear case $p = 2$ such a relation is well-documented, see e.g. [4]. We start with formulating the well–known Picone's Identity for p-Laplacian (see e.g. [6, 17, 42]).

Proposition A.1 (Picone's Identity)**.** *Let $w, \phi \in W_{loc}^{1,p}(G) \cap C(G)$ be such that $w \geq 0$ and $\phi > 0$. Set*

$$\begin{aligned}
\mathcal{L}(w,\phi) &:= |\nabla w|^p + (p-1)\left(\frac{w}{\phi}\right)^p |\nabla\phi|^p - p\left(\frac{w}{\phi}\right)^{p-1} \nabla w |\nabla\phi|^{p-2}\nabla\phi, \\
\mathcal{R}(w,\phi) &:= |\nabla w|^p - \nabla\left(\frac{w^p}{\phi^{p-1}}\right)|\nabla\phi|^{p-2}\nabla\phi.
\end{aligned}$$

Then $\mathcal{L}(w,\phi) = \mathcal{R}(w,\phi) \geq 0$ a.e. in G. Moreover, $\mathcal{L}(w,\phi) = \mathcal{R}(w,\phi) = 0$ a.e. in G if and only if $w = c\phi$ in G for a constant $c > 0$.

An immediate consequence of Picone's identity is that the existence of a positive super-solution to (2.3) implies positivity of the form $\mathcal{E}_V$, as the following proposition shows.

Proposition A.2. *Let $\phi > 0$ be a super-solution (sub-solution) to equation* (2.3)*. Then the form $\mathcal{E}_V$ satisfies the following inequality*

$$\mathcal{E}_V(u) \geq (\leq) \int_G \mathcal{R}(u,\phi)\,dx + \int_G \frac{f}{\phi^{p-1}}|u|^p\,dx, \qquad \forall\, u \in W_c^{1,p}(G) \cap C(G). \tag{A.1}$$

Proof. Let $\phi > 0$ be a super-solution (sub-solution) to (2.3). Testing (2.3) by $\xi = \frac{|u|^p}{\phi^{p-1}} \in W_c^{1,p}(G) \cap C(G)$ we obtain

$$\int_G V u^p\,dx \;\leq (\geq)\, p\int_G \frac{u\nabla\phi}{\phi}\left|\frac{u\nabla\phi}{\phi}\right|^{p-2}\nabla u\,dx - (p-1)\int_G |\nabla\phi|^p\left|\frac{u}{\phi}\right|^p dx - \int_G \frac{f}{\phi^{p-1}}|u|^p\,dx,$$

which implies (A.1). □

Remark A.3. (*i*) If $\phi > 0$ is a super-solution to (2.3) then $\mathcal{R}(u,\phi) \geq 0$ a.e. in G and, in particular,

$$\mathcal{E}_V(u) \geq \int_G \frac{f}{\phi^{p-1}}|u|^p\,dx \geq 0, \qquad \forall\, u \in W_c^{1,p}(G) \cap C(G).$$

If, in addition, $f > 0$ then

$$\mathcal{E}_V(u) > 0, \qquad \forall\, u \in W_c^{1,p}(G) \cap C(G), \quad u \neq 0.$$

(*ii*) If $\phi > 0$ is a solution to (2.3) then inequality (A.1) becomes an identity.

The following straightforward corollary of Proposition A.2 is our main tool in proving nonexistence of positive solutions to nonlinear equation (1.1).

Corollary A.4 (Nonexistence principle). *Assume that there exists* $u \in W^{1,p}_c(G) \cap C(G)$ *such that* $\mathcal{E}_V(u) < 0$. *Then equation* (2.2) *has no positive super-solution.*

Another interesting application of Proposition A.2 is a version of Barta's inequality (cf. [6]).

Corollary A.5 (Barta's inequality). *Assume that equation* (2.2) *admits a positive super-solution. Then for every* $0 < \varphi \in W^{1,p}_{loc}(G) \cap C(G)$ *such that* $-\Delta_p \varphi - V\varphi^{p-1} \in L^1_{loc}(G)$ *the following inequality holds*

$$\inf_{x \in G} \frac{-\Delta_p \varphi - V\varphi^{p-1}}{\varphi^{p-1}} \le \inf_{0 \lneq u \in W^{1,p}_c \cap C(G)} \frac{\int_G (|\nabla u|^p - Vu^p)dx}{\int_G u^p\, dx}. \tag{A.2}$$

Proof. Set $F(x) := -\Delta_p \varphi - V\varphi^{p-1} \in L^1_{loc}(G)$. We may assume that $F \ge 0$ (otherwise inequality (A.2) is trivial). Proposition A.2 implies that

$$\mathcal{E}_V(u) \ge \int_G \frac{F}{\varphi^{p-1}} u^p dx \ge \inf_{x \in G} \frac{F}{\varphi^{p-1}} \int_G u^p dx, \qquad \forall\, u \in W^{1,p}_c(G) \cap C(G).$$

So the assertion follows. □

We need the following version of the Caccioppoli inequality, which is a consequence of Proposition A.2.

Corollary A.6 (Caccioppoli-type Inequality). *Let* $u > 0$ *be a sub-solution to* (2.2). *Then*

$$\int_G |\theta\, \nabla u|^p dx \le p \int_G V u^p |\theta|^p dx + p^p \int_G u^p |\nabla \theta|^p dx, \qquad \forall\, \theta \in W^{1,\infty}_c(G). \tag{A.3}$$

Proof. From (A.1) we have

$$\begin{aligned} \mathcal{E}_V(u\theta) &\le \int_G |\nabla(u\theta)|^p dx - p\int_G \theta\nabla u\, |\theta\nabla u|^{p-2}\nabla(u\theta)\, dx + (p-1)\int_G |\theta \nabla u|^p dx \\ &\le \int_G |\nabla(u\theta)|^p dx + p\int_G |\theta\, \nabla u|^{p-1} u |\nabla\theta|\, dx - \int_G |\theta\, \nabla u|^p dx. \end{aligned}$$

Using the Young's Inequality and (2.1) we obtain

$$\int_G |\theta\nabla u|^p dx \le \int_G V u^p |\theta|^p dx + p^{p-1}\int_G |u\, \nabla\theta|^p dx + \frac{p-1}{p}\int_G |\theta\, \nabla u|^p dx,$$

so the assertion follows. □

B Sample sub- and super-solutions

Below we construct explicit super- and sub-solutions to the homogeneous equation of the form

$$-\Delta_p u - \frac{\mu}{|x|^p} u^{p-1} - \frac{\epsilon}{|x|^p \log^{m_*} |x|} u^{p-1} = 0 \quad \text{in} \quad B^c_\rho, \tag{B.1}$$

where $\rho \ge 2$. In what follows we assume that $\mu \le C_H$ and $\epsilon \in [0, C_*)$, where C_H, C_* and m_* are defined in (3.1). When u is radially symmetric we loosely write $u(|x|) = u(r)$ instead of $u(x)$.

In this case in the polar coordinates (r,ω) on $\mathbb{R}^N$ equation (B.1) transforms into the ordinary differential equation

$$-r^{1-N}(r^{N-1}|u_r|^{p-2}u_r)_r - \frac{\mu}{r^p}u^{p-1} - \frac{\epsilon}{r^p \log^{m_*} r}u^{p-1} = 0 \qquad (r > \rho). \tag{B.2}$$

Let $\mu \le C_H$. Set $\gamma_* := \frac{p-N}{p}$. By $\gamma_- \le \gamma_+$ we denote the real roots of the equation

$$-\gamma|\gamma|^{p-2}\left(\gamma(p-1)+N-p\right) = \mu. \tag{B.3}$$

If $\mu < C_H$ then $\gamma_- < \gamma_* < \gamma_+$. If $\mu = C_H$ then $\gamma_\pm = \gamma_*$. It is straightforward to see that if $\mu \le C_H$ and $\epsilon = 0$ then *the function* $u = r^\gamma$ *is a sub-solution to equation* (B.2) *if* $\gamma \in (-\infty, \gamma_-] \cup [\gamma_+, +\infty)$ *and a super-solution if* $\gamma \in [\gamma_-, \gamma_+]$.

Let $p = N$ and $\epsilon \in [0, C_*]$. Then $\beta_- \le \beta_+$ denote the real roots of the equation

$$\beta^{N-1}(1-\beta)(N-1) = \epsilon. \tag{B.4}$$

Notice that $0 \le \beta_- \le \frac{N-1}{N} \le \beta_+ \le 1$. It is simple to verify that *the function* $u := \log^\beta r$ *is a sub-solution to* (B.2) *if* $\beta \in (-\infty, \beta_-] \cup [\beta_+, +\infty)$ *and a super-solution if* $\beta \in [\beta_-, \beta_+]$.

When $p \ne N$, $\mu = C_H$ and $\epsilon \in [0, C_*]$ the situation becomes more delicate. We denote by $\beta_- \le \beta_+$ the real roots of the equation

$$\frac{1}{2}|\gamma^*|^{p-2}(p-1)(2-\beta p)\beta = \epsilon, \tag{B.5}$$

If $\epsilon < C_*$ then $0 \le \beta_- < \frac{1}{p} < \beta_+ \le \frac{2}{p}$. If $\epsilon = C_*$ then $\beta_- = \beta_+ = \frac{1}{p}$.

Lemma B.1. *Let* $p \ne N$, $\mu = C_H$ *and* $\varepsilon \in [0, C_*]$. *Let* $u_{\beta,\tau}(r) := r^{\gamma_*}(\log r)^\beta(\log\log r)^\tau$, *where* $\beta \ge 0$ *and* $\tau \in \mathbb{R}$. *The following assertions are valid.*

(i) Let $\epsilon \in [0, C_*)$. *Then there exists* $\rho = \rho(p, N, \beta, \tau) > 1$ *such that*

- *(a)* $u_{\beta,\tau}$ *is a super-solution to* (B.2) *if* $\beta \in (\beta_-, \beta_+)$ *and a sub-solution if* $\beta < \beta_-$ *or* $\beta > \beta_+$*;*
- *(b)* $u_{\beta_-,\tau}$ *is a super-solution to* (B.2) *if* $\tau > 0$ *and a sub-solution if* $\tau < 0$*;*
- *(c)* $u_{\beta_+,\tau}$ *is a super-solution to* (B.2) *if* $\tau < 0$ *and a sub-solution if* $\tau > 0$*;*
- *(d)* $u_{\beta_\pm,0}$ *is a super-solution to* (B.2) *if* $p \in (1,2] \cup (N, +\infty)$ *and a sub-solutions if* $p \in [2, N)$.

(ii) Let $\epsilon = C_*$. *Then there exists* $\rho = \rho(p, N, \beta, \tau) > 1$ *such that*

- *(a)* $u_{\beta,\tau}$ *is a sub-solution to* (B.2) *if* $\beta \ne 1/p$*;*
- *(b)* $u_{1/p,\tau}$ *is a super-solution to* (B.2) *if* $\tau \in (0, \frac{2}{p})$ *and a sub-solution if* $\tau < 0$ *or* $\tau > 2/p$*;*
- *(c)* $u_{1/p,0}$ *is a super-solution to* (B.2) *if* $p \in (1,2] \cup (N, +\infty)$ *and a sub-solution if* $p \in [2, N)$.

Proof. Observe that for every $\beta, \tau \in \mathbb{R}$ there exists $\rho > 2$ such that u_r does not change sign on (ρ, ∞). Then a direct computation similar to [40, Lemmas 2.1, 2.2] verifies that

$$\begin{aligned}
-\Delta_p u_{\beta,\tau} \;&=\; \frac{|\gamma_*|^{p-2}}{r^p}u^{p-1}\Bigg(|\gamma_*|^2 + \frac{\beta(p-1)(2-\beta p)}{2}\frac{1}{\log^2 r} \\
&+\; \tau(p-1)(1-\beta p)\frac{1}{\log^2 r \log\log r} + \frac{\tau(p-1)(2-\tau p)}{2}\frac{1}{\log^2 r\, \log\log^2 r} \\
&+\; \frac{p(p-1)(p-2)}{3(N-p)}\beta^2(\beta p - 3)\frac{1}{\log^3 r} + R(r)\Bigg),
\end{aligned} \tag{B.6}$$

	Sub-solution	*Super-solution*	*Sub-solution*
$p = N$, $\varepsilon = 0$	$\beta \leq \beta_-$, $\tau = 0$	$\beta \in [\beta_-, \beta_+]$, $\tau = 0$	$\beta \geq \beta_+$, $\tau = 0$
$p \neq N$, $\varepsilon = 0$	$\beta \leq 0$, $\tau = 0$	$\beta \in [0, 2/p]$, $\tau = 0$	$\beta > 2/p$, $\tau = 0$ $\beta = 2/p$, $\tau > 0$
$p \neq N$, $\varepsilon \in (0, C_*)$	$\beta < \beta_-$, $\tau = 0$ $\beta = \beta_-$, $\tau < 0$	$\beta \in (\beta_-, \beta_+)$, $\tau = 0$ $\beta = \beta_-$, $\tau > 0$ or $\beta = \beta_+$, $\tau < 0$	$\beta > \beta_+$, $\tau = 0$ $\beta = \beta_+$, $\tau > 0$
$p \neq N$, $\varepsilon = C_*$	$\beta < 1/p$, $\tau = 0$ $\beta = 1/p$, $\tau < 0$	$\beta = 1/p$, $\tau \in (0, 2/p)$	$\beta > 1/p$, $\tau = 0$ $\beta = 1/p$, $\tau > 2/p$

Table 1: Case $\mu = C_H$. Properties of $u_{\beta,\tau} = r^{\gamma_*}(\log r)^\beta(\log\log r)^\tau$, for a large $\rho > 1$.

where

$$R(r) = O\left(\frac{1}{\log^3 r \log\log r} + \frac{1}{\log^4 r}\right) \quad \text{as} \quad r \to \infty.$$

The rest of the proof is straightforward. □

Remark B.2. Table 1 summarizes some values of the parameters $\beta, \tau \in \mathbb{R}$ which make the function $u_{\beta,\tau} = r^{\gamma_*}(\log r)^\beta(\log\log r)^\tau$ a sub- or super-solution to (B.2) with $\mu = C_H$ and $\epsilon \in [0, C_*]$, for a sufficiently large radius $\rho > 1$. Observe that the radius $\rho > 1$ depends on the data and, in general, can not be determined explicitly. Similar calculations with $\tau = 0$ were provided in [40, 41] for interior domains.

C Small sub-solutions

A *small* (sub) solution to equation (B.1) is a (sub) solution $v > 0$ to (B.1) that satisfies the condition:

($\mathcal{S}$) there exists a sequence $(\theta_n) \in W_c^{1,\infty}(\mathbb{R}^N)$ such that $\theta_n \to 1$ a.e. in $\mathbb{R}^n$ and

$$\int_{B_\rho^c} \mathcal{R}(\theta_n v, v)\, dx \to 0 \quad \text{as} \quad R \to +\infty,$$

where $\mathcal{R}(w, v) = |\nabla w|^p - \nabla\left(\frac{w^p}{v^{p-1}}\right)|\nabla v|^{p-2}\nabla v$ is defined as in Proposition A.1. In order to apply Theorem 2.1 to equation (B.1) we need to verify that (B.1) has small sub-solutions, which is done in the following proposition.

Proposition C.1. *Set $v = r^\gamma(\log r)^\beta(\log\log r)^\tau$. The following assertions are valid.*

(i) *Let $\gamma \leq \gamma_*$, $\beta = 0$, $\tau = 0$. Then v is a small sub-solution to* (B.1) *with $\mu \leq C_H$ and $\epsilon = 0$;*

(ii) *Let $p \neq N$, $\gamma = \gamma_*$, $\beta = 1/p$, $\tau < 0$. Then v is a small sub-solution to* (B.1) *with $\mu = C_H$ and $\epsilon \in (0, C_*)$;*

(iii) *Let $p = N$, $\gamma = \gamma_*$, $\beta = \frac{N-1}{N}$, $\tau < 0$. Then v is a small sub-solution to* (B.1) *with $\mu = 0$ and $\epsilon \in (0, C_*)$.*

Proof. Lemma B.1 in case (ii) and direct computations in cases (i), (iii) show that v is a sub-solution to (B.1) for corresponding μ and ϵ. Below we show that $\int_{B_\rho^c} \mathcal{R}(\theta_R^\alpha v, v)\, dx \to 0$ as $R \to +\infty$, where $\theta_R \in C^{1,1}(0,\infty)$ is defined by

$$\theta_R(r) := \begin{cases} 1, & 0 \le r \le R, \\ \frac{\log \frac{R^2}{r}}{\log R}, & R \le r \le R^2, \\ 0, & r \ge R, \end{cases}$$

and $\alpha \ge 1$ will be chosen later. By Proposition A.1, for $R > \rho$ we have

$$\begin{aligned} \int_{B_\rho^c} \mathcal{R}(\theta_R^\alpha v, v)\, dx &= \int_{A_{\rho,R}} \mathcal{R}(\theta_R^\alpha v, v)\, dx + \int_{A_{R,R^2}} \mathcal{R}(\theta_R^\alpha v, v)\, dx \\ &= \int_{A_{\rho,R}} \mathcal{R}(v, v)\, dx + \int_{A_{R,R^2}} \mathcal{R}(\theta_R^\alpha v, v)\, dx \\ &= c_N \int_R^{R^2} \mathcal{R}(\theta_R^\alpha v, v) r^{N-1} dr \end{aligned}$$

Below we estimate the latter integral.

(i) Using the inequalities (see, e.g., [42, Lemma 7.4])

$$\begin{aligned} \mathcal{R}(\theta_R v, v) &\le c_1 |\theta_R v_r'|^{p-2} |v(\theta_R)_r'|^2 + c_2 |v\,(\theta_R)_r'|^p, && (p > 2), && \text{(C.1)} \\ \mathcal{R}(\theta_R v, v) &\le c_3 |v\,(\theta_R)_r'|^p, && (1 < p \le 2), && \text{(C.2)} \end{aligned}$$

we obtain directly that there exists $c > 0$ such that

$$\int_R^{R^2} \mathcal{R}(\theta_R v, v) r^{N-1} dr \le c \frac{R^{\gamma p + N - p}}{(\log R)^p}. \tag{C.3}$$

(ii) Set $Q(r) := -\gamma_* \log r \log\log r - \beta \log\log r - \tau$. Then direct computations give

$$\mathcal{R}(\theta_R^\alpha v, v) = \frac{(\log r)^{(\beta-1)p} (\log\log r)^{(\tau-1)p}}{r^N (\log R)^{\alpha p}} \left(\log \frac{R^2}{r}\right)^{\alpha p - p} Q^p(r) \times$$
$$\times \left\{ \left| \log \frac{R^2}{r} + \alpha \frac{\log r \log\log r}{Q(r)} \right|^p - \left(\log \frac{R^2}{r}\right)^{p-1} \left(\log \frac{R^2}{r} + \alpha p \frac{\log r \log\log r}{Q(r)}\right) \right\}.$$

Let $p \ge 2$. Choose $\alpha = 1$. We use the inequality (see, e.g., [42, Lemma 7.4])

$$|z_1 + z_2|^p - |z_1|^p - p|z_1|^{p-2} z_1 z_2 \le \frac{p(p-1)}{2} \left(|z_1| + |z_2|\right)^{p-2} |z_2|^2, \qquad \forall z_1, z_2 \in \mathbb{R} \tag{C.4}$$

with

$$z_1 = \log \frac{R^2}{r}, \qquad z_2 = \frac{\log r \log\log r}{Q(r)}$$

to obtain that

$$\begin{aligned} \mathcal{R}(\theta_R v, v) &\le c \frac{(\log r)^{(\beta-1)p+2} (\log\log r)^{(\tau-1)p+2}}{r^N \log^p R} Q^{p-2}(r) \left| \log \frac{R^2}{r} + \frac{\log r \log\log r}{Q(r)} \right|^{p-2} \\ &= c \frac{(\log r)^{(\beta-1)p+2} (\log\log r)^{(\tau-1)p+2}}{r^N \log^p R} \left| Q(r) \log \frac{R^2}{r} + \log r \log\log r \right|^{p-2}. \end{aligned}$$

Thus we arrive at

$$\int_{A_{R,R^2}} \mathcal{R}(\theta_R v, v)\, dx \leq \frac{c}{\log^2 R} \int_R^{R^2} \frac{(\log r)^{\beta p} (\log\log r)^{\tau p}}{r}\, dr$$
$$\leq c(\log R)^{\beta p - 1} (\log\log R)^{\tau p}. \tag{C.5}$$

If $1 < p < 2$, choose $\alpha > \frac{2}{p}$. Observe that the Taylor expansion applied to the function $f(t) = |z_1 + tz_2|^p$ with $0 < t < 1$, $z_1, z_2 \in \mathbb{R}$, $z_1 \neq 0$ and $z_1 z_2 \geq 0$ leads to

$$|z_1 + z_2|^p - |z_1|^p - p|z_1|^{p-2} z_1 z_2 = \frac{p(p-1)}{2} |z_1 + t_0 z_2|^{p-2} |z_2|^2 \leq \frac{p(p-1)}{2} |z_1|^{p-2} |z_2|^2,$$

for some $t_0 \in (0, 1)$. Using the above inequality with

$$z_1 = \log \frac{R^2}{r}, \qquad z_2 = \alpha \frac{\log r \log\log r}{Q(r)},$$

we obtain

$$\mathcal{R}(\theta_R v, v) \leq c \frac{(\log r)^{(\beta-1)p+2} (\log\log r)^{(\tau-1)p+2}}{r^N (\log R)^{\alpha p}} \left(\log \frac{R^2}{r}\right)^{\alpha p - 2} Q^{p-2}(r).$$

Since $\alpha p - 2 > 0$ we conclude that

$$\int_{A_{R,R^2}} \mathcal{R}(\theta_R v, v)\, dx \leq \frac{c}{\log^2 R} \int_R^{R^2} \frac{(\log r)^{\beta p} (\log\log r)^{\tau p}}{r}\, dr$$
$$\leq c(\log R)^{\beta p - 1} (\log\log R)^{\tau p}. \tag{C.6}$$

(iii) An easy computations shows that

$$\mathcal{R}(\theta_R v, v) \leq \frac{c}{r^N \log^N R} (\log r)^{\beta N} (\log\log r)^{\tau N}.$$

Therefore

$$\int_{A_{R,R^2}} \mathcal{R}(\theta_R v, v)\, dx \leq \frac{c}{\log^N R} \int_R^{R^2} \frac{(\log r)^{\beta N} (\log\log r)^{\tau N}}{r}\, dr$$
$$\leq c(\log R)^{\beta N - N + 1} (\log\log r)^{\tau N}. \tag{C.7}$$

This completes the proof. □

D Large sub-solutions

A *large* (sub) solution to equation (B.1) is a positive (sub) solution of the problem

$$-\Delta_p u - \frac{\mu}{|x|^p} u^{p-1} - \frac{\epsilon}{|x|^p \log^{m_*} |x|} u^{p-1} = 0 \quad \text{in } B_R^c, \qquad u = 0 \quad \text{on } S_R, \tag{D.1}$$

with a sufficiently large $R > 1$. Below we establish the existence and asymptotic behavior of large sub-solutions.

Theorem D.1. *Let $\mu \leq 0$ and $\epsilon = 0$. The following assertions are valid.*

(*i*) *if* $p \neq N$ *or* $\mu < 0$ *then* $u = |x|^{\gamma_+} - R^{\gamma_+}$ *is a positive sub-solution to* (D.1).

(*ii*) *if* $\mu = 0$ *and* $p = N$ *then* $u = \log|x| - \log R$ *is a positive sub-solution to* (D.1).

Proof. Note that if $\mu \leq 0$ then $0 \in [\gamma_-, \gamma_+]$. Hence positive constants are super-solutions to (D.1). Then a direct computation verifies that $u = r^{\gamma_+} - R^{\gamma_+}$ or $u = \log|x| - \log R$ are sub-solutions to (D.1). □

Theorem D.2. *The following assertions are valid.*

(*i*) *Let* $p \neq N$, $\mu \in (0, C_H)$ *and* $\epsilon = 0$. *Then* (D.1) *admits a solution* $u > 0$ *such that*

$$u = c|x|^{\gamma_+(1+o(1))} \quad as \quad r \to +\infty.$$

(*ii*) *Let* $p \neq N$, $\mu = C_H$ *and* $\epsilon = 0$. *Then* (D.1) *admits a solution* $u > 0$ *such that*

$$u = c|x|^{\gamma_*} (\log|x|)^{\frac{2}{p}(1+o(1))} \quad as \quad r \to +\infty.$$

(*iii*) *Let* $p = N$, $\mu = C_H$ *and* $\epsilon \in (0, C_*)$. *Then* (D.1) *admits a solution* $u > 0$ *such that*

$$u = c\,(\log|x|)^{\beta_+(1+o(1))} \quad as \quad r \to +\infty.$$

(*iv*) *Let* $p \neq N$, $\mu = C_H$ *and* $\epsilon \in (0, C_*)$. *Then* (D.1) *admits a solution* $u > 0$ *such that for every* $\delta \in (0, \min\{\beta_+ - \frac{1}{p}, \frac{2}{p} - \beta_+\})$ *there exists* $c_\delta > 0$ *and* $R_\delta > e$ *and* u *satisfies*

$$c_\delta^{-1}|x|^{\gamma_*} (\log|x|)^{\beta_+ - \delta} \leq u \leq c_\delta |x|^{\gamma_*} (\log|x|)^{\beta_+ + \delta} \quad in \quad (R_\delta, +\infty).$$

Our proof of Theorem D.2 employs the generalized Prüfer Transformation. The classical Prüfer transformation is a well-known tool in the theory of linear second-order elliptic equations, cf. [15, Chapter 8]. Its generalization to the context of p-Laplace equations was recently introduced by Reichel and Walter [39], see also [10, 14]. For the readers' convenience we collect below required facts for the generalized sine functions and Prüfer transformation.

D.1 Generalized sine function

The generalized sine function $S_p(\psi)$ ($p > 1$) was introduced in [31] as the solution to the problem

$$|w'|^p + \frac{|w|^p}{p-1} = 1, \quad w(0) = 0, \quad w'(0) = 1. \tag{D.2}$$

Equation (D.2) arises as a first integral of $(w'|w'|^{p-2})' + w|w|^{p-2} = 0$. The solution of (D.2) defines the function $S_p(\psi) = \sin_p(\psi)$ as long as it is increasing, that is, for $\psi \in [0, \pi_p/2]$, where

$$\frac{\pi_p}{2} = \int_0^{(p-1)^{1/p}} \frac{dt}{1 - t^p/(p-1)^{1/p}} = \frac{(p-1)^{1/p}}{p\sin(\pi/p)}\,\pi. \tag{D.3}$$

Since $S_p'(\pi_p/2) = 0$, we define S_p on the interval $[\pi_p/2, \pi_p]$ by $S_p(\psi) = S_p(\pi_p - \psi)$, and for $\psi \in (\pi_p, 2\pi_p]$ we put $S_p(\psi) = -S_p(2\pi_p - \psi)$ and extend S_p as a $2\pi_p$ – periodic function on $\mathbb{R}$. The following properties of S_p will be used frequently (see [31]).

Lemma D.3. *The generalized sine function* S_p *satisfies the following properties.*

(i) S_p *satisfies* (D.2) *on* $\mathbb{R}$; $S_p \in C^1(\mathbb{R})$ *and* $\|S_p\|_\infty = (p-1)^{1/p}$;

(ii) $S_p'|S_p'|^{p-2} \in C^1(\mathbb{R})$, $\|S_p'\|_\infty = 1$ *and* $\|(S_p'|S_p'|^{p-2})'\|_\infty = (p-1)^{(p-1)/p}$;

(iii) *if* $p \le 2$ *then* $S_p' \in C^1(\mathbb{R})$, *while if* $p \ge 2$ *then* $S_p' \in C^{1,1/(p-1)}(\mathbb{R})$;

(iv) $(p-1)S_p''(\psi) = -S_p^{p-1}|S_p'|^{2-p}$, $\psi \in (0,\pi_p)$, $\psi \neq \pi_p/2$.

Clearly, $S_2(\psi) = \sin(\psi)$ and $\pi_2 = \pi$. Notice also that $S_p(t) \to 1 - |t-1|$ as $p \to \infty$, and $S_p(t) \to 0$ as $p \to 1$. The generalized sine function was discussed in great detail by Lindquist in [31].

D.2 Generalized Prüfer transformation

In order to construct a positive solution of (D.1) it is sufficient to solve the initial value problem

$$\begin{cases} -r^{1-N}(r^{N-1}|u_r|^{p-2}u_r)_r - V(r)u^{p-1} = 0 \quad \text{in } (R,+\infty), \\ u(R) = 0, \qquad u'(R) > 0, \end{cases} \tag{D.4}$$

where we set

$$V(r) := \frac{\mu}{r^p} + \frac{\epsilon}{r^p \log^{m_*} r}.$$

Following [14], we use the generalized sine function to transform (D.4) into phase space via the generalized polar coordinates (ρ,ψ) defined by

$$\begin{cases} r^{N-1}u'|u'|^{p-2} &= \rho(r)S_p'(\psi(r))|S_p'(\psi(r))|^{p-2}, \\ Q(r)^{(p-1)/p}u^{p-1} &= \rho(r)S_p^{p-1}(\psi(r)), \end{cases} \tag{D.5}$$

where the function $0 < Q \in C^1(R,+\infty)$ will bee chosen later. A calculation similar to [39, Lemma 2] shows that by means of the generalized polar coordinates (D.5) equation (D.4) transforms into the Cauchy problem

$$\begin{cases} \psi' = V_1|S_p'(\psi)|^p + V_2\frac{S_p^p(\psi)}{p-1} + \frac{1}{p}\frac{Q'}{Q}\,S_p(\psi)S_p'(\psi)|S_p'(\psi)|^{p-2}, & \psi(R) = 0, \\ \rho' = \rho\left\{(V_1 - V_2)\,S_p^{p-1}(\psi)S_p'(\psi) + \frac{1}{p}\frac{Q'}{Q}S_p^p(\psi)\right\}, & \rho(R) > 0, \end{cases} \tag{D.6}$$

in $(R,+\infty)$, where V_1 and V_2 are defined by

$$V_1(r) := r^{\frac{1-N}{p-1}}Q^{\frac{1}{p}}(r), \qquad V_2(r) := r^{N-1}V(r)Q^{\frac{1-p}{p}}(r).$$

Notice also that by means of (D.5) a pair of C^2–functions (ρ,ψ) satisfying (D.6) transforms into a positive solution u to (D.4).

The main feature of system (D.6) is the fact that its first equation is independent of ρ. Notice also that the second equation is linear in ρ and is completely integrable provided the solution ψ of the first equation is given.

For the choice of $Q(r)$ we distinguish between the cases $V(r) > 0$ and $V(r) < 0$. If $V(r) > 0$ then we set

$$Q(r) = V(r)r^{\frac{p(N-1)}{p-1}}. \tag{D.7}$$

Then $V_1 = V_2 = V^{1/p}$ and using Lemma D.3 we rewrite (D.6) in the form

$$\begin{cases} \psi' = V^{1/p} + \left(\frac{1}{p}\frac{V'}{V} + \frac{N-1}{p-1}\frac{1}{r}\right)S_p(\psi)S_p'(\psi)|S_p'(\psi)|^{p-2}, & \psi(R) = 0, \\ \rho' = \rho\left(\frac{1}{p}\frac{V'}{V} + \frac{N-1}{p-1}\frac{1}{r}\right)S_p^p(\psi), & \rho(R) > 0, \end{cases} \tag{D.8}$$

in $(R,+\infty)$. In the case $V(r)<0$ one can choose $Q(r)=-V(r)r^{\frac{p(N-1)}{p-1}}$, however we are not interested in this case below.

The main tools of our analysis of (D.8) will be a simple comparison principle between sub- and super-solutions and a stabilization argument for a time–dependent one-dimensional ODEs. The comparison principle below can be found in [39].

Lemma D.4 (Comparison principle). *Let $f:(R,\infty)\times\mathbb{R}\to\mathbb{R}$ be locally Lipschitz–continuous in $(R,\infty)\times\mathbb{R}$. Let ϕ,φ be C^1–functions on (R,∞), continuous in $[R,\infty)$, and such that*

$$\phi'(r)\le f(r,\phi),\qquad \varphi'(r)\ge f(r,\varphi),\qquad \phi(R)\le\varphi(R).$$

Then $\phi(r)\le\varphi(r)$ in $[R,\infty)$.

Lemma D.5 (Stabilization principle). *Let $f:(R,\infty)\times\mathbb{R}\to\mathbb{R}$ be locally Lipschitz–continuous in $(R,\infty)\times\mathbb{R}$, and $\lim_{r\to\infty}f(r,\xi)=f_*(\xi)$, uniformly on compact subsets of $\mathbb{R}$. Let $0<\eta\in C^1(R,\infty)$ and $\int_R^\infty\eta^{-1}(r)dr=\infty$. Let ψ be a C^1–function on (R,∞) such that*

$$\psi'=\frac{f(r,\psi)}{\eta(r)}\qquad (r>R).$$

Assume that $f(r,\psi(r))>0$ for all $r>R$ and ψ is bounded above. Then $f_*(\psi_*)=0$, where $\psi_*=\lim_{r\to\infty}\psi(r)$.

Proof. Observe that $\psi(r)$ is monotone increasing and uniformly bounded, so the limit ψ_* exists. Assume for a contradiction that $f_*(\psi_*)>0$. Then there exist $\delta>0$ and $R_1>R$ such that $f(r,\psi(r))>\delta$ for all $r>R_1+1$. Then

$$\psi(r)=\psi(R_1)+\int_{R_1}^r\frac{f(s,\psi(s))}{\eta(s)}ds\ge c_1+\int_{R_1+1}^r\frac{\delta}{\eta(s)}ds\to\infty\quad\text{as } r\to+\infty,$$

which contradicts to the boundedness of ψ. Thus the assertion follows. □

D.3 Proof of Theorem D.2

Below we establish the existence and asymptotic behavior of a solution (ψ,ρ) to system (D.8). Then the existence and asymptotic of a positive solution to (D.4) can be computed directly from the asymptotic of ψ and ρ via (D.5) and (D.7).

(i) Case $\mu\in(0,C_H)$, $\epsilon=0$, $p\neq N$. We consider in detail only the case $p>N$, the case $p<N$ being similar.

System (D.8) can be written in the form

$$\psi'=\frac{F(\psi)}{r},\qquad \frac{\rho'}{\rho}=\frac{G(\psi)}{r}\quad\text{in}\quad(R,+\infty),\tag{D.9}$$

where

$$F(\psi):=\mu^{1/p}+\frac{N-p}{p-1}S_p(\psi)S_p'(\psi)|S_p'(\psi)|^{p-2},\qquad G(\psi):=\frac{N-p}{p-1}S_p^p(\psi).$$

Notice that $0<\gamma_-<\gamma_+$. An elementary calculation involving (D.2) shows that $F(\psi)=0$ if and only if ψ satisfies

$$S_p(\psi)=\left((\gamma_\pm(p-1)+(N-p))\frac{p-1}{N-p}\right)^{1/p}\quad\text{and}\quad S_p'(\psi)=\left(\gamma_\pm\frac{p-1}{p-N}\right)^{1/p}.\tag{D.10}$$

Then it follows from the definition and properties of $S_p(\psi)$ that the solutions $\psi_\pm \in (0, \pi_p)$ of $F(\psi) = 0$ are uniquely (modulo $2\pi_p$) determined by $\gamma_\pm$ via (D.10). One can also see that

$$0 < \psi_+ < \psi_- < \frac{\pi_p}{2}.$$

Moreover, $F(\psi)$ is strictly positive for $\psi \in (0, \psi_+)$.

Let $\psi(r)$ be the solution to the problem

$$\psi' = \frac{F(\psi)}{r} \quad \text{in} \quad (R, +\infty), \qquad \psi(R) = 0, \tag{D.11}$$

for some $R > 1$. Observe that the right hand side of (D.11) is bounded and smooth for all $(r, \psi) \in (1, \infty) \times \mathbb{R}$, so $\psi(r)$ exists for all $r > R$. Note also that $\psi_+(r) \equiv \psi_+$ is a stationary solution to (D.11). So, $\psi(r) \le \psi_+$ for all $r > R$, by Lemma D.4. Moreover $\psi(r)$ is monotonically increasing and $F(\psi(r)) > 0$ for all $r > 0$. Thus, by Lemma D.5 we conclude that $\lim_{r\to\infty} \psi(r) = \psi_+$.

Lemma D.6. *Let ψ be the solution to* (D.11). *Then $\psi(r) = \psi_+ + \omega(r)$ where $\omega(r) < 0$ in $[R, +\infty)$ and*

$$\omega(r) = cr^{-(\gamma_+ p + N - p)(1+o(1))} \quad \textit{as} \quad r \to +\infty,$$

for some $c < 0$.

Proof. Since

$$F(\psi) = F'(\psi_+)(\psi - \psi_+) + \Theta(\psi - \psi_+), \tag{D.12}$$

where $\Theta(\psi - \psi_+) = O((\psi - \psi_+)^2)$ as $\psi \to \psi_+$, by Lemma D.3 (iv) we obtain that

$$F'(\psi) = \frac{N-p}{p-1}\left(|S_p'(\psi)|^p + (p-1)S_p(\psi)|S_p'(\psi)|^{p-2}S_p''(\psi)\right) = \frac{N-p}{p-1}\left(|S_p'(\psi)|^p - S_p^p(\psi)\right).$$

Using (D.10) we arrive at $F'(\psi_+) = -(\gamma_+ p + N - p) < 0$. Set $\omega(r) := \psi(r) - \psi_+$. Thus $\omega(R) = -\psi_+$ and ω satisfies

$$\frac{\omega'}{\omega} = \frac{F'(\psi_+)}{r} + \frac{\Theta(\omega)}{r\omega}, \qquad r \in (R, +\infty).$$

Therefore we infer that

$$\log \frac{\omega(r)}{\omega(R)} = \log\left(\frac{r}{R}\right)^{-(\gamma_+ p + N - p)} + \int_R^r \frac{\Theta(\omega)}{\omega}\frac{ds}{s}.$$

So, the assertion follows by the L'Hopital's Rule. □

Given the solution $\psi(r)$ to (D.11), let $\rho(r)$ be the solution to the problem

$$\frac{\rho'}{\rho} = \frac{G(\psi)}{r} \quad \text{in} \quad (R, +\infty), \qquad \rho(R) = 1. \tag{D.13}$$

Observe that the right hand side of (D.13) is bounded and smooth for all $(r, \psi) \in (R, \infty) \times \mathbb{R}$, so $\rho(r)$ exists for all $r > R$.

Lemma D.7. *Let ρ be the solution to* (D.13). *Then $\rho(r) = cr^{(\gamma_+(p-1)+N-p)(1+o(1))}$ as $r \to +\infty$, for some $c > 0$.*

Proof. Observe that ρ satisfies

$$\frac{\rho'}{\rho} = \frac{G(\psi_+)}{r} + \frac{\Xi(\omega(r))}{r}, \qquad r \in (R, +\infty), \tag{D.14}$$

where $\Xi(\psi - \psi_+) = o(\psi - \psi_+)$ as $\psi \to \psi_+$ and $\omega(r) := \psi - \psi_+$ is given by Lemma D.6. Using the definition of G, (D.10) and (B.3) we conclude that $G(\psi_+) = \gamma_+(p-1) + N - p$. Therefore

$$\log \frac{\rho(r)}{\rho(R)} = \log \left(\frac{r}{R}\right)^{\gamma_+(p-1)+N-p} + \int_R^r \Xi(\omega) \frac{ds}{s}.$$

So, the assertion follows by the l'Hopital Rule. □

Remark D.8. The case $\mu \in (0, C_H)$, $\epsilon = 0$ and $p < N$ is similar, the only difference being that if $p < N$ then $\gamma_- < \gamma_+ < 0$ and hence $\pi_p/2 < \psi_+ < \psi_- < \pi_p$.

(ii) Case $\mu = C_H$, $\epsilon = 0$, $p \neq N$. We consider in detail only the case $p > N$, the case $p < N$ being similar. System (D.8) can be written in the form (D.9), where

$$F(\psi) := |\gamma_*| + \frac{N-p}{p-1} S_p(\psi) S_p'(\psi) |S_p'(\psi)|^{p-2}, \qquad G(\psi) := \frac{N-p}{p-1} S_p^p(\psi). \tag{D.15}$$

Notice that $\gamma_* = \frac{p-N}{p} > 0$. A simple analysis shows that $F(\psi) = 0$ if and only if $\psi_* = (\pi/4)_p$ modulo $2\pi_p$, where $(\pi/4)_p \in (0, \pi_p/2)$ denotes the unique solution to the equation

$$S_p(\psi) = S_p'(\psi) = \left(\frac{p-1}{p}\right)^{1/p}. \tag{D.16}$$

It is clear that $(\pi/4)_2 = \pi/4$. Observe that $F(\psi)$ is nonnegative for all $\psi \in \mathbb{R}$ and strictly positive for $\psi \in (0, \psi_*)$.

Let $\psi(r)$ be the solution to (D.11), for some $R > 1$. Clearly $\psi(r)$ exists for all $r > R$. Note also that $\psi_*(r) \equiv \psi_*$ is a stationary solution to (D.11). So, $\psi(r) \leq \psi_*$ for all $r > R$ by Lemma D.4. Moreover, by Lemma D.5 we conclude that $\lim_{r\to\infty} \psi(r) = \psi_*$.

Lemma D.9. *Let ψ be the solution to* (D.11). *Then $\psi(r)$ admits a representation $\psi(r) = \psi_* + \omega(r)$ where $\omega(r) < 0$ in $[R, +\infty)$ and*

$$\omega(r) = -\frac{2}{p}\frac{p-1}{p-N}\frac{1+o(1)}{\log r} \quad \textit{as} \quad r \to +\infty. \tag{D.17}$$

Proof is similar to the arguments in the proof of Lemma D.6. Notice only that $F'(\psi_*) = F(\psi_*) = 0$ and $F''(\psi_*) = \frac{(p-N)\,p}{p-1}$, so use $F(\psi) = \frac{1}{2}F''(\psi_*)(\psi - \psi_*)^2 + o((\psi - \psi_*)^2)$ instead of (D.12).

Lemma D.10. *Let ρ be the solution to* (D.11). *Then $\rho(r) = cr^{-\gamma_*}(\log r)^{\left(\frac{2(p-1)}{p}\right)(1+o(1))}$ as $r \to +\infty$, for some $c > 0$.*

Proof is essentially the same as the one for Lemma D.7, the only difference being that instead of (D.14) one uses

$$\frac{\rho'}{\rho} = \frac{G(\psi_*)}{r} + \frac{G'(\psi_*)\omega(r)}{r} + \frac{o(\omega(r))}{r}, \qquad r \in (R, +\infty),$$

where $G(\psi_*) = -\gamma_*$, $G'(\psi_*) = N - p$.

Remark D.11. The case $\mu = C_H$, $\epsilon = 0$ and $p < N$ is similar, the only difference being that $\gamma_* < 0$ and hence $\psi_* = (3\pi/4)_p := \pi_p - (\pi/4)_p$.

(iii) Case $\mu = C_H$, $\epsilon \in (0, C_*)$, $p = N$. In this case system (D.8) can be written in the form

$$\psi' = \frac{F(\psi)}{r \log r}, \qquad \frac{\rho'}{\rho} = \frac{G(\psi)}{r \log r} \quad \text{in} \quad (R, +\infty),$$

where

$$F(\psi) := \epsilon^{1/N} - S_N S_N' |S_N'|^{N-2}, \qquad G(\psi) := -S_N^N.$$

A simple calculation shows that $F(\psi) = 0$ if and only if

$$S_N(\psi) = (1 - \beta_\pm)^{2/N}, \qquad S_N'(\psi) = \beta_\pm^{1/N}, \tag{D.18}$$

where $\beta_\pm$ are roots of (B.4). Note that $0 < \beta_- < \frac{N-1}{N} < \beta_+ < 1$ and hence the solutions $\psi_\pm \in (0, \pi_N)$ of (D.18) are uniquely (modulo $2\pi_N$) determined and satisfy

$$0 < \psi_+ < \frac{\pi_N}{2} < \psi_- < \pi_N.$$

Observe that $F(\psi)$ is smooth, bounded and nonnegative for all $\psi \in \mathbb{R}$ and strictly positive for $\psi \in (0, \psi_+)$. Let $\psi(r)$ be the solution to the problem

$$\psi' = \frac{(\psi)}{r \log r}, \qquad \psi(R) = 0, \tag{D.19}$$

in $(R, +\infty)$, for some $R > e$. Note also that $\psi_+(r) \equiv \psi_+$ is a stationary solution to (D.19). So, $\psi(r) \le \psi_+$ for all $r > R$ by Lemma D.4. Thus, by Lemma D.5 we conclude that $\lim_{r\to\infty} \psi(r) = \psi_+$.

Lemma D.12. *Let ψ be the solution to* (D.19). *Then $\psi(r)$ admits a representation $\psi(r) = \psi_+ + \omega(r)$ where $\omega(r) < 0$ in $[R, +\infty)$ and*

$$\omega = c(\log r)^{(N(1-\beta_+)-1)(1+o(1))} \quad \textit{as} \quad r \to +\infty,$$

for some $c < 0$.

The proof is the literary repetition of the arguments in the proof of Lemma D.6. Note only that $F(\psi_+) = 0$, $F_N'(\psi_+) = (1 - \beta_+)N - 1$.

Given the solution $\psi(r)$ to (D.19), let $\rho(r)$ be the solution to the problem

$$\frac{\rho'}{\rho} = \frac{G(\psi)}{r \log r} \quad \text{in} \quad (R, +\infty), \qquad \rho(R) = 1. \tag{D.20}$$

Observe that the right hand side of (D.20) is bounded and smooth for all $(r, \psi) \in (R, \infty) \times \mathbb{R}$, so $\rho(r)$ exists for all $r > R$.

Lemma D.13. *Let ρ be the solution to* (D.20). *Then $\rho(r) = c(\log r)^{(\beta_+-1)(N-1)(1+o(1))}$ as $r \to +\infty$, for some $c > 0$.*

The proof is the literary repetition of the arguments in Lemma D.7. Notice only that $G(\psi_+) = (1 - \beta_+)(N - 1)$.

(iv) Case $\mu = C_H$, $\epsilon \in (0, C_*)$, $p \neq N$. We consider in detail only the case $p > N$, the case $p < N$ being similar.

The equations in system (D.8) can be written in the form

$$\psi' = \frac{F_\varepsilon(r,\psi)}{r}, \qquad \frac{\rho'}{\rho} = \frac{G_\varepsilon(r,\psi)}{r} \quad \text{in} \quad (R, +\infty), \tag{D.21}$$

where we use the notation

$$F_\varepsilon(r,\psi) := U(r) + W(r) S_p(\psi) S_p'(\psi) |S_p'(\psi)|^{p-2}, \qquad G_\varepsilon(r,\psi) := W(r) S_p^p(\psi),$$

$$U(r) := \left(C_H + \frac{\epsilon}{\log^2 r}\right)^{1/p}, \qquad W(r) := \frac{N-p}{p-1} - \frac{2\epsilon}{p\,(C_H \log^2 r + \epsilon)\log r}.$$

Observe that $\frac{U(r)}{W(r)} \neq const$, so the first equation in (D.8) has no stationary solutions. For $\beta > 0$, denote

$$A(r) := \gamma_* + \frac{\beta}{\log r}.$$

Below we suppress the dependence on r in $U(r)$ and $A(r)$ writing simply U and A. For $r > e$, let $\psi_\beta(r)$ be defined as the solution to the system

$$S_p(\psi) = \frac{U}{\left(|A|^p + \frac{U^p}{p-1}\right)^{1/p}} \quad \text{and} \quad S_p'(\psi) = \frac{A}{\left(|A|^p + \frac{U^p}{p-1}\right)^{1/p}}, \tag{D.22}$$

satisfying $0 \leq \psi_\beta < \frac{\pi_p}{2}$. From the definition of U and A one can see that $\lim_{r\to\infty} \psi_\beta(r) = (\pi/4)_p$, where $(\pi/4)_p$ is defined by (D.16).

Lemma D.14. *Let $\beta > 0$. The following assertions are valid.*

(i) If $\beta \in (\beta_-, \beta_+)$ then there exists $R_\beta > e$ such that $\psi_\beta(r)$ is a positive super-solution to the equation

$$\psi' = \frac{F_\varepsilon(r,\psi)}{r} \quad \textit{in} \quad (R_\beta, +\infty). \tag{D.23}$$

(ii) If $\beta \notin [\beta_-, \beta_+]$ then there exists $R_\beta > e$ such that $\psi_\beta(r)$ is a sub-solution to (D.23)*.*

Proof. A routine calculation based on (D.22) gives that

$$\psi_\beta' = \frac{1}{r}\left(F_\varepsilon(r,\psi_\beta) + \frac{U}{A^p + \frac{U^p}{p-1}}\,\Theta(r)\right),$$

where

$$\begin{aligned}
\Theta(r) &:= A^{p-2}\frac{\beta}{\log^2 r} - A^p - \frac{U^p}{p-1} - \frac{N-p}{p-1}A^{p-1} \\
&= \frac{\beta}{\log^2 r}\left|\gamma_* + \frac{\beta}{\log r}\right|^{p-2} + \left(\frac{\gamma_*}{p-1} - \frac{\beta}{\log r}\right)\left|\gamma_* + \frac{\beta}{\log r}\right|^{p-1} \\
&\quad - \frac{\epsilon}{p-1}\frac{1}{\log^2 r} - \frac{|\gamma_*|^p}{p-1}.
\end{aligned}$$

For $r \to +\infty$ we obtain

$$\Theta(r) = -\frac{|\gamma_*|^{p-2}}{2p}\frac{(\beta-\beta_+)(\beta-\beta_-)}{\log^2(r)} + o\left(\frac{1}{\log^3 r}\right).$$

Thus the assertion follows. □

Set $b := \beta_+ - \delta$, $B := \beta_+ + \delta$, where $\delta > 0$ is chosen such that $\frac{1}{p} < b < \beta_+ < B < \frac{2}{p}$. By Lemma D.14 there exist R_b and R_B such that ψ_b and ψ_B are sub- and super-solution to (D.23), respectively. Set $R_\delta := \max\{R_b, R_B\}$. It follows from (D.22) that $\psi_B(R_\delta) < \psi_b(R_\delta)$. Let $\psi_*(r)$ be the solution to the problem

$$\psi_*' = \frac{F_\varepsilon(r, \psi_*)}{r} \quad \text{in} \quad (R_\delta, +\infty), \qquad \psi_*(R_\delta) = \psi_0, \tag{D.24}$$

where $\psi_0 \in (\psi_B(R_\delta), \psi_b(R_\delta))$. Observe that $F_\varepsilon(r, \psi)$ is smooth and bounded, so ψ_* exists for all $r > R_\delta$. Moreover, by Lemma D.4 we conclude that

$$\psi_B(r) \le \psi_*(r) \le \psi_b(r), \qquad r \in [R_\delta, +\infty), \tag{D.25}$$

and, one can see that $F_\varepsilon(r, \psi_*(r)) > 0$ in $[R_\delta, +\infty)$. Observe also that $\lim_{r\to\infty} \psi_*(r) = (\pi/4)_p$.

Let $\psi(r)$ be the solution to the problem

$$\psi' = \frac{F_\varepsilon(r, \psi)}{r}, \qquad \psi(R_\delta) = 0. \tag{D.26}$$

Clearly, $\psi(r)$ exists for all $r > R_\delta$. By Lemma D.4 one has $0 \le \psi(r) \le \psi_*(r)$. Hence using the definitions of F_ε, S_p and S_p' one can see that $F_\varepsilon(r, \psi(r))$ is strictly positive in $[R_\delta, +\infty)$. Notice that

$$\lim_{r\to\infty} F_\varepsilon(r, \psi) = F(\psi),$$

uniformly in ψ, where F is defined by (D.15). Thus, by Lemma D.5 we conclude that $\lim_{r\to\infty} \psi(r) = (\pi/4)_p$.

Lemma D.15. *Let ψ be the solution to* (D.26) *and ψ_* be the solution to* (D.24). *Then $\omega(r) := \psi(r) - \psi_*(r) < 0$ in $[R_\delta, +\infty)$ and satisfies the inequality*

$$c_1\omega_+(r) \le \omega(r) \le c_2\omega_-(r),$$

for some $c_1 < 0$, $c_2 < 0$, where $\omega_+(r) = (\log r)^{-bp(1+o(1))}$, $\omega_-(r) = (\log r)^{-Bp(1+o(1))}$ as $r \to +\infty$.

Proof. Note that $\omega(R_\delta) = -\psi_0$ and $\omega(r) \to 0$ as $r \to +\infty$. Fix $r > R_\delta$. Near $\psi_*(r)$ we have

$$F_\varepsilon(r, \psi) = F_\varepsilon(\psi_*) + (F_\varepsilon)_\psi'(r, \psi_*)(\psi - \psi_*) + \frac{1}{2}(F_\varepsilon)_\psi''(r, \psi_*)(\psi - \psi_*)^2 + \Theta(r, \psi - \psi_*) \tag{D.27}$$

where $\Theta(r, \psi - \psi_*) = o((\psi - \psi_*)^2)$ as $\psi \to \psi_*$. A direct computation gives

$$(F_\varepsilon(r, \psi))_\psi' = W(r)\,(|S_p'(\psi)|^p - |S_p(\psi)|^p); \tag{D.28}$$

$$(F_\varepsilon(r, \psi))_\psi'' = -\frac{p^2}{p-1}\, W(r)|S_p(\psi)|^{p-1}S_p'(\psi). \tag{D.29}$$

Since $\psi = \psi_* + \omega$, and ψ_* solves the same equation, from (D.27) we obtain

$$\omega' = (F_*)_\psi'(\psi_*)\frac{\omega}{r} + \frac{(F_*)_\psi''(\psi_*)}{2}\,\frac{\omega^2}{r} + \frac{\Psi(\omega)}{r}. \tag{D.30}$$

Using (D.25), (D.22), (D.28) and (D.29) we conclude that

$$-\frac{Bp}{\log r} + o\left(\frac{1}{\log^2 r}\right) \le (F_*)_\psi'(\psi_*(r)) \le -\frac{bp}{\log r} + o\left(\frac{1}{\log^2 r}\right),$$

$$\frac{bp^2}{p-1}\frac{1}{\log r}+o\left(\frac{1}{\log^2 r}\right)\le (F_*)''_{\psi}(\psi_*(r))-\frac{(p-N)p}{p-1}\le \frac{Bp^2}{p-1}\frac{1}{\log r}+o\left(\frac{1}{\log^2 r}\right),$$

as $r\to+\infty$. We substitute the above estimates into (D.30). Then

$$\omega'\le -Bp\frac{\omega}{r\log r}+\frac{(p-N)p}{2(p-1)}\frac{\omega^2}{r}+\frac{Bp^2}{2(p-1)}\frac{\omega^2}{r\log r}+O\left(\frac{\omega+\omega^2}{r\log^2 r}+\frac{\omega^3}{r}\right), \tag{D.31}$$

$$\omega'\ge -bp\frac{\omega}{r\log r}+\frac{(p-N)p}{2(p-1)}\frac{\omega^2}{r}+\frac{bp^2}{2(p-1)}\frac{\omega^2}{r\log r}+O\left(\frac{\omega+\omega^2}{r\log^2 r}+\frac{\omega^3}{r}\right), \tag{D.32}$$

as $r\to+\infty$. From (D.32) we infer that

$$\frac{\omega'}{\omega}\le -\frac{bp}{r\log r}+\frac{bp^2}{p-1}\frac{\omega}{r\log r}+O\left(\frac{1+\omega}{r\log^2 r}+\frac{\omega^2}{r}\right),$$

and, hence,

$$\log\frac{\omega(r)}{\omega(R)}\le c\log(\log r)^{-bp(1+o(1))},$$

or, equivalently,

$$\omega(r)\ge -c(\log r)^{-bp(1+o(1))},$$

as $r\to+\infty$. Therefore by (D.31) we infer that ω satisfies

$$\frac{\omega'}{\omega}\ge -\frac{Bp}{r\log r}-\frac{c}{r(\log r)^{bp(1+o(1))}}+\frac{Bp^2}{2(p-1)}\frac{\omega}{r\log r}+O\left(\frac{1+\omega}{r\log^2 r}+\frac{\omega^2}{r}\right),$$

and, hence,

$$\log\frac{\omega(r)}{\omega(R)}\ge \log(\log r)^{-Bp(1+o(1))}-c(\log r)^{-bp(1+o(1))+1},$$

as $r\to+\infty$. Since $b>\frac{1}{p}$, one has

$$\omega(r)\le -c(\log r)^{-Bp(1+o(1))},$$

as $r\to+\infty$. The assertion follows. □

Given the solution $\psi(r)$ to the problem (D.26), let $\rho(r)$ be the solution to the problem

$$\frac{\rho'}{\rho}=\frac{G_\varepsilon(r,\psi)}{r}\quad\text{in}\quad (R_\delta,+\infty),\qquad \rho(R_\delta)=1. \tag{D.33}$$

Clearly the right hand side of (D.33) is bounded and smooth for all $(r,\psi)\in(R_\beta,\infty)\times\mathbb{R}$, so $\rho(r)$ exists for all $r>R_\beta$.

Lemma D.16. *Let ρ be the solution to (D.33). Then ρ satisfies the estimate*

$$c_1\rho_-(r)\le\rho(r)\le c_2\rho_+(r)$$

for some $c_1>0$, $c_2>0$, where

$$\rho_-(r)=r^{-\gamma_*}(\log r)^{b(p-1)(1+o(1))},\quad \rho_+(r)=r^{-\gamma_*}(\log r)^{B(p-1)(1+o(1))}$$

as $r\to+\infty$.

Proof. Near $\psi_*(r)$ we have $G_\varepsilon(r,\psi) = G_\varepsilon(r,\psi_*) + \Xi(r,\psi-\psi_*)$, where $\Xi(r,\psi-\psi_*) = O(\psi-\psi_*)$ as $r \to +\infty$. Using (D.25) we obtain

$$W(r)S_p^p(\psi_b) \le G_\varepsilon(r,\psi_*) \le W(r)S_p^p(\psi_B).$$

By a simple computation we conclude that

$$\frac{b(p-1)}{\log r} + \frac{o(1)}{\log^2 r} \le G_\varepsilon(r,\psi_*) + \gamma_* \le \frac{B(p-1)}{\log r} + \frac{o(1)}{\log^2 r},$$

as $r \to +\infty$. Therefore ρ satisfies

$$\frac{b(p-1)}{r\log r} + O\left(\frac{1}{r\log^2 r} + \frac{\omega}{r}\right) \le \frac{\rho'}{\rho} + \frac{\gamma_*}{r} \le \frac{B(p-1)}{r\log r} + O\left(\frac{1}{r\log^2 r} + \frac{\omega}{r}\right),$$

as $r \to +\infty$. Thus we have

$$\log\left(\frac{\log r}{\log R}\right)^{b(p-1)} + \frac{O(1)}{\log r} + c(\log r)^{-bp(1+o(1))+1}$$
$$\le \log\frac{\rho(r)}{\rho(R)} - \log\left(\frac{r}{R}\right)^{-\gamma_*} \le \log\left(\frac{\log r}{\log R}\right)^{B(p-1)} + \frac{O(1)}{\log r} + c(\log r)^{-Bp(1+o(1))+1},$$

as $r \to +\infty$. So, the assertion follows from $\frac{1}{p} < b < B$. □

Remark D.17. The case $\mu = C_H$, $\epsilon \in (0, C_*)$ and $p < N$ is similar, the only difference being $\gamma_* < 0$ and hence $\lim_{r\to+\infty}\psi_*(r) = (3\pi/4)_p$.

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